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ON A VARIATION OF SANDS' METHOD

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ABSTRACT. A subset of a of a finite additive abelian group G is a Z-set if for all $a \in G$, $na \in G$ for all $n \in Z$. The group G is called "Z-good" if in every factorization $G = A \oplus B$, where A and B are Z-sets at least one factor is periodic. Otherwise G is called "Z-bad."

The purpose of this paper is to investigate factorizations of finite ablian groups which arise from a variation of Sands' method. A necessary condition is given for a factorization $G = A \oplus B$, where A and B are Z-sets, to be obtained by this variation. An example is provided to show that this condition is not sufficient. It is also shown that in general all factorizations $G = A \oplus B$, where A and B are Z-sets, of a "Z-good" group do not arise from this variation of Sands' method.

KEY WORDS AND PHRASES. Finite abelian group, factorization, Z-set, good group, bad group.

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I. INTRODUCTION.

Let G be a finite additive abelian group and let A and B be subsets of G. If every element $g \in G$ can be uniquely represented in the form $g = a + b$, where $a \in A$, $b \in B$, then we write $G = A \oplus B$ and call this a factorization of G. A subset A of G is said to be periodic if there exists an element $g \neq 0$ such that $g + A = A$. Such an element g is called a period of A. The set of all periods of A together with 0 forms a subgroup of G. A subset of G is a Z-set if for all $a \in A$, $na \in A$ for all $n \in Z$. We say G is "good" ("Z-good") if in every factorization $G = A \oplus B$, where A and B are sets (Z-sets) at least one factor is periodic. Otherwise G is called "bad" ("Z-bad").

The problem of classifying a finite abelian group as either "good" or "bad" arose from the solution of G. Hajos [1] to a group-theoretical interpretation of a conjecture of H. Minkowski on homogeneous linear forms. Hajos [1-3], Redei [4-5], de Bruijn [6-7], and Sands [8-11] have completely solved this problem of classification. C. Okuda

[12] classified all finite abelian groups as either "Z-good" of "Z-bad," obtaining quite different results from the "good" - "bad" classification.

Sands [8] gave a method which yields all factorizations of a finite abelian "good" group. His method corrects one given previously by Hajos [2].

The purpose of this paper is to investigate factorizations of finite abelian groups which arise from a variation of Sands' method. A necessary condition is given for a factorization $G = A \oplus B$, where A and B are Z-sets, to be obtained by this variation. An example is provided to show that this condition is not sufficient. It is also shown that in general all factorizations $G = A \oplus B$, where A and B are Z-sets, of a "Z-good" group do not arise from this variation of Sands' method.

2. PRELIMINARIES.

This section provides some basic unpublished results on Okuda's [12] "Z-good" - "Z-bad" classification of finite abelian groups as well as an elementary result concerning factorizations $G = S \oplus A$, where S is a subgroup of G and A is a Z-set. For completeness, we state Sands' Theorem on the factorizations of finite abelian "good" groups.

LEMMA 1 (Okuda [12]). A finite abelian group G is "Z-good" if and only if at least one Sylow p-subgroup of G is "Z-good."

LEMMA 2 (Okuda [12]). Every cyclic group is "Z-good."

LEMMA 3 (Okuda [12]). If $G = A \oplus B$, where A and B are Z-sets, then A and B are pure in G.

LEMMA 4 (Okuda [12]). Let G be a group isomorphic to $Z_p \oplus Z_p + Z_p + Z_p$, where p is an odd prime. Let $\{a_1, a_2, b_1, b_2\}$ be a basis of G and define

$$A = (\langle a_1, a_2\rangle \setminus \langle a_2\rangle) \cup \langle a_2 + b_2\rangle,$$

$$B = (\langle b_1, b_2\rangle \setminus \bigcup_{i=1}^{p-1} \langle b_1 + ib_2\rangle) \cup (\bigcup_{i=1}^{p-1} \langle b_1 + ib_2 + 2a_2\rangle).$$

Then A and B are non-periodic Z-sets and $G = A + B$.

PROPOSITION 1. Let S be a subgroup of G. G has a factorization $G = S + A$, A a Z-set, if and only if S is pure in G.

PROOF. This is a direct consequence of Lemma 3 and the fact that a pure subgroup of a finite abelian group G is a direct summand of G.

