

COMPLETE ANALYSIS OF $MAP/G/1/N$ QUEUE WITH SINGLE (MULTIPLE) VACATION(S) UNDER LIMITED SERVICE DISCIPLINE

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We consider a finite-buffer single-server queue with Markovian arrival process (MAP) where the server serves a limited number of customers, and when the limit is reached it goes on vacation. Both single- and multiple-vacation policies are analyzed and the queue length distributions at various epochs, such as pre-arrival, arbitrary, departure, have been obtained. The effect of certain model parameters on some important performance measures, like probability of loss, mean queue lengths, mean waiting time, is discussed. The model can be applied in computer communication and networking, for example, performance analysis of token passing ring of LAN and SVC (switched virtual connection) of ATM.

1. Introduction

In B-ISDN/ATM network, IP packets or cells of voice, video, data are sent over a common transmission channel on statistical multiplexing basis. The performance analysis of statistical multiplexer whose input consists of a superposition of several packetized sources is not a straightforward one. The difficulty in modelling this type of traffic is due to the correlated structure of arrivals. A common approach is to approximate this complex non-renewal input process by analytically tractable arrival process, namely, Markovian arrival process (MAP). This type of arrival process includes many familiar input processes, such as Markov modulated Poisson process (MMPP), PH-type renewal process, interrupted Poisson process (IPP), Poisson process. It was first introduced by Lucantoni et al. [13].

Vacation queue is an efficient and easy way to analyze queues in cases where a single channel is serving more than one queue. Many complex queueing systems like priority queue, cyclic service polling station, and so forth can be described as vacation queue. For more details and versatile implementation of vacation queue, one can refer to the comprehensive survey by Doshi [3] and the book by Takagi [18]. In fact, there exists a wealth of article on various policies with vacations. Brill and Harris [2] studied $M/G/1$ queue with server vacation where each vacation taken by the server will depend on the immediately prior vacation type via a Markov chain. Vacation queues have also been studied for the case of bulk arrival and bulk service, for example, by Dshalalow and

Yellen [8], Lee et al. [12], and Selvam and Sivasankaran [17], whereas in [12] the analysis of $M/G^b/1$ queue with multiple and single vacations has been carried out, in [8] the analysis of $M^X/G/1$ queue with (r, N) -policy has been presented, and in [17] a two-phase (customers receive a batch service in the first phase and individual services in the second phase) queueing model with vacations has been studied. Besides these, Dshalalow [4] studied $M^X/G^Y/1$ queueing system, where the server takes vacations each time the queue level falls to $r (\geq 1)$. The analysis of the queueing systems with bulk input, batch state-dependent service with server vacations, and three post-vacation disciplines have been performed in Dshalalow's [5, 6]. Here the server leaves the system whenever the queue falls below $r (\geq 1)$, and resumes service when during his absence the system replenishes to $N (\geq r)$ upon one of his returns, and also "the post-vacation" period is characterized by three different disciplines: waiting, or leaving on multiple vacations with or without emergency. Further in [7], he discussed the time-dependent analysis of multivariate delayed marked renewal processes which has wide application in a queueing system with vacations under N- and D-policy.

In recent years there has been a great interest in analyzing queueing systems with vacations and *MAP* as input process: *MAP/G/1* queue, see, for example, Lucantoni et al. [13]. Blondia [1] analyzed *MAP/G/1/N* queue with multiple vacations for two types of service disciplines: (i) exhaustive service discipline and (ii) limited service discipline. A more general study of *MAP/G/1/N* queue with single (multiple) vacation(s) along with setup and close-down time can be found in Niu and Takahashi [16].

In this paper, we analyze *MAP/G/1/N* queue with single (multiple) vacation(s) under limited service discipline, that is, a fixed limit L is placed before the server and the server will at most serve L customers during a busy period before going for vacation(s). The model was earlier analyzed by Blondia, however he only considered multiple-vacation policy, but in several applications single-vacation policy is more effective as it utilizes the server more efficiently. In this paper, we use imbedded Markov chain technique and supplementary variable technique to obtain queue length distributions at various epochs. Thereafter we obtain the LST of actual waiting time distribution. We present an unified approach to analyze both single- and multiple-vacation models together and for that we define an indicator function (δ_S) as follows:

$$\delta_S = \begin{cases} 1, & \text{for single-vacation policy,} \\ 0, & \text{for multiple-vacation policy.} \end{cases} \quad (1.1)$$

That is, by fixing $\delta_S = 1$, one can get the results for single-vacation policy and, similarly, $\delta_S = 0$ gives the results for multiple-vacation policy. For the sake of notational convenience the model is denoted by *MAP/G/1/N/LS*, *SV*, *MV*, where *LS* stands for "limited service discipline," *SV* stands for "single vacation policy," and *MV* stands for "multiple-vacation policy." Finally it may be remarked here that results of special case of our model such as, *M/G/1/N* queue with multiple vacations and limited service discipline earlier analyzed by Lee [11], can be obtained from our analysis.

It may be remarked here that though the model was previously analyzed in [1] for the case of multiple-vacation policy. Our analysis differs from that in several ways; for

example, (i) we analyze both single- and multiple-vacation policies, whereas he considered only multiple-vacation policy, (ii) we provide simple and straightforward relations among various epoch probabilities, and finally, (iii) we suggest a computational procedure for evaluating state probabilities.

2. Description of the model

We consider a *MAP/G/1/N* queue with vacations and limited service disciplines where N is the capacity of the queue. The server is allowed to serve a maximum of L customers during each visit to the queue, that is, the server goes for a vacation if either the queue has been emptied or L customers have been served, whichever occurs earlier. Obviously, when $L \rightarrow \infty$ the service process is equivalent to exhaustive service discipline.

The arrival process is *MAP* which is a rich class of arrival processes and arrivals are governed by an underlying M -state Markov chain. It is characterized by the matrices $\mathbf{C} = [c_{ij}]$, and $\mathbf{D} = [d_{ij}]$, $1 \leq i, j \leq M$, where c_{ij} is the state transition rate from state i to state j in the underlying Markov chain without an arrival and d_{ij} is the state transition rate from state i to state j in the underlying Markov chain with an arrival. The matrix \mathbf{C} has nonnegative off-diagonal and negative diagonal elements, and the matrix \mathbf{D} has nonnegative elements. Let $N(t)$ denote the number of customers arriving in $(0, t]$ and $J(t)$ the state of the underlying Markov chain at time t with state space $\{i : 1 \leq i \leq M\}$. Then $\{N(t), J(t)\}$ is a two-dimensional Markov process with state space $\{(n, i) : n \geq 0, 1 \leq i \leq M\}$. The infinitesimal generator of the above Markov process is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{C} & \mathbf{D} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{C} & \mathbf{D} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{D} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.1}$$

Then $\{N(t), J(t)\}$ is called the *MAP*. Since \mathbf{Q} is the infinitesimal generator of the *MAP*, we have $(\mathbf{C} + \mathbf{D})\mathbf{e} = \mathbf{0}$, where \mathbf{e} is an $M \times 1$ vector with all its elements equal to 1. Since $(\mathbf{C} + \mathbf{D})$ is the infinitesimal generator of the underlying Markov chain $\{J(t)\}$, there exists a stationary probability vector $\bar{\pi}$ such that $\bar{\pi}(\mathbf{C} + \mathbf{D}) = \mathbf{0}$, $\bar{\pi}\mathbf{e} = 1$. The fundamental arrival rate of the above Markov process is given by $\lambda^* = \bar{\pi}\mathbf{D}\mathbf{e}$. For more details on this topic see Lucantoni et al. [13].

Let $S(x)\{s(x)\}[S^*(\theta)]$ be the distribution function (DF) {probability density function (pdf)} [Laplace-Stieltjes transform (LST)] of the service time S of a typical customer. Let $V(x)\{v(x)\}[V^*(\theta)]$ be the DF {pdf} [LST] of a typical vacation time V of the server. The mean service (resp., vacation) time is $E(S) = -S^{*(1)}(0)$ (resp., $E(V) = -V^{*(1)}(0)$), where $f^{*(j)}(\zeta)$ is the j^{th} ($j \geq 1$) derivative of $f^*(\theta)$ at $\theta = \zeta$. The service and vacation times are assumed to be i.i.d. random variables and are independent of the arrival process. The traffic intensity is given by $\rho = \lambda^*E(S)$. Further, let ρ' be the probability that the server is busy. The state of the system at time t is described by the r.v.s., namely, $\xi(t) = \{l\}(0)[d]$ if the server is {serving l th ($1 \leq l \leq L$) customer in the present busy period}{on vacation}[on dormancy], $N_q(t)$ = number of customers present in the queue, $J(t)$ = state

of the underlying Markov chain of MAP, $\tilde{S}(t)$ = remaining service time of the customer in service, $\tilde{V}(t)$ = remaining vacation time of the server.

