We study uniform finite-difference method for solving first-order singularly perturbed boundary value problem (BVP) depending on a parameter. Uniform error estimates in the discrete maximum norm are obtained for the numerical solution. Numerical results support the theoretical analysis.

1. Introduction

In this paper, we are going to devise a finite-difference method for the following parameter-dependant singularly perturbed boundary value problem (BVP):

\[ Lu := \epsilon u'(x) + a(x)u(x) = f(x, \lambda), \quad x \in \Omega = (0, l), \]
\[ u(0) = A, \quad u(l) = B, \]

where \( A, B \) are given constants and \( a(x), f(x, \lambda) \) are sufficiently smooth functions such that

\[ a(x) \geq \alpha > 0 \quad \text{in} \; \bar{\Omega} = [0, l], \]
\[ 0 < m_1 \leq \left| \frac{\partial f}{\partial \lambda} \right| \leq M_1 < \infty \quad \text{in} \; \bar{\Omega} \times \mathbb{R}. \]  

\( \epsilon > 0 \) is a small parameter and \( \{u(x, \lambda)\} \) is a solution.

For \( \epsilon \ll 1 \), the function \( u(x) \) has a boundary layer of thickness \( O(\epsilon) \) near \( x = 0 \).

Under the above conditions, there exists a unique solution to problem (1.1), (1.2) (see [7, 12]). An overview of some existence and uniqueness results and applications of parameterized equations may be obtained, for example, in [6, 7, 8, 9, 12, 13, 15, 16]. In [7, 9, 12], have also been considered some approximating aspects of this kind of problems. But designed in the above-mentioned papers, algorithms are only concerned with the regular cases (i.e., when the boundary layers are absent).

The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid
changes within very thin layers near the boundary or inside the problem domain [4, 10, 11]. It is well known that standard numerical methods on uniform meshes for solving such problems are unstable and fail to give accurate results when the perturbation parameter $\varepsilon$ is small. Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value $\varepsilon$, that is, methods that are convergent $\varepsilon$-uniformly. For the various approaches on the numerical solution of differential equations with steep, continuous solutions, we may refer to the monographs [4, 5, 14].

Here we analyze a fitted difference scheme on a uniform mesh for the numerical solution of the problem (1.1), (1.2). In Section 2, we state some important properties of the exact solution. In Section 3, we present the difference scheme and obtain uniform error estimates for the truncation term and appropriate solution on a uniform mesh. Uniform convergence is proved in the discrete maximum norm. In Section 4, we formulate the iterative algorithm for solving the discrete problem and give the illustrative numerical results. The approach to construct discrete problem and error analysis for approximate solution is similar to those ones from [1, 2, 3].

Henceforth, $C$ and $c$ denote the generic positive constants independent of $\varepsilon$ and of the mesh parameter. A subscripted such constant is also independent of $\varepsilon$ and mesh parameter, but whose value is fixed.

2. The continuous problem

In this section, we give uniform bounds of the solution of the BVP (1.1), (1.2), which will be used to analyze properties of the appropriate difference problem.

**Lemma 2.1.** For the solution $\{u(x), \lambda\}$ of the problem (1.1), (1.2),

\[
|\lambda| \leq c_0, \quad (2.1) \\
\|u\|_\infty \leq c_1, \quad (2.2)
\]

where

\[
c_0 = \frac{\|a\|_\infty}{m_1(1 - \exp(-\|a\|_\infty l))} (|A| + |B|) + m_1^{-1}\|F\|_\infty,
\]

\[
c_1 = |A| + \alpha^{-1}\|F\|_\infty + c_0\alpha^{-1}M_1,
\]

\[
\left( F(x) = f(x, 0), \|a\|_\infty \equiv \|a\|_{\infty, \Omega} := \max_{\Omega} |a(x)| \right).
\]

**Proof.** We rewrite (1.1) as

\[
\varepsilon u'(x) + a(x)u(x) = f(x, 0) + \frac{\partial \tilde{f}}{\partial \lambda} \lambda, \quad (2.4)
\]

where

\[
\frac{\partial \tilde{f}}{\partial \lambda} = \frac{\partial f}{\partial \lambda}(x, \lambda^*), \quad \lambda^* = y\lambda, \quad 0 < y < 1.
\]