THEOREM 1 (Sands [8]). Let G be a finite abelian "good" group. $G = A + B$ if and only if there exists subsets $H_1, H_2, \ldots, H_n$ such that $H_i + H_{i+1} + \cdots + H_n = K_i$ is a subgroup of G, $1 \leq i \leq n$, $K_1 = G$, and

$$A = \langle 0\rangle + H_1 \circ H_2 + H_3 \circ H_4 + \ldots,$$

$$B = \langle 0\rangle \circ H_1 + H_2 \circ H_3 + H_4 \circ \ldots,$$

where the notation $C \circ D$ indicates any of the sets formed by adding to each element of C some element of D.

Let us note that the subgroups, K_i, in Sands' Theorem yield the following series for G

$$G = K_1 \supset K_2 \supset K_3 \supset \cdots \supset K_n \supset K_{n+1} = \langle 0\rangle$$

where $K_i = H_i \oplus K_{i+1}$, $1 \le i \le n$, $K_n = H_n$. We shall say that the factorization $G = A \dotplus B$ arises from the above series if

$$A = H_1 \dotplus H_3 \dotplus \cdots + h_2 + h_4 + \cdots ,$$

$$B = H_2 + H_4 \dotplus \cdots + h_1 + h_3 + \cdots ,$$

where H_i is a set of coset representatives for K_i modulo K_{i+1}, and $h_i \in H_i$, $1 \le i \le n$.

Factorizations which arise from the above series can be obtained from Sands' method if one computes $C \circ D$ by adding a fixed element of D to the set C. However, as shown in Example 1, there are factorizations which are obtained from Sands' method which do not arise from the corresponding series of subgroups K_i, $1 \le i \le n$.

EXAMPLE 1. Let G be the cyclic group of order 81. Consider the series $G = K_1 \supset K_2 \supset K_3 \supset K_4 \supset \langle 0\rangle$, where $K_4 = \langle 27\rangle$, $K_3 = \langle 9\rangle$, $K_2 = \langle 3\rangle$. If we choose $H_3 = \{0, 9, 18\}$, $H_2 = \{0, 3, 6\}$, $H_1 = \{0, 1, 2\}$ then the following factorization, $G = A \dotplus B$, can be obtained from Sands' method.

$$A = \langle 0\rangle \oplus H_1 \circ H_2 \dotplus H_3 \circ H_4 = (H_1 \dotplus H_3) \circ H_4$$

$$= (\{0,1,2\} \dotplus \{0,9,18\}) \circ \{0,27,54\} = \{0,1,2,9,10,11,18,19,47\}$$

$$B = \langle 0\rangle \circ H_1 \oplus H_2 \circ H_3 \dotplus H_4 = H_2 \dotplus H_4 = \{0,3,6\} \oplus \{0,27,54\}.$$

Clearly for every choice of the sets H_i, $1 \le i \le 3$, the set A cannot be written in the form $A = H_1 \dotplus H_3 + h_2 + h_4$.

3. TRANSLATIONS.

THEOREM 2. Let $G = A + B$ a factorization of G arising from the series $G = K_1 \supset K_2 \supset \cdots \supset K_n \supset \langle 0\rangle$ with coset representatives H_i so that $K_i = H_i + K_{i+1}$, $1 \le i \le n-1$, $K_n = H_n$. Then $G = A' \dotplus B'$ with $A' = A + g_1$, $B' = B + g_2$, $g_1, g_2 \in G$ arises from the same series with coset representatives $H_i' = H_i + u_i$, $u_i \in H_i$, $1 \le i \le n$.

PROOF. We will proceed by induction on the length of the series, n. For $n = 2$, we may assume $A = H_1 + h_2$, $B = H_2 + h_1$, $h_i \in H_i$, $i = 1,2$. Suppose $G = A' + B'$ with $A' = A + g_1$, $B' = B + g_2$, $g_1, g_2 \in G$. We can write $g_1 = u_1 + u_2$, $u_i \in H_i$, $i = 1,2$. Since $G = (H_1 + u_1) \dotplus H_2$ we have $h_1 + g_2 = z_1 + u_1 + z_2$, $z_i \in H_i$, $i = 1,2$. Let $H_1' = H_1 + u_1$, $H_2' = H_2 + u_2$. Then

$$A' = A + g_1 = H_1 + u_1 + h_2 + u_2 = H_1' + h_2' ,$$

$$B' = B + g_2 = H_2 + z_2 + z_1 + u_1 = H_2 + u_2 + z_1 + u_1 = H_2' + h_1' ,$$

where we have used the fact that $H_2 = H_2 + u_2 = H_2 + z_2$ since $H_2 + K_2$ is a subgroup.