We define for $1 \leq i \leq M$ the joint probability densities of queue length $N_q(t)$, the state of the server $J(t)$, and the remaining service (vacation) time $\tilde{S}(\tilde{V})$, respectively, by

$$\begin{aligned} \pi_{i,l}(n,x;t)\Delta x &= P\{N_q(t) = n, J(t) = i, x < \tilde{S}(t) < x + \Delta x, \xi(t) = l\}, \\ &0 \leq n \leq N, \quad 1 \leq l \leq L, \quad x \geq 0, \\ \omega_i(n,x;t)\Delta x &= P\{N_q(t) = n, J(t) = i, x < V(t) < x + \Delta x, \xi(t) = 0\}, \\ &0 \leq n \leq N, \quad x \geq 0, \\ \nu_i(0;t) &= P\{N_q(t) = 0, J(t) = i, \xi(t) = d\}. \end{aligned} \tag{2.2}$$

As we will discuss the model in steady state, in limiting case, that is, when $t \rightarrow \infty$ the above probabilities will be denoted by $\pi_{i,l}(n,x)$, $\omega_i(n,x)$, and $\nu_i(0)$. We further define the row vectors of order $1 \times M$,

$$\begin{aligned} \boldsymbol{\pi}_i(n,x) &= [\pi_{i,l}(n,x)], \quad \boldsymbol{\omega}(n,x) = [\omega_i(n,x)], \quad \boldsymbol{\nu}(0) = [\nu_i(0)], \\ &1 \leq i \leq M, \quad 0 \leq n \leq N, \quad 1 \leq l \leq L, \end{aligned} \tag{2.3}$$

where $\boldsymbol{\pi}_i(n)$ is the $1 \times M$ vector whose i th component is $\pi_{i,l}(n)$ and it denotes the joint probability that there are $n(0 \leq n \leq N)$ customers in the queue and the state of the arrival process is $i(1 \leq i \leq M)$ when the server is serving the l th ($1 \leq l \leq L$) customer in the present busy period at arbitrary time. Similarly, $\boldsymbol{\omega}(n)$ is also $1 \times M$ vector whose i th component is $\omega_i(n)$ and it denotes the arbitrary epoch probability that there are n customers in the queue and state of the arrival process is i when the server is on vacation. Also $\boldsymbol{\nu}(0)$ is a vector of order $1 \times M$ whose i th component is $\nu_i(0)$ and it denotes the probability that the server is in dormancy state with zero customer in the queue and phase of the arrival process is i .

3. Queue length distributions at various epochs

3.1. Queue length distributions at service completion and vacation termination epochs.

Consider the system at service completion/vacation termination epochs which are taken as imbedded points. Let t_0, t_1, t_2, \dots be the time epochs at which either service completion or vacation termination occurs. The state of the system at t_i is defined as $\{N_q(t_i), \xi(t_i), J(t_i)\}$, where $N_q(t_i)$ is the number of customers in the queue at time t_i , $\xi(t_i)$ denotes the nature of the imbedded point at time t_i , and $J(t_i)$ is the phase of the arrival process. $\xi(t_i) = 0$ indicates that the imbedded point is a vacation termination instant and $\xi(t_i) = l$ ($1 \leq l \leq L$) indicates the imbedded point is a service completion instant of the l th customer in the present busy period. Let the limiting probability distributions exist,

then we have

$$\begin{aligned} \pi_{m,l}^+(n) &= \lim_{i \rightarrow \infty} P(N_q(t_i) = n, \xi(t_i) = l, J(t_i) = m), \quad 0 \leq n \leq N, 1 \leq l \leq L, 1 \leq m \leq M, \\ \omega_m^+(n) &= \lim_{i \rightarrow \infty} P(N_q(t_i) = n, \xi(t_i) = 0, J(t_i) = m), \quad 0 \leq n \leq N, 1 \leq m \leq M, \end{aligned} \tag{3.1}$$

$\pi_{m,l}^+(n)$ represents the probability that there are n customers in the queue and the state of the arrival process is $m(1 \leq m \leq M)$ at service completion epoch of the l th ($1 \leq l \leq L$) customer in the present busy period. Similarly, let $\omega_m^+(n)$ represent the probability that there are n customers in the queue and the state of the arrival process is $m(1 \leq m \leq M)$ at vacation termination epoch. Further we denote the row vectors of order $1 \times M$,

$$\boldsymbol{\pi}_l^+(n) = [\pi_{m,l}^+(n)], \quad \boldsymbol{\omega}^+(n) = [\omega_m^+(n)], \quad 1 \leq m \leq M. \tag{3.2}$$

Let $\mathbf{A}_n(\mathbf{V}_n), n \geq 0$, denote an $M \times M$ matrix whose (i, j) th element represents the conditional probability that n customers have been accepted during a service (vacation) time of a customer and the underlying Markov chain is in phase j at the end of the service (vacation) time given that the underlying Markov chain was in phase i at the beginning of the service (vacation). Further, we denote \mathbf{A}'_n and \mathbf{V}'_n by

$$\mathbf{A}'_n = \sum_{k=n}^N \mathbf{A}_k, \quad \mathbf{V}'_n = \sum_{k=n}^N \mathbf{V}_k, \quad 0 \leq n \leq N. \tag{3.3}$$

Observing the system immediately after each imbedded point, we have the transition probability matrix (TPM) \mathcal{P} with four block matrices of the form

$$\mathcal{P} = \begin{bmatrix} \boldsymbol{\Xi}_{(N+1)LM \times (N+1)LM} & \boldsymbol{\Psi}_{(N+1)LM \times (N+1)M} \\ \boldsymbol{\Delta}_{(N+1)M \times (N+1)LM} & \boldsymbol{\Phi}_{(N+1)M \times (N+1)M} \end{bmatrix}_{(N+1)(L+1)M \times (N+1)(L+1)M}, \tag{3.4}$$

where $\boldsymbol{\Xi}$ describes the probability of transitions among the service completion epochs. The elements of this block are of the form as follows:

$$\boldsymbol{\Xi}_{i,j} = \begin{cases} \mathbf{Q}_{j-i+1} & 1 \leq i \leq N, i-1 \leq j \leq N-1, \\ \mathbf{Q}_{j-i+1}^c & 1 \leq i \leq N, j = N, \\ \mathbf{Z} & \text{otherwise,} \end{cases} \tag{3.5}$$

where $\mathbf{Z}, \mathbf{Q}_r, \mathbf{Q}_r^c, 0 \leq r \leq N$ are all matrices of order $L \times L$. The elements of \mathbf{Z} matrix are null matrix of order $M \times M$. The matrices \mathbf{Q}_r and \mathbf{Q}_r^c are given by

$$\begin{aligned} (\mathbf{Q}_r)_{l,l'} &= \begin{cases} \mathbf{A}_r, & 1 \leq l \leq L, l' = l+1, \\ \mathbf{0}, & \text{otherwise,} \end{cases} \\ (\mathbf{Q}_r^c)_{l,l'} &= \begin{cases} \mathbf{A}'_r, & 1 \leq l \leq L, l' = l+1, \\ \mathbf{0}, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.6}$$

where $\mathbf{0}$ is a null matrix of order $M \times M$. Ψ gives the probability of transition from any service completion epoch to the next vacation termination epochs. The structure of this block is given by

$$\Psi_{(i,l)(j,0)} = \begin{cases} \mathbf{V}_j, & i = 0, 1 \leq l \leq L, 0 \leq j \leq N - 1, \\ \mathbf{V}'_j, & i = 0, 1 \leq l \leq L, j = N, \\ \mathbf{V}_{j-i}, & 1 \leq i \leq N - 1, l = L, i \leq j \leq N - 1, \\ \mathbf{V}'_{j-i}, & 1 \leq i \leq N, l = L, j = N, \\ \mathbf{0}, & \text{otherwise.} \end{cases} \quad (3.7)$$

Δ of TPM gives the probability of transition from every vacation termination epoch to the next service completion epochs. This block is of the form given below:

$$\Delta_{(i,0)(j,l)} = \begin{cases} \delta_S \bar{\mathbf{D}} \mathbf{A}_j, & i = 0, l = 1, 0 \leq j \leq N - 1, \\ \delta_S \bar{\mathbf{D}} \mathbf{A}'_j, & i = 0, l = 1, j = N, \\ \mathbf{A}_{j-i+1}, & 1 \leq i \leq N - 1, l = 1, i - 1 \leq j \leq N - 1, \\ \mathbf{A}'_{j-i+1}, & 1 \leq i \leq N, l = 1, j = N, \\ \mathbf{0}, & \text{otherwise.} \end{cases} \quad (3.8)$$