(2.5)
Integrating (2.4), we get
\[ u(x) = A \exp \left( -\frac{1}{\varepsilon} \int_0^x a(t) dt \right) \]
\[ + \frac{1}{\varepsilon} \int_0^x F(\tau) \exp \left( -\frac{1}{\varepsilon} \int_\tau^x a(t) dt \right) d\tau \]
\[ + \frac{1}{\varepsilon} \lambda \int_0^x \frac{\partial f}{\partial \lambda}(\tau, \lambda^*) \exp \left( -\frac{1}{\varepsilon} \int_\tau^x a(t) dt \right) d\tau, \]
(2.6)
from which, by setting the boundary condition \( u(l) = B \), we have
\[ \lambda = \frac{B - A \exp \left( -\frac{1}{\varepsilon} \int_0^l a(t) dt \right) - \frac{1}{\varepsilon} \int_0^l F(\tau) \exp \left( -\frac{1}{\varepsilon} \int_\tau^l a(t) dt \right) d\tau}{\frac{1}{\varepsilon} \int_0^l \frac{\partial f}{\partial \lambda}(\tau, \lambda^*) \exp \left( -\frac{1}{\varepsilon} \int_\tau^l a(t) dt \right) d\tau}. \]
(2.7)
Applying the mean-value theorem for integrals, we deduce that
\[ \left| \frac{(1/\varepsilon) \int_0^l F(\tau) \exp \left( -\frac{1}{\varepsilon} \int_\tau^l a(t) dt \right) d\tau}{(1/\varepsilon) \int_0^l \frac{\partial f}{\partial \lambda}(\tau, \lambda^*) \exp \left( -\frac{1}{\varepsilon} \int_\tau^l a(t) dt \right) d\tau} \right| \leq m_1^{-1} \|F\|_\infty, \quad 0 < x^* < l. \]
(2.8)
It then follows from (2.7) that
\[ |\lambda| \leq \frac{|B - A \exp \left( -\frac{1}{\varepsilon} \int_0^l a(t) dt \right)|}{m_1 (1/\varepsilon) \int_0^l \exp \left( -\frac{1}{\varepsilon} \int_\tau^l a(t) dt \right) d\tau} + m_1^{-1} \|F\|_\infty, \]
(2.9)
which, for \( \varepsilon \leq 1 \), immediately leads to (2.1).

Next, from (2.6), we see that
\[ |u(x)| \leq |A| \exp \left( -\frac{\alpha x}{\varepsilon} \right) + \alpha^{-1} \left\{ 1 - \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right\} (\|F\|_\infty + |\lambda|M_1) \]
(2.10)
and using the estimate (2.1), we obtain (2.2).

3. Discrete problem and convergence

3.1. Derivation of the difference scheme. In what follows, we denote by \( \omega_h \) the uniform mesh on \( \Omega \):
\[ \omega_h = \{ x_i = ih, \ i = 1, \ldots, N; \ Nh = l \}, \quad \widehat{\omega}_h = \omega_h \cup \{ x = 0 \}. \]
To simplify the notation, we set \( g_i = g(x_i) \) for any function \( g(x) \), while \( g^h \) denotes an approximation of \( g(x) \) at \( x_i \).

For any mesh function \( \{ w_i \} \) defined on \( \bar{\omega}_h \), we use the discrete maximum norm
\[
\| w \|_{\infty} \equiv \| w \|_{\infty, \bar{\omega}_h} := \max_{0 \leq i \leq N} | w_i |. \tag{3.2}
\]

The approach of generating difference method is through the integral identity
\[
\chi_i h^{-1} \int_{x_{i-1}}^{x_i} L u \varphi_i(x) dx = \chi_i h^{-1} \int_{x_{i-1}}^{x_i} f(x, \lambda) \varphi_i(x) dx, \quad 1 \leq i \leq N, \tag{3.3}
\]
with the exponential basis functions
\[
\varphi_i(x) = \exp \left( - \frac{a_i (x_i - x)}{\epsilon} \right), \quad x_{i-1} \leq x \leq x_i, \tag{3.4}
\]
where
\[
\chi_i = \left( h^{-1} \int_{x_{i-1}}^{x_i} \varphi_i(x) dx \right)^{-1} = \frac{a_i \rho}{1 - \exp \left( -a_i \rho \right)}, \quad \rho = \frac{h}{\epsilon}. \tag{3.5}
\]

We note that function \( \varphi_i(x) \) is the solution of the problem
\[
-\epsilon \varphi_i'(x) + a_i \varphi_i(x) = 0, \quad x_{i-1} \leq x < x_i, \quad \varphi_i(x_i) = 1. \tag{3.6}
\]