Let us assume the theorem is true for series of length less than n. Let $G = A + B$ be a factorization arising from a series of length n, say $G = B + g_2$, $K_1 \supset K_2 \supset \cdots \supset K_n \supset \langle 0\rangle$. We may assume that

$$A = H_1 \oplus H_3 \oplus \cdots + h_2 + h_4 + \cdots ,$$
$$B = H_2 \oplus H_4 \oplus \cdots + h_1 + h_3 + \cdots .$$

Define

$$A_1 = H_3 \oplus H_5 \oplus \cdots + h_2 + h_4 + \cdots ,$$
$$B_1 = H_2 \oplus H_4 \oplus \cdots + h_3 + h_5 + \cdots ,$$

so that $A = A_1 \oplus H_1$, $B = B_1 + h_1$. Suppose $G = A' \oplus B'$, where

$$A' = A + g_1,\ g_1 = s_1 + s_2,\ s_1 \in H_1,\ s_2 \in K_2,$$
$$B' = B + g_2,\ g_2 = t_1 + t_2,\ t_1 \in H_1,\ t_2 \in K_2.$$

Setting $H_1' = H_1 + s_1$ we have

$$A' = H_1' \oplus (A_1 + s_2),$$
$$B' = (h_1 + t_1) + (B_1 + t_2) = h_1' + (B_1 + t_2 + k_2),$$

where $h_1 + t_1 = h_1 + t_1 - s_1 + s_1 = \tilde{h}_1 + k_2 + s_1 = h_1' + k_2$, $k_2 \in K_2$, $\tilde{h}_1 \in H_1$, and $h_1' = \tilde{h}_1 + s_1 \in H_1'$.

Note that $K_2 = A_1 \oplus B_1$ arises from the series $K_2 \supset K_3 \supset \cdots \supset K_n \supset \langle 0\rangle$. Therefore the factorization $K_2 = (A_1 + s_2) \oplus (B_1 + t_2 + k_2)$ arises from $K_2 \supset K_3 \supset \cdots \supset K_n \supset \langle 0\rangle$ with coset representatives $H_i' = H_i + u_i$, $u_i \in H_i$, $2 \le i \le n$, i.e.,

$$A_1 + s_2 = H_3' \oplus H_5' \oplus \cdots + h_2' + h_4' + \cdots,$$
$$B_1 + t_2 + k_2 = H_2' \oplus H_4' + \cdots + h_3' + h_5' + \cdots .$$

Consequently we have

$$A' = H_1' \oplus (A_1 + s_2) = H_1' \oplus H_3' \oplus \cdots + h_2' + h_4' + \cdots ,$$
$$B' = h_1' + (B_1 + t_2 + k_2) = H_2' \oplus H_4' \oplus \cdots + h_1' + h_3' + \cdots ,$$

which completes the proof.

THEOREM 3. Let $G = A \oplus B$ be a factorization of G arising from the series $G = K_1 \supset K_2 \supset \cdots \supset K_n \supset \langle 0\rangle$ with coset representatives H_i so that $K_i = H_i \oplus K_{i+1}$, $1 \le i \le n-1$, $K_n = H_n$. Then each H_i may be translated to obtain H_i', $1 \le i \le n$, in such a way that $0 \in H_i'$, $1 \le i \le n$, and the factorization $G = A \oplus B$ arises from the original series with coset representatives H_i', $1 \le i \le n$, $K_n = H_n'$.

PROOF. We will use induction on the length of the series, n. For $n = 2$, we may assume $A = H_1 + h_2$, $B = H_2 + h_1$, $h_i \in H_i$, $i = 1,2$. We can write $0 = y_1 + y_2$, $y_i \in H_i$, $i = 1,2$. Define $H_1' = H_1 + y_2$, $H_2' = H_2$, and let $h_1' = h_1 + y_2$. Note that $H_2 = H_2 + y_2$ and $h_2 - y_2 \in H_2$ since $H_2 = K_2$ is a subgroup. Thus,

$$A = H_1 + y_2 + h_2 - y_2 = H_1' + h_2' ,$$
$$B = H_2 + h_1 = H_2 + h_1 + y_2 = H_2' + h_1' .$$