Note that the factor $\bar{\mathbf{D}} = (-\mathbf{C})^{-1} \mathbf{D}$ represents the phase transition matrix during an inter-arrival time, for details see Lucantoni et al. [13]. Φ of the TPM describes the probability of transitions among vacation termination epochs. This block matrix is of the form

$$\Phi_{(i,0)(j,0)} = \begin{cases} (1 - \delta_S) \mathbf{V}_j, & i = 0, 0 \leq j \leq N, \\ \mathbf{0}, & \text{otherwise.} \end{cases} \quad (3.9)$$

Now we can obtain the unknown probability vectors $\pi_l^+(n) (0 \leq n \leq N, 1 \leq l \leq L)$ and $\omega^+(n) (0 \leq n \leq N)$ by solving the system of equations

$$(\pi_l^+(n), \omega^+(n)) = (\pi_l^+(n), \omega^+(n)) \mathcal{P}. \quad (3.10)$$

3.2. Queue length distributions at departure epochs. In this sequel we present queue length distributions at departure epoch through the relations between distributions of number of customers in the queue at service completion and departure epochs. Let $\mathbf{p}_l^+(n) (0 \leq n \leq N, 1 \leq l \leq L)$ denote the row vector whose i th element represents steady-state probability that there are $n (0 \leq n \leq N)$ customers in the queue and the phase of the arrival process is $i (1 \leq i \leq M)$ at departure epoch of the l th $(1 \leq l \leq L)$ customer in the present busy period, and since $\mathbf{p}_l^+(n)$ is proportional to $\pi_l^+(n)$ and $\sum_{n=0}^N \sum_{l=1}^L \mathbf{p}_l^+(n) \mathbf{e} = \mathbf{1}$, we get

$$\mathbf{p}_l^+(n) = \frac{\pi_l^+(n)}{\sum_{i=0}^N \sum_{l=1}^L \pi_l^+(n) \mathbf{e}}. \quad (3.11)$$

3.3. Queue length distributions at arbitrary epochs. To obtain queue length distributions at arbitrary epoch we will develop relations between distributions of number of customers in the queue at service completion (vacation termination) and arbitrary epochs. Relating the states of the system at two consecutive time epochs t and $t + \Delta t$, and using probabilistic arguments, we get a set of partial differential equations for each phase i ($1 \leq i \leq M$). Taking limit as $t \rightarrow \infty$ and using matrices and vector notations, we obtain

$$-\frac{d}{dx}\pi_1(0,x) = \pi_1(0,x)\mathbf{C} + \omega(1,0)s(x) + \delta_S \mathbf{v}(0)\mathbf{D}s(x), \tag{3.12}$$

$$-\frac{d}{dx}\pi_l(0,x) = \pi_l(0,x)\mathbf{C} + \pi_{l-1}(1,0)s(x), \quad 2 \leq l \leq L, \tag{3.13}$$

$$-\frac{d}{dx}\pi_1(n,x) = \pi_1(n,x)\mathbf{C} + \pi_1(n-1,x)\mathbf{D} + \omega(n+1,0)s(x), \tag{3.14}$$

$$1 \leq n \leq N-1,$$

$$-\frac{d}{dx}\pi_l(n,x) = \pi_l(n,x)\mathbf{C} + \pi_l(n-1,x)\mathbf{D} + \pi_{l-1}(n+1,0)s(x), \tag{3.15}$$

$$1 \leq n \leq N-1, \quad 2 \leq l \leq L,$$

$$-\frac{d}{dx}\pi_l(N,x) = \pi_l(N,x)(\mathbf{C} + \mathbf{D}) + \pi_l(N-1,x)\mathbf{D}, \tag{3.16}$$

$$1 \leq l \leq L,$$

$$-\frac{d}{dx}\omega(0,x) = \omega(0,x)\mathbf{C} + \left(\sum_{l=1}^L \pi_l(0,0) + (1 - \delta_S)\omega(0,0) \right) \mathbf{v}(x), \tag{3.17}$$

$$-\frac{d}{dx}\omega(n,x) = \omega(n,x)\mathbf{C} + \omega(n-1,x)\mathbf{D} + \pi_L(n,0)\mathbf{v}(x), \tag{3.18}$$

$$1 \leq n \leq N-1,$$

$$-\frac{d}{dx}\omega(N,x) = \omega(N,x)(\mathbf{C} + \mathbf{D}) + \omega(N-1,x)\mathbf{D} + \pi_L(N,0)\mathbf{v}(x), \tag{3.19}$$

$$\mathbf{0} = \delta_S \mathbf{v}(0)\mathbf{C} + \delta_S \omega(0,0). \tag{3.20}$$

We define the Laplace transform of $\pi_l(n,x)$ and $\omega(n,x)$ as

$$\pi_l^*(n,\theta) = \int_0^\infty e^{-\theta x} \pi_l(n,x) dx, \tag{3.21}$$

$$\omega^*(n,\theta) = \int_0^\infty e^{-\theta x} \omega(n,x) dx, \quad 0 \leq n \leq N, \quad 1 \leq l \leq L, \quad \text{Re } \theta \geq 0,$$

so that

$$\pi_l(n) \equiv \pi_l^*(n,0) = \int_0^\infty \pi_l(n,x) dx, \tag{3.22}$$

$$\omega(n) \equiv \omega^*(n,0) = \int_0^\infty \omega(n,x) dx, \quad 0 \leq n \leq N, \quad 1 \leq l \leq L.$$

One may note here that $\pi_l(n,0)$ ($\omega(n,0)$) are jumping epoch probabilities at service completion (vacation termination) epoch. Multiplying equations (3.12)–(3.19) by $e^{-\theta x}$ and

integrating with respect to x over 0 to ∞ , we have

$$-\theta\boldsymbol{\pi}_1^*(0, \theta) + \boldsymbol{\pi}_1(0, 0) = \boldsymbol{\pi}_1^*(0, \theta)\mathbf{C} + \boldsymbol{\omega}(1, 0)S^*(\theta) + \delta_S \boldsymbol{\nu}(0)\mathbf{D}S^*(\theta), \tag{3.23}$$

$$-\theta\boldsymbol{\pi}_l^*(0, \theta) + \boldsymbol{\pi}_l(0, 0) = \boldsymbol{\pi}_l^*(0, \theta)\mathbf{C} + \boldsymbol{\pi}_{l-1}^*(1, 0)S^*(\theta), \quad 2 \leq l \leq L, \tag{3.24}$$

$$-\theta\boldsymbol{\pi}_1^*(n, \theta) + \boldsymbol{\pi}_1(n, 0) = \boldsymbol{\pi}_1^*(n, \theta)\mathbf{C} + \boldsymbol{\pi}_1^*(n-1, \theta)\mathbf{D} + \boldsymbol{\omega}(n+1, 0)S^*(\theta), \tag{3.25}$$

$$1 \leq n \leq N-1,$$

$$-\theta\boldsymbol{\pi}_l^*(n, \theta) + \boldsymbol{\pi}_l(n, 0) = \boldsymbol{\pi}_l^*(n, \theta)\mathbf{C} + \boldsymbol{\pi}_l^*(n-1, \theta)\mathbf{D} + \boldsymbol{\pi}_{l-1}(n+1, 0)S^*(\theta), \tag{3.26}$$

$$1 \leq n \leq N-1, \quad 2 \leq l \leq L,$$

$$-\theta\boldsymbol{\pi}_l^*(N, \theta) + \boldsymbol{\pi}_l(N, 0) = \boldsymbol{\pi}_l^*(N, \theta)(\mathbf{C} + \mathbf{D}) + \boldsymbol{\pi}_l^*(N-1, \theta)\mathbf{D}, \quad 1 \leq l \leq L, \tag{3.27}$$

$$-\theta\boldsymbol{\omega}^*(0, \theta) + \boldsymbol{\omega}(0, 0) = \boldsymbol{\omega}^*(0, \theta)\mathbf{C} + \left(\sum_{l=1}^L \boldsymbol{\pi}_l(0, 0) + (1 - \delta_S)\boldsymbol{\omega}(0, 0) \right) V^*(\theta), \tag{3.28}$$

$$-\theta\boldsymbol{\omega}^*(n, \theta) + \boldsymbol{\omega}(n, 0) = \boldsymbol{\omega}^*(n, \theta)\mathbf{C} + \boldsymbol{\omega}^*(n-1, \theta)\mathbf{D} + \boldsymbol{\pi}_L(n, 0)V^*(\theta), \tag{3.29}$$

$$1 \leq n \leq N-1,$$

$$-\theta\boldsymbol{\omega}^*(N, \theta) + \boldsymbol{\omega}(N, 0) = \boldsymbol{\omega}^*(N, \theta)(\mathbf{C} + \mathbf{D}) + \boldsymbol{\omega}^*(N-1, \theta)\mathbf{D} + \boldsymbol{\pi}_L(N, 0)V^*(\theta). \tag{3.30}$$

Now using equations (3.23)–(3.30), we will first derive certain results in the form of lemmas. These lemmas are true for both single- and multiple-vacation models.