The relation (3.3) is rewritten as
\[
\chi_i h^{-1} \epsilon \int_{x_{i-1}}^{x_i} u'(x) \varphi_i(x) dx + a_i \chi_i h^{-1} \int_{x_{i-1}}^{x_i} u(x) \varphi_i(x) dx + R_i = f(x_i, \lambda) \tag{3.7}
\]
with the remainder term
\[
R_i = \chi_i h^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a(x_i)] \varphi_i(x) dx + \chi_i h^{-1} \int_{x_{i-1}}^{x_i} f(x, \lambda) \varphi_i(x) dx. \tag{3.8}
\]

To be consistent with [1, 2, 3], we then obtain
\[
\epsilon \theta_i u_{x,i} + a_i u_i + R_i = f(x_i, \lambda), \quad 1 \leq i \leq N. \tag{3.9}
\]
where

\[
\theta_i = 1 + \chi_i h^{-1} a_i \epsilon^{-1} \int_{x_{i-1}}^{x_i} (x - x_i) \varphi_i(x) dx,
\]

\[
u_{\pm, i} = \frac{u_i - u_{i-1}}{h},
\]

and a simple calculation gives us

\[
\theta_i = \frac{a_i \rho}{1 - \exp(-a_i \rho)} \exp(-a_i \rho).
\]

As a consequence of (3.9), we propose the following difference scheme for approximating (1.1), (1.2):

\[
L^h u_i^h := \epsilon \theta_i \nu_{\pm, i} u_i^h + a_i u_i^h = f(x_i, \lambda^h), \quad 1 \leq i \leq N,
\]

\[
u_0^h = A, \quad u_N^h = B,
\]

where \(\theta_i\) is defined by (3.11).

3.2. Uniform error estimates. To investigate the convergence of the method, note that the error functions \(z_i^h = u_i^h - u_i, 0 \leq i \leq N, \mu^h = \lambda^h - \lambda\) are the solution of the discrete problem

\[
\epsilon \theta_i z_{\pm, i}^h + a_i z_i^h = f(x_i, \mu^h + \lambda) - f(x_i, \lambda) + R_i, \quad 1 \leq i \leq N,
\]

\[
z_0^h = 0, \quad z_N^h = 0,
\]

where \(\theta_i\) and \(R_i\) are given by (3.11) and (3.8), respectively.

Lemma 3.1. For the error function \(R_i\),

\[
\|R\|_{\infty, \Omega} \leq C h,
\]

provided \(a \in C^1(\bar{\Omega})\) and \(|\partial f / \partial x| \leq C\) for \(x \in \bar{\Omega}\) and \(\lambda\) satisfying (2.1).

The proof easily follows from the explicit expression of \(R_i\) defined by (3.8).

Lemma 3.2. The solution \(\{z_i^h, \mu^h\}\) of the problem (3.13), (3.14) satisfies

\[
|m^h| \leq m_1^{-1} \|R\|_{\infty, \Omega},
\]

\[
\|z_i^h\|_{\infty, \Omega} \leq \alpha^{-1} (1 + m_1^{-1} M_1) \|R\|_{\infty, \Omega},
\]
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Proof. From (3.13), we obtain

\[ z_i^h = \frac{\varepsilon \theta_i}{\varepsilon \theta_i + h a_i} z_{i-1}^h + \frac{h (\partial \tilde{f} / \partial \lambda)_i}{\varepsilon \theta_i + h a_i} \mu^h + \frac{h R_i}{\varepsilon \theta_i + h a_i}, \tag{3.18} \]

where

\[ \left( \frac{\partial \tilde{f}}{\partial \lambda} \right)_i = \frac{\partial}{\partial \lambda} (x_i, \lambda + \gamma \mu^h), \quad 0 < \gamma < 1. \tag{3.19} \]

Solving the first-order difference equation with respect to \( z_i^h \) and setting the boundary condition \( z_0^h = 0 \), we get

\[ z_i^h = \mu^h h \sum_{k=1}^{i} \frac{(\partial f / \partial \lambda)_k}{\varepsilon \theta_k + h a_k} Q_{ik} + h \sum_{k=1}^{i} \frac{R_k}{\varepsilon \theta_k + h a_k} Q_{ik}, \tag{3.20} \]

where

\[ Q_{ik} = \begin{cases} 1, & k = i, \\ \prod_{j=k+1}^{i} \frac{\varepsilon \theta_j}{\varepsilon \theta_j + h a_j}, & 1 \leq k \leq i - 1. \end{cases} \tag{3.21} \]

For \( i = N \), taking into consideration that \( z_N^h = 0 \), we have

\[ \mu^h = -\frac{\sum_{k=1}^{N} \left( R_k / (\varepsilon \theta_k + h a_k) \right) Q_{N,k}}{\sum_{k=1}^{N} \left( (\partial f / \partial \lambda)_k / (\varepsilon \theta_k + h a_k) \right) Q_{N,k}}, \tag{3.22} \]

from which, since \( \varepsilon \theta_i + h a_i > 0 \) (1 ≤ i ≤ N), the required result (3.16) easily follows.