Let us assume the theorem is true for series of length less than n. Let the factorization $G = A \oplus B$ arise from a series of length n, say, $G = K_1 \supset K_2 \supset \ldots \supset K_n \supset \langle 0 \rangle$. We may assume that

$$A = H_1 \oplus H_3 \oplus \ldots + h_2 + h_4 + \ldots ,$$

$$B = H_2 \oplus H_4 \oplus \ldots + h_1 + h_3 + \ldots .$$

Define

$$A_1 = H_3 \oplus H_5 \oplus \ldots + h_2 + h_4 + \ldots ,$$

$$B_1 = H_2 \oplus H_4 \oplus \ldots + h_3 + h_5 + \ldots .$$

so that $A = A_1 \oplus H_1$, $B = B_1 + h_1$, and $K_2 + A_1 \oplus B_1$.

We can write $0 = y_1 + y_2 + \ldots + y_n$, $y_i \in H_i$, $1 \leq i \leq n$. Set $x_2 = y_2 + y_3 + \ldots + y_n$. Then $x_2 \in K_2$. Define $H_1' = H_1 + x_2$ and let $h_1 = h_1' + z_2$, $z_2 \in K_2$. Note that $0 \in H_1'$. We have $K_2 = (A_1 - x_2) \oplus (B_1 + z_2)$. By Theorem 2 there exists coset representatives $H_2', H_3', \ldots, H_n'$ translates of $H_2, H_3, \ldots, H_n$ respectively such that

$$A_1 - x_2 = H_3' \oplus H_5' \oplus \ldots + h_2' + h_4' + \ldots ,$$

$$B_1 + z_2 = H_2' \oplus H_4' \oplus \ldots + h_3' + h_5' + \ldots .$$

By the inductive hypothesis, $0 \in H_i'$, $2 \leq i \leq n$. Hence,

$$A = H_1 \oplus A_1 = H_1' - x_2 \oplus A_1 = H_1' \oplus H_3' \oplus \ldots + h_2' + h_4' + \ldots$$

$$B = B_1 + h_1 = B_1 + h_1' + z_2 = H_2' \oplus H_4' \oplus \ldots + h_1' + h_3' + \ldots .$$

This completes the proof.

4. Z-FACTORIZATIONS.

We shall use the term "Z-factorization" when refering to a factorization of the form $G = A \oplus B$, where A and B are Z-sets.

LEMMA 5. Let $G = A \oplus B$ be a Z-factorization of G arising from the series $G = K_1 \supset K_2 \supset \ldots \supset K_n \supset \langle 0 \rangle$. Then we may choose the coset representatives, H_i', $1 \leq i \leq n$, appearing in the expressions for A and B such that $0 \in H_i'$, $1 \leq i \leq n$, and $h_i' = 0$, $1 \leq i \leq n$.

PROOF. We may assume

$$A = H_1 \oplus H_3 \oplus \ldots + h_2 + h_4 + \ldots ,$$

$$B = H_2 \oplus H_4 \oplus \ldots + h_1 + h_3 + \ldots .$$

By theorem 3 there exist coset representatives H_i', $1 \leq i \leq n$, such that $0 \in H_i'$, $1 \leq i \leq n$, and

$$A = H_1' \oplus H_3' \oplus \ldots + h_2' + h_4' + \ldots ,$$

$$B = H_2' \oplus H_4' \oplus \ldots + h_1' + h_3' + \ldots .$$

Observe that $0 \in H_1' \oplus H_3' \oplus \ldots$ and $0 \in H_2' \oplus H_4' \oplus \ldots$. Consequently

$h'_2 + h'_4 + \dots \in A$. Since A is a Z-set we have $2(h'_2 + h'_4 + \dots) \in A$. Therefore

$$2(h'_2 + h'_4 + \dots) = \tilde{h}'_2 + \tilde{h}'_3 + \dots + h'_2\, h'_4 + \dots ,$$

where $\tilde{h}'_i \in H'_i$, $i = 1, 3, 5, \dots$. Thus,

$$h'_2 + h'_4 + \dots = \tilde{h}'_1 + \tilde{h}'_3 + \dots .$$

But $G = H'_1 \oplus H'_2 \oplus H'_3 \oplus \dots$ and $0 \in H'_i$, $1 \le i \le n$. Hence $h'_2 = h'_4 = \dots = 0$. Similarly we have $h'_1 = h'_3 = \dots = 0$, so establishing the lemma.

We shall assume throughout the rest of the paper that whenever a Z-factorization $G = A \oplus B$ arises