LEMMA 3.1.

$$\sum_{l=1}^{L-1} \boldsymbol{\pi}_l(0, 0)\mathbf{e} + \sum_{n=0}^N \boldsymbol{\pi}_L(n, 0)\mathbf{e} = \delta_S \boldsymbol{\omega}(0, 0)\mathbf{e} + \sum_{n=1}^N \boldsymbol{\omega}(n, 0)\mathbf{e}. \tag{3.31}$$

This result can be interpreted like that the left-hand side denotes the mean number of entrances to the vacation states per unit of time and the right-hand side denotes the mean number of departures from the vacation states per unit of time.

Proof. Setting $\theta = 0$ in (3.23)–(3.27) and using (3.22), we get

$$\boldsymbol{\pi}_1(0, 0) = \boldsymbol{\pi}_1(0)\mathbf{C} + \boldsymbol{\omega}(1, 0) + \delta_S \boldsymbol{\nu}(0)\mathbf{D}, \tag{3.32}$$

$$\boldsymbol{\pi}_l(0, 0) = \boldsymbol{\pi}_l(0)\mathbf{C} + \boldsymbol{\pi}_{l-1}(1), \quad 2 \leq l \leq L, \tag{3.33}$$

$$\boldsymbol{\pi}_1(n, 0) = \boldsymbol{\pi}_1(n)\mathbf{C} + \boldsymbol{\pi}_1(n-1)\mathbf{D} + \boldsymbol{\omega}(n+1, 0), \quad 1 \leq n \leq N-1, \tag{3.34}$$

$$\boldsymbol{\pi}_l(n, 0) = \boldsymbol{\pi}_l(n)\mathbf{C} + \boldsymbol{\pi}_l(n-1)\mathbf{D} + \boldsymbol{\pi}_{l-1}(n+1, 0), \quad 1 \leq n \leq N-1, \quad 2 \leq l \leq L, \tag{3.35}$$

$$\boldsymbol{\pi}_l(N, 0) = \boldsymbol{\pi}_l(N)(\mathbf{C} + \mathbf{D}) + \boldsymbol{\pi}_l(N-1)\mathbf{D}, \quad 1 \leq l \leq L. \tag{3.36}$$

Post-multiplying by the vector \mathbf{e} in equations (3.32)–(3.36), adding them, using (3.20) and $(\mathbf{C} + \mathbf{D})\mathbf{e} = \mathbf{0}$, after simplification we obtain the result of Lemma 3.1. \square

LEMMA 3.2.

$$E(S) \sum_{n=0}^N \sum_{l=1}^L \pi_l(n, 0) \mathbf{e} = \sum_{n=0}^N \sum_{l=1}^L \pi_l(n) \mathbf{e} = \rho', \tag{3.37}$$

$$E(V) \sum_{n=0}^N \omega(n, 0) \mathbf{e} + \delta_S \nu(0) \mathbf{e} = \sum_{n=0}^N \omega(n) \mathbf{e} + \delta_S \nu(0) \mathbf{e} = 1 - \rho'. \tag{3.38}$$

$\sum_{n=0}^N \sum_{l=1}^L \pi_l(n, 0) \mathbf{e}$ denotes the mean number of service completions per unit of time, and multiplying this by $E(S)$ will give ρ' . Similarly, the other result can be interpreted.

Proof. Post-multiplying (3.23)–(3.27) by \mathbf{e} , differentiating these with respect to θ , and using $(\mathbf{C} + \mathbf{D})\mathbf{e} = \mathbf{0}$, we get

$$-\pi_1^*(0, \theta) \mathbf{e} - \theta \pi_1^{*(1)}(0, \theta) \mathbf{e} = \pi_1^{*(1)}(0, \theta) \mathbf{C} \mathbf{e} + S^{*(1)}(\theta) \omega(1, 0) \mathbf{e} + \delta_S S^{*(1)}(\theta) \nu(0) \mathbf{D} \mathbf{e}, \tag{3.39}$$

$$-\pi_l^*(0, \theta) \mathbf{e} - \theta \pi_l^{*(1)}(0, \theta) \mathbf{e} = \pi_l^{*(1)}(0, \theta) \mathbf{C} \mathbf{e} + S^{*(1)}(\theta) \pi_{l-1}(1, 0) \mathbf{e}, \quad 2 \leq l \leq L, \tag{3.40}$$

$$-\pi_1^*(n, \theta) \mathbf{e} - \theta \pi_1^{*(1)}(n, \theta) \mathbf{e} = \pi_1^{*(1)}(n, \theta) \mathbf{C} \mathbf{e} + \pi_1^{*(1)}(n-1, \theta) \mathbf{D} \mathbf{e} + S^{*(1)}(\theta) \omega(n+1, 0) \mathbf{e}, \quad 1 \leq n \leq N-1, \tag{3.41}$$

$$-\pi_l^*(n, \theta) \mathbf{e} - \theta \pi_l^{*(1)}(n, \theta) \mathbf{e} = \pi_l^{*(1)}(n, \theta) \mathbf{C} \mathbf{e} + \pi_l^{*(1)}(n-1, \theta) \mathbf{D} \mathbf{e} + S^{*(1)}(\theta) \pi_{l-1}(n+1, 0) \mathbf{e}, \quad 1 \leq n \leq N-1, 2 \leq l \leq L \tag{3.42}$$

$$-\pi_l(N, \theta) \mathbf{e} - \theta \pi_l^{*(1)}(N, \theta) \mathbf{e} = \pi_l^{*(1)}(N, \theta) (\mathbf{C} + \mathbf{D}) \mathbf{e} + \pi^{*(1)}(N-1, \theta) \mathbf{D} \mathbf{e}, \quad 1 \leq l \leq L. \tag{3.43}$$

Setting $\theta = 0$ in (3.39)–(3.43), adding them, and using (3.20), $(\mathbf{C} + \mathbf{D})\mathbf{e} = \mathbf{0}$, and Lemma 3.1, after simplification we obtain (3.37). Similarly, post-multiplying (3.28)–(3.30) by \mathbf{e} , differentiating these equations with respect to θ , and setting $\theta = 0$, after simple algebraic manipulation we obtain $E(V) \sum_{n=0}^N \omega(n, 0) \mathbf{e} = \sum_{n=0}^N \omega(n) \mathbf{e}$. Adding $\delta_S \nu(0) \mathbf{e}$ to both sides, we have the result. \square

3.3.1. *Relations between queue length distributions at arbitrary and service completion (vacation termination) epochs.* We first relate the service completion (vacation termination) epoch probabilities $\pi_l^+(n)$ and $\omega^+(n)$ with the probabilities $\pi_l(n, 0)$ and $\omega(n, 0)$ which are given by

$$\begin{aligned} \pi_{i,l}^+(n) &= P(n \text{ customers in the queue just prior to service completion epoch of the } l\text{th} \\ &\quad \text{customer and state of the arrival process is } i \mid \leq N \text{ customers in the queue} \\ &\quad \text{just prior to service completion or vacation termination epoch}) \\ &= \frac{\pi_{i,l}(n, 0)}{\sum_{n=0}^N \sum_{l=1}^L \pi_l(n, 0) \mathbf{e} + \sum_{n=0}^N \omega(n, 0) \mathbf{e}} \\ &= \frac{1}{\sigma} \pi_{i,l}(n, 0), \quad 0 \leq n \leq N, 1 \leq l \leq L, 1 \leq i \leq M, \end{aligned} \tag{3.44}$$

and similarly,

$$\omega_i^+(n) = \frac{1}{\sigma} \omega_i(n, 0), \quad 0 \leq n \leq N, \quad 1 \leq i \leq M, \tag{3.45}$$

where $\sigma = \sum_{n=0}^N \sum_{l=1}^L \pi_l(n, 0)\mathbf{e} + \sum_{n=0}^N \omega(n, 0)\mathbf{e}$. Now we can get the relations by which we can determine arbitrary epoch probabilities. Setting $\theta = 0$ in the equations (3.23)–(3.26), (3.28)–(3.29) and using (3.44)–(3.45), we get the relations