Finally, applying the maximum principle for difference operator \( L^h z_i^h = \varepsilon \theta_i (x_i, \mu^h) + a_i z_i^h, 1 \leq i \leq N \), to (3.13) yields

\[ \| z_i^h \|_{\infty, \tilde{\omega}_h} \leq \alpha^{-1} (M_1 \| \mu^h \| + \| R \|_{\infty, \tilde{\omega}_h}), \tag{3.23} \]

which, along with (3.16), leads to (3.17).

Combining the two previous lemmas gives us the following convergence result.

**Theorem 3.3.** Let \( \{ u(x), \lambda \} \) be the solution of (1.1), (1.2) and \( \{ u_h^h, \lambda_h \} \) the solution of (3.13), (3.14). Then

\[ | \lambda - \lambda_h | \leq C h, \quad \| u - u_h \|_{\infty, \tilde{\omega}_h} \leq C h. \tag{3.24} \]
4. Numerical results

In this section, we present some numerical experiments in order to illustrate the present method.

(a) We solve the nonlinear problem (3.12) using the following quasilinearization technique:

\[
\varepsilon \theta_i u_{i,n}^{(n)} + a_i u_i^{(n)} = f(x_i, \lambda^{(n-1)}), \quad 1 \leq i < N, \\
u_0^{(n)} = A, \\
\lambda^{(n)} = \lambda^{(n-1)} - \frac{f(l, \lambda^{(n-1)}) - \theta_N \rho^{-1} (B - u_{N-1}^{(n)}) - a_N B}{(\partial f/\partial \lambda)(l, \lambda^{(n-1)})},
\]

\[n = 1, 2, \ldots; \lambda^{(0)} \text{ given. (For simplicity, the } h \text{ on } u_i \text{ is omitted.) The initial guess } \lambda^{(0)} \text{ is being chosen by condition (2.1).}
\]

(b) Consider the test problem

\[
\varepsilon u'(x) + \frac{1}{1 + x^2} u(x) = 2\lambda + \sin \frac{\lambda x}{2}, \quad 0 \leq x \leq 1, \\
u(0) = 1, \quad u(1) = 0.
\]

The initial guess in (4.2) is taken as \( \lambda^{(0)} = 0.00039 \) and stopping criterion is

\[
\max_i |u_i^{(n)} - u_i^{(n-1)}| \leq 10^{-5}; \quad |\lambda^{(n)} - \lambda^{(n-1)}| \leq 10^{-5}.
\]

We use a double-mesh method (see, e.g., [5]) to compute the experimental rates of convergence:

\[
p_{\varepsilon,h}^{u} = \frac{\ln \left( e_{u}^{\varepsilon,h}/e_{u}^{\varepsilon,h/2} \right)}{\ln 2}, \quad e_{u}^{\varepsilon,h} = \max_{0 \leq i \leq N} |u_i^h - u_i^{h/2}| 
\]

for \( u_i^h \), and

\[
p_{\varepsilon,h}^{\lambda} = \frac{\ln \left( e_{\lambda}^{\varepsilon,h}/e_{\lambda}^{\varepsilon,h/2} \right)}{\ln 2}, \quad e_{\lambda}^{\varepsilon,h} = |\lambda^h - \lambda^{h/2}|
\]

for \( \lambda^h \).

Tables 4.1 and 4.2 contain some numerical results for different values of \( \varepsilon \) and \( h \), based on the double-mesh principle. The result established here is that the discrete solution is uniformly convergent with respect to the perturbation parameter, and also clearly, that we obtain first-order convergence, so Theorem 3.3 is sharp.
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Table 4.1. Errors \( \{e_{u}^{\epsilon,h}, e_{A}^{\epsilon,h}\} \) and convergence rates \( \{p_{u}^{\epsilon,h}, p_{A}^{\epsilon,h}\} \) on \( \omega_{h} \) for \( h = 1/8 \) and \( h = 1/16 \).

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Table 4.2. Errors \( \{e_{u}^{\epsilon,h}, e_{A}^{\epsilon,h}\} \) and convergence rates \( \{p_{u}^{\epsilon,h}, p_{A}^{\epsilon,h}\} \) on \( \omega_{h} \) for \( h = 1/16 \) and \( h = 1/32 \).

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Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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