$$\pi_1(0) = [\sigma(\omega^+(1) - \pi_1^+(0)) + \delta_S \nu(0)\mathbf{D}](-\mathbf{C})^{-1}, \tag{3.46}$$

$$\pi_l(0) = [\sigma(\pi_{l-1}^+(1) - \pi_l^+(0))](-\mathbf{C})^{-1}, \quad 2 \leq l \leq L, \tag{3.47}$$

$$\pi_1(n) = [\pi_1(n-1)\mathbf{D} + \sigma(\omega^+(n+1) - \pi_1^+(n))](-\mathbf{C})^{-1}, \quad 1 \leq n \leq N-1, \tag{3.48}$$

$$\pi_l(n) = [\pi_l(n-1)\mathbf{D} + \sigma(\pi_{l-1}^+(n+1) - \pi_l^+(n))](-\mathbf{C})^{-1}, \quad 1 \leq n \leq N-1, \quad 2 \leq l \leq L, \tag{3.49}$$

$$\omega(0) = \sigma \left[\sum_{l=1}^L \pi_l^+(0) - (1 - \delta_S)\omega^+(0) \right] (-\mathbf{C})^{-1}, \tag{3.50}$$

$$\omega(n) = [\omega(n-1)\mathbf{D} + \sigma(\pi_L^+(n) - \omega^+(n))](-\mathbf{C})^{-1}, \quad 1 \leq n \leq N-1. \tag{3.51}$$

It may be noted here that we do not have such definite expression for $\pi_l(N)$ ($1 \leq l \leq L$) and $\omega(N)$. However, one can compute them by using Lemma 3.2 as $\sum_{l=1}^L \pi_l(N)\mathbf{e} = \rho' - \sum_{n=0}^{N-1} \sum_{l=1}^L \pi_l(n)\mathbf{e}$ and $\omega(N)\mathbf{e} = (1 - \rho') - \sum_{n=0}^{N-1} \omega(n)\mathbf{e} - \delta_S \nu(0)\mathbf{e}$. Though we are not getting componentwise $\pi_l(N)$ ($1 \leq l \leq L$) and $\omega(N)$, instead of that we are obtaining $\sum_{l=1}^L \pi_l(N)\mathbf{e}$ and $\omega(N)\mathbf{e}$, which are sufficient to determine key performance measures in Section 4. The unknown quantities ρ' and σ present in the above expressions can be evaluated with the help of following two Lemmas.

LEMMA 3.3. *The expression of ρ' (probability that the server is busy) is given by*

$$\rho' = \frac{E(S) \sum_{n=0}^N \sum_{l=1}^L \pi_l^+(n)\mathbf{e}}{E(S) \sum_{n=0}^N \sum_{l=1}^L \pi_l^+(n)\mathbf{e} + E(V) \sum_{n=0}^N \omega^+(n)\mathbf{e} + \delta_S \omega^+(0)(-\mathbf{C})^{-1}\mathbf{e}}. \tag{3.52}$$

Proof. Let Θ_b $\{\Theta_i\}$ be the random variable denoting the length of busy {idle} period and θ_b $\{\theta_i\}$ the mean length of a busy {idle} period, then we have

$$\rho' = \frac{\theta_b}{\theta_b + \theta_i}. \quad \text{Also } \frac{\theta_b}{\theta_i} = \frac{\sum_{n=0}^N \sum_{l=1}^L \pi_l(n)\mathbf{e}}{\sum_{n=0}^N \omega(n)\mathbf{e} + \delta_S \nu(0)\mathbf{e}}. \tag{3.53}$$

Applying Lemma 3.2, using (3.20) and then (3.44)–(3.45), we obtain

$$\frac{\theta_b}{\theta_i} = \frac{E(S) \sum_{n=0}^N \sum_{l=1}^L \pi_l^+(n) \mathbf{e}}{E(V) \sum_{n=0}^N \omega^+(n) \mathbf{e} + \delta_S \omega^+(0) (-\mathbf{C})^{-1} \mathbf{e}}. \tag{3.54}$$

The above ratio yields the result. □

LEMMA 3.4. *The probability that the server is in dormancy is given by*

$$P(\text{server is in dormancy}) = \boldsymbol{\nu}(0) \mathbf{e} = \sigma \omega^+(0) (-\mathbf{C})^{-1} \mathbf{e}. \tag{3.55}$$

Proof. Putting $\delta_S = 1$ and then multiplying (3.20) by σ^{-1} , using (3.45) and then post-multiplying by \mathbf{e} , we obtain the result. One may note here that σ is frequently needed for computation purpose and it can be obtained by using ρ' and (3.44) in (3.37).

Let $\mathbf{p}(n)$ denote the row vector of order $1 \times M$ whose i th component is the probability distributions of $n(0 \leq n \leq N)$ customers in the queue at arbitrary epoch and state of the arrival process is $i(1 \leq i \leq M)$. $\mathbf{p}(n)$ can be obtained in terms of arbitrary epoch probabilities as

$$\mathbf{p}(0) = \delta_S \boldsymbol{\nu}(0) + \sum_{l=1}^L \boldsymbol{\pi}_l(0) + \boldsymbol{\omega}(0), \tag{3.56}$$

$$\mathbf{p}(n) = \sum_{l=1}^L \boldsymbol{\pi}_l(n) + \boldsymbol{\omega}(n), \quad 1 \leq n \leq N - 1, \tag{3.57}$$

$$\mathbf{p}(N) = \bar{\boldsymbol{\pi}} - \delta_S \boldsymbol{\nu}(0) - \sum_{n=0}^{N-1} \left(\sum_{l=1}^L \boldsymbol{\pi}_l(n) + \boldsymbol{\omega}(n) \right). \tag{3.58}$$

□

3.4. Queue length distributions at pre-arrival epochs. Let $\mathbf{p}^-(n)$ be the $1 \times M$ vectors whose j th components are given by $p_j^-(n)$ and which gives the steady-state probability that an arrival finds $n(0 \leq n \leq N)$ customers in the queue and the arrival process is in state j . Then the vectors $\mathbf{p}^-(n)$ are given by

$$\mathbf{p}^-(n) = \frac{\mathbf{p}(n) \mathbf{D}}{\lambda^*}, \quad 0 \leq n \leq N. \tag{3.59}$$

One can easily evaluate the pre-arrival epoch probabilities using (3.59) and also the loss probability (P_{loss}) is equivalent to $\mathbf{p}^-(N) \mathbf{e} = \mathbf{p}(N) \mathbf{D} \mathbf{e} / \lambda^*$.

3.5. Waiting time. Here we obtain the LST of actual waiting time distribution in the queue of an arrived customer. Consider an arrival epoch sequence $t_k : k = 0, 1, 2, \dots$ of

MAP arrival process and define the probability density vectors as

$$\begin{aligned} \pi_{i,l}^-(n,x;t_k)\Delta x &= P\{N_q(t_k) = n, J(t_k) = i, x < \tilde{S}(t_k) < x + \Delta x, \xi(t_k) = l\}, \\ &0 \leq n \leq N, 1 \leq l \leq L, x \geq 0, \\ \omega_i^-(n,x;t_k)\Delta x &= P\{N_q(t_k) = n, J(t_k) = i, x < \tilde{V}(t_k) < x + \Delta x, \xi(t_k) = 0\}, \\ &0 \leq n \leq N, x \geq 0, \\ \nu_i^-(0;t_k) &= P\{N_q(t_k) = 0, J(t_k) = i, \xi(t_k) = d\}. \end{aligned} \tag{3.60}$$

Now applying steady-state argument and using vector notations, we get

$$\begin{aligned} \pi_l^-(n,x) &= [\pi_{i,l}^-(n,x)], \omega^-(n,x) = [\omega_i^-(n,x)], \nu^-(0) = [\nu_i^-(0)], \\ &1 \leq i \leq M, 0 \leq n \leq N - 1, 1 \leq l \leq L. \end{aligned} \tag{3.61}$$

The LSTs of $\pi_l^-(n,x)$ and $\omega^-(n,x)$ are given by

$$\begin{aligned} \pi_l^{*-}(n,\theta) &= \frac{\pi_l^*(n,\theta)\mathbf{D}}{\lambda^*}, \quad 0 \leq n \leq N, 1 \leq l \leq L, \\ \omega^{*-}(n,\theta) &= \frac{\omega^*(n,\theta)\mathbf{D}}{\lambda^*}, \quad 0 \leq n \leq N. \end{aligned} \tag{3.62}$$

Thus the LST of the actual waiting time distribution in the queue is given by

$$\begin{aligned} \mathbf{W}_q^*(\theta) &= \frac{1}{1 - P_{\text{loss}}} \left\{ \delta_S \nu^-(0) + \sum_{k=0}^{N-1} \omega^{*-}(k,\theta) (S^*(\theta))^k (V^*(\theta))^{[k/L]} \right. \\ &\quad \left. + \sum_{n=0}^{N-1} \sum_{l=1}^L \pi_l^{*-}(n,\theta) (S^*(\theta))^n (V^*(\theta))^{[(n+l)/L]} \right\} \\ &= \frac{1}{\lambda^* (1 - P_{\text{loss}})} \left\{ \delta_S \nu(0)\mathbf{D} + \sum_{k=0}^{N-1} \omega^*(k,\theta)\mathbf{D} (S^*(\theta))^k (V^*(\theta))^{[k/L]} \right. \\ &\quad \left. + \sum_{n=0}^{N-1} \sum_{l=1}^L \pi_l^*(n,\theta)\mathbf{D} (S^*(\theta))^n (V^*(\theta))^{[(n+l)/L]} \right\}, \end{aligned} \tag{3.63}$$

where $[k/L]$ denotes the integer part of k/L . One can obtain average waiting time in the queue of an arbitrary customer using

$$\begin{aligned} W_q &= -\mathbf{W}_q^{*(1)}(0)\mathbf{e} \\ &= -\frac{1}{\lambda^* (1 - P_{\text{loss}})} \left\{ \sum_{k=0}^{N-1} \left(\omega^{*(1)}(k,0)\mathbf{D} - kE(S)\omega(k)\mathbf{D} - \left[\frac{k}{L} \right] E(V)\omega(k)\mathbf{D} \right) \right. \\ &\quad \left. + \sum_{n=0}^{N-1} \sum_{l=1}^L \left(\pi_l^{*(1)}(n,0)\mathbf{D} - (n)E(S)\pi_l(n)\mathbf{D} \right. \right. \\ &\quad \left. \left. - \left[\frac{n+l}{L} \right] E(V)\pi_l(n)\mathbf{D} \right) \right\} \mathbf{e}, \end{aligned} \tag{3.64}$$

where the unknown vector quantities $(\boldsymbol{\pi}_l^{*(1)}(n,0)$ and $\boldsymbol{\omega}^{*(1)}(n,0)$) can be obtained by differentiating (3.23)–(3.26) and (3.28)–(3.29), and then setting $\theta = 0$. They are given by

$$\boldsymbol{\pi}_1^{*(1)}(0,0) = [\boldsymbol{\pi}_1(0) - E(S)\sigma\boldsymbol{\omega}^+(1) - \delta_S E(S)\boldsymbol{\nu}(0)\mathbf{D}](-\mathbf{C})^{-1}, \tag{3.65}$$

$$\boldsymbol{\pi}_l^{*(1)}(0,0) = [\boldsymbol{\pi}_l(0) - E(S)\sigma\boldsymbol{\pi}_{l-1}^+(1)](-\mathbf{C})^{-1}, \quad 2 \leq l \leq L, \tag{3.66}$$

$$\boldsymbol{\pi}_1^{*(1)}(n,0) = [\boldsymbol{\pi}_1(n)\mathbf{D} + \boldsymbol{\pi}_1^{*(1)}(n-1,0)\mathbf{D} - E(S)\sigma\boldsymbol{\omega}^+(n+1)](-\mathbf{C})^{-1}, \tag{3.67}$$

$$1 \leq n \leq N-1,$$

$$\boldsymbol{\pi}_l^{*(1)}(n,0) = [\boldsymbol{\pi}_l(n) + \boldsymbol{\pi}_l^{*(1)}(n,0)\mathbf{D} - E(S)\sigma\boldsymbol{\pi}_{l-1}^+(n+1)](-\mathbf{C})^{-1}, \tag{3.68}$$

$$1 \leq n \leq N-1, \quad 2 \leq l \leq L,$$

$$\boldsymbol{\omega}^{*(1)}(0,0) = \left[\boldsymbol{\omega}(0) - E(V)\sigma \left(\sum_{l=1}^L \boldsymbol{\pi}_l^+(0) - (1 - \delta_S)\boldsymbol{\omega}^+(0) \right) \right](-\mathbf{C})^{-1}, \tag{3.69}$$

$$\boldsymbol{\omega}^{*(1)}(n,0) = [\boldsymbol{\omega}(n) + \boldsymbol{\omega}^{*(1)}(n-1,0)\mathbf{D} - E(V)\sigma\boldsymbol{\pi}_L^+(n)](-\mathbf{C})^{-1}, \quad 1 \leq n \leq N-1. \tag{3.70}$$

Remark 3.5. One may note that we can also obtain W_q using Little’s rule and it is given by $W_q = Lq/\lambda'$, $\lambda' =$ effective arrival rate $= \lambda^*(1 - P_{\text{loss}})$.

4. Performance measures

Performance measures are important features of queueing systems as they reflect the efficiency of the queueing system under consideration. The steady-state probabilities at service completion, vacation termination, departure, and arbitrary epochs are known, various performance measures of the queue can be easily obtained such as the average number of customers in the queue at any arbitrary epoch ($Lq = \sum_{i=0}^N i\mathbf{p}(i)\mathbf{e}$), the average number of customers in the queue when the server is busy ($Lq_1 = \sum_{i=0}^N i[\sum_{l=1}^L \boldsymbol{\pi}_l(i)]\mathbf{e}$), and the average number of customers in the queue when the server is on vacation ($Lq_2 = \sum_{i=0}^N i\boldsymbol{\omega}(i)\mathbf{e}$). Other performance measures such as probability of loss, and average waiting time are given in Sections 3.4 and 3.5, respectively.

This completes analytic analysis of *MAP/G/1/N/LS*, *SV*, *MV* queue. Now we present computational procedures and discussion of numerical results in Sections 5 and 6, respectively.

5. Computational procedures

In this section, we will briefly discuss the necessary steps required for the computation of the matrices $\mathbf{A}_n, \mathbf{V}_n$ of TPM \mathcal{P} . The evaluation of $\mathbf{A}_n(\mathbf{V}_n)$, in general, for arbitrary service (vacation) time distribution requires numerical integration and can be carried out along the lines proposed by Lucantoni and Ramaswami [14]. However, when the service- and vacation-time distributions are of phase type (PH-distribution), these matrices can be evaluated without any numerical integration, Nuets [15, pages 67–70]. It may be noted here that various service- and vacation-time distributions arising in practical applications can be approximated by PH-distribution. The following theorem gives a procedure for

the computation of the matrices \mathbf{A}_n and \mathbf{V}_n . Thereafter we can construct the TPM as described in Section 3.1 and then solve the system of equations through GTH (Grassmann, Taksar and Heyman) algorithm given by Latouche and Ramaswami [10].

THEOREM 5.1. *Let $S(x)$ follow a PH-distribution with irreducible representation $(\boldsymbol{\beta}, \mathbf{S})$, where $\boldsymbol{\beta}$ and \mathbf{S} are of dimension γ , then the matrices \mathbf{A}_n are given by*

$$\mathbf{A}_n = \mathbf{B}_n(\mathbf{I}_M \otimes \mathbf{S}^0), \quad 0 \leq n \leq N - 1, \tag{5.1}$$

where

$$\begin{aligned} \mathbf{B}_0 &= -(\mathbf{I}_M \otimes \boldsymbol{\beta})[\mathbf{C} \otimes \mathbf{I}_\gamma + \mathbf{I}_M \otimes \mathbf{S}]^{-1}, \\ \mathbf{B}_n &= -\mathbf{B}_{n-1}(\mathbf{D} \otimes \mathbf{I}_\gamma)[\mathbf{C} \otimes \mathbf{I}_\gamma + \mathbf{I}_M \otimes \mathbf{S}]^{-1}, \quad 1 \leq n \leq N - 1, \\ \mathbf{B}_N &= -\mathbf{B}_{N-1}(\mathbf{D} \otimes \mathbf{I}_\gamma)[(\mathbf{C} + \mathbf{D}) \otimes \mathbf{I}_\gamma + \mathbf{I}_M \otimes \mathbf{S}]^{-1}, \end{aligned} \tag{5.2}$$

where $\mathbf{S}^0 = -\mathbf{S}\mathbf{e}$ and the symbol \otimes denotes the Kronecker product of two matrices. Similarly, let $V(x)$ follow a PH-distribution with irreducible representation $(\boldsymbol{\alpha}, \mathbf{T})$, where $\boldsymbol{\alpha}$ and \mathbf{T} are dimension μ , then the matrices \mathbf{V}_n are given by

$$\mathbf{V}_n = \mathbf{R}_n(\mathbf{I}_M \otimes \mathbf{T}^0), \quad 0 \leq n \leq N - 1, \tag{5.3}$$

where

$$\begin{aligned} \mathbf{R}_0 &= -(\mathbf{I}_M \otimes \boldsymbol{\alpha})[\mathbf{C} \otimes \mathbf{I}_\mu + \mathbf{I}_M \otimes \mathbf{T}]^{-1}, \\ \mathbf{R}_n &= -\mathbf{R}_{n-1}(\mathbf{D} \otimes \mathbf{I}_\mu)[\mathbf{C} \otimes \mathbf{I}_\mu + \mathbf{I}_M \otimes \mathbf{T}]^{-1}, \quad 1 \leq n \leq N - 1, \\ \mathbf{R}_N &= -\mathbf{R}_{N-1}(\mathbf{D} \otimes \mathbf{I}_\mu)[(\mathbf{C} + \mathbf{D}) \otimes \mathbf{I}_\mu + \mathbf{I}_M \otimes \mathbf{T}]^{-1}, \end{aligned} \tag{5.4}$$

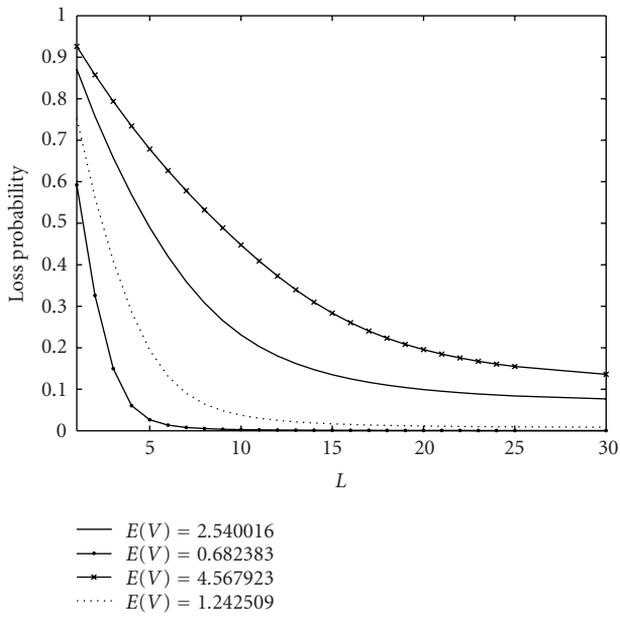
where $\mathbf{T}^0 = -\mathbf{T}\mathbf{e}$ and the symbol \otimes denotes the Kronecker product of two matrices.

For proof see Neuts [15], Gupta and Vijaya Laxmi [9].

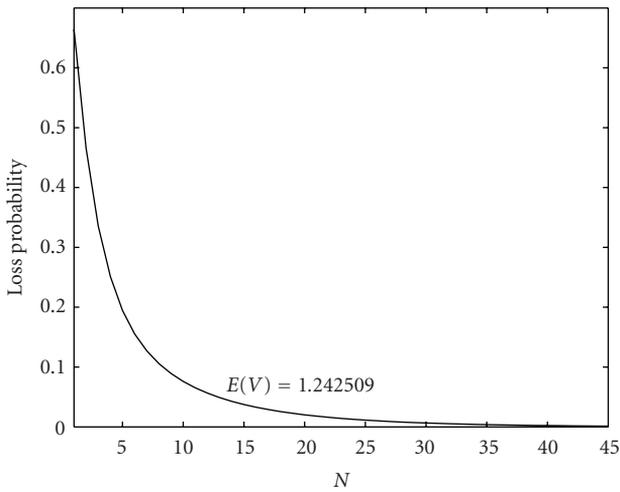
6. Numerical result

To demonstrate the applicability of the results obtained in the previous sections, some numerical results have been presented in the form of graphs showing the nature of some performance measures against the variation of some model parameters. Numerical work has been carried out in LINUX environment using C++ language. Higher values of N and L with reasonable $\rho < 1$ (ρ can take greater value than 1) will lead ρ' (the probability that the server is busy) to asymptotically converge to ρ (offered load), which is a popular check of asymptotic property, this is shown graphically in Figure 6.3(b).

In Figure 6.1(a) the effect of limit L (L varies from 1 to 30) on P_{loss} has been studied for MAP/PH/1/15 single-vacation (vacation time follows PH-distribution) queue with the following input parameters: MAP representation is taken as $\mathbf{C} = \begin{bmatrix} -4.657 & 1.761 \\ 1.128 & -3.941 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1.657 & 1.239 \\ 0.872 & 1.941 \end{bmatrix}$. For service- and vacation-time, PH-type representation of service time is taken as $\boldsymbol{\beta} = [0.4 \ 0.6]$, $\mathbf{S} = \begin{bmatrix} -6.683 & 2.453 \\ 1.367 & -7.986 \end{bmatrix}$ with $E(S) = 0.180050$, PH-type representation of vacation time is taken as $\boldsymbol{\alpha} = [0.7 \ 0.3]$, $\mathbf{T} = \begin{bmatrix} -1.098 & 0.864 \\ 0.071 & -0.532 \end{bmatrix}$ with $E(V) = 2.540016$. PH-type



(a)



(b)

Figure 6.1. (a) Effect of L on loss probability. (b) Effect of N on loss probability.

representations of other vacation times are taken as $\alpha = [0.7 \ 0.3]$, $\mathbf{T} = \begin{bmatrix} -2.098 & 1.899 \\ 0.071 & -2.832 \end{bmatrix}$ with $E(V) = 0.682383$; $\alpha = [0.7 \ 0.3]$, $\mathbf{T} = \begin{bmatrix} -1.098 & 1.099 \\ 0.071 & -1.832 \end{bmatrix}$ with $E(V) = 1.242509$; and $\alpha = [0.7 \ 0.3]$, $\mathbf{T} = \begin{bmatrix} -1.098 & 1.899 \\ 0.071 & -0.532 \end{bmatrix}$ with $E(V) = 4.567923$. For this model $M = 2$, $\lambda^* = 2.846200$, and $\rho = 0.512459$. The effect of N on P_{loss} is studied in Figure 6.1(b) for the above-described MAP/G/1/N queue with fixed $L = 10$ and a vacation time from the above list where N varies from 1 to 45. It can be seen from the Figures 6.1(a) and 6.1(b) that P_{loss} asymptotically approaches to its minimum value with the increase of N and L . Also the minimum value of P_{loss} is heavily dependent on the mean vacation time.

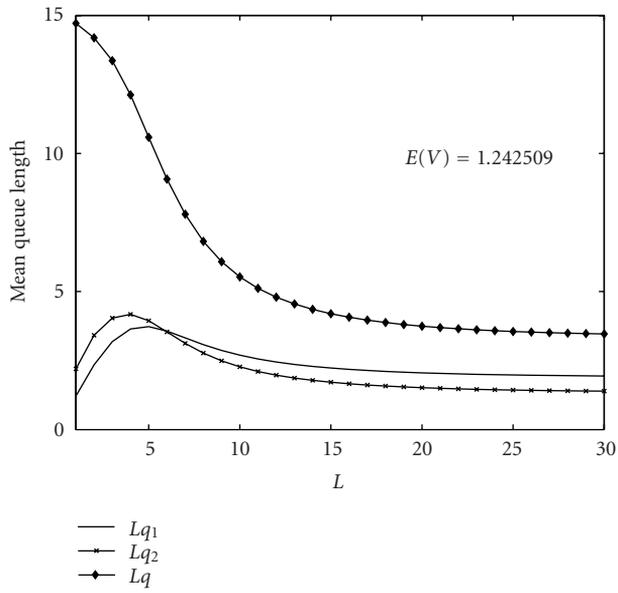
Next we study the effect of L against various mean queue lengths (L_{q1}, L_{q2}, L_q) and average waiting time for MAP/PH/1/15 single-vacation queue with PH-type vacation time representation and the model parameters are the same as above. This is shown in the Figures 6.2(a) and 6.2(b), respectively. Here also mean queue lengths and mean waiting time asymptotically converge to a minimum value as L increases.

In Figure 6.3, the nature of ρ' has been shown for MAP/PH/1/40 single- and multiple-vacation (vacation time follows PH-type distribution) queues with L varying from 1 to 12 and the other input parameters are given below. The MAP representation is taken as $\mathbf{C} = \begin{bmatrix} -3.119 & 0.961 \\ 1.985 & -3.343 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1.119 & 1.039 \\ 1.015 & 0.343 \end{bmatrix}$. For service and vacation times, PH-type representation of service time is taken as $\beta = [0.4 \ 0.6]$, $\mathbf{S} = \begin{bmatrix} -6.683 & 2.453 \\ 1.367 & -7.986 \end{bmatrix}$ with $E(S) = 0.180050$, PH-type representation of vacation time is taken as $\alpha = [0.7 \ 0.3]$, $\mathbf{T} = \begin{bmatrix} -1.098 & 0.864 \\ 0.071 & -0.532 \end{bmatrix}$ with $E(V) = 2.540016$. Here $M = 2$, $\lambda^* = 1.838000$, and $\rho = 0.330932$. In a similar way in Figure 6.3(b), we have plotted N versus ρ' for fixed $L = 10$ and N varies from 1 to 45 for the above parameters. ρ' is almost equal for single and multiple vacations under the above set of model parameters. Here we have plotted the results for single vacation. Clearly we see from the Figure 6.3 that as L and N increase ρ' asymptotically approaches towards ρ .

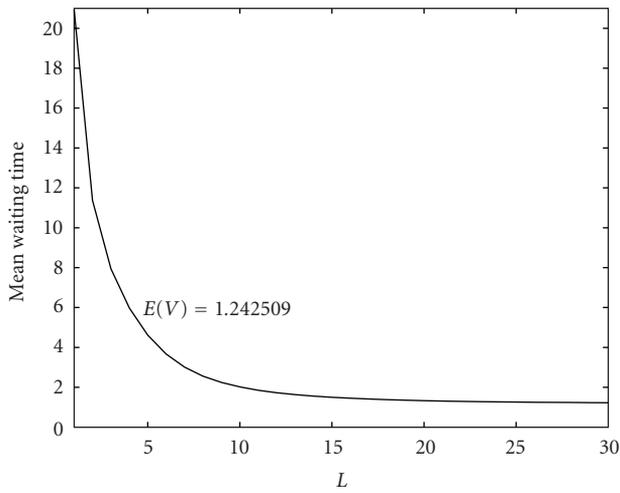
Figure 6.4 describes the influence of ρ on P_{loss} , mean waiting time (Wq), respectively, for both single- and multiple-vacation (vacation time follows PH-type distribution) MAP/E₂/1/20 queues with the following input parameters: MAP representation is taken as $\mathbf{C} = \begin{bmatrix} -1.625 & 0.250 \\ 0.875 & -1.375 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 0.875 & 0.500 \\ 0.125 & 0.375 \end{bmatrix}$. For E₂ service time, PH-type representation of service time is taken as $\beta = [1.0 \ 0.0]$, $\mathbf{S} = \begin{bmatrix} -\gamma & \gamma \\ 0.0 & -\gamma \end{bmatrix}$ with $E(S) = 2.0/\gamma$ and γ varies from 20.0 to 1.8182. As a result ρ varies from 0.1 to 1.1. PH-type representation of vacation time is taken as $\alpha = [0.3 \ 0.7]$, $\mathbf{T} = \begin{bmatrix} -1.098 & 0.864 \\ 0.071 & -0.532 \end{bmatrix}$ with $E(V) = 2.366333$. Here $M = 2$, $\lambda^* = 1.000000$. In Figure 6.4, we only present multiple-vacation results, as both results are numerically close. Figure 6.4(b) distinguishes mean waiting time between single- and multiple-vacation models for low traffic load. However, for high traffic load there is no significant difference between single- and multiple-vacation models.

7. Conclusion and future scopes

In this paper, we have discussed analytical and computational aspects of MAP/G/1/N limited-service queue with single and multiple vacations. The queue length distributions at service completion, vacation termination, departure, arbitrary, and pre-arrival epochs have been obtained. Finally, it may be mentioned here that the method of analysis adopted in this paper can be used to analyze other complex models such as MAP/G/1/N queue with single (multiple) vacation(s) under probabilistically limited service discipline.

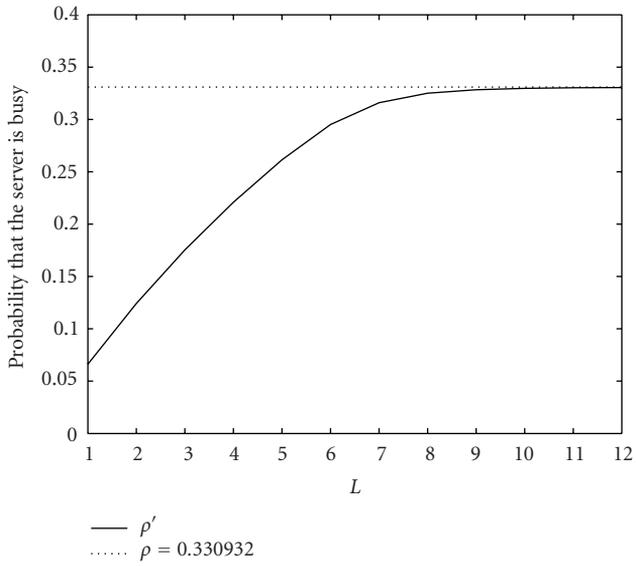


(a)

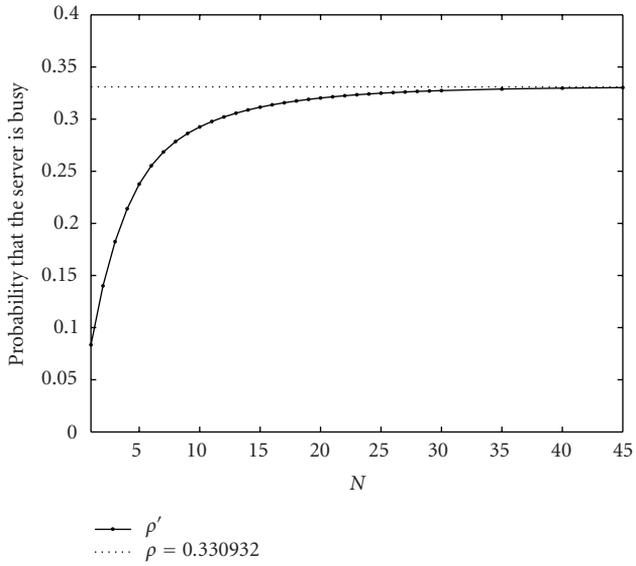


(b)

Figure 6.2. (a) Effect of L on mean queue lengths. (b) Effect of L on mean waiting time.

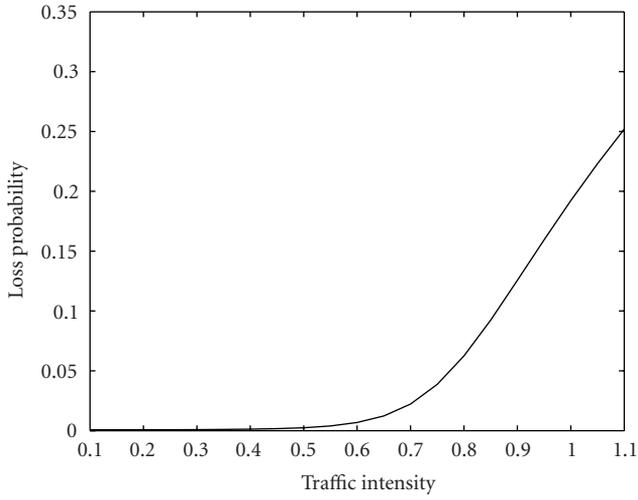


(a)

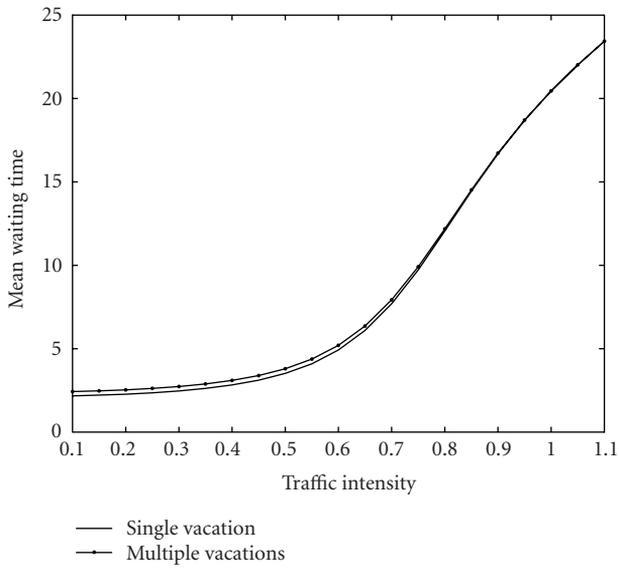


(b)

Figure 6.3. (a) Effect of L on ρ' . (b) Effect of N on ρ' .



(a)



(b)

Figure 6.4. (a) Effect of ρ on probability of loss. (b) Effect of ρ on mean waiting time.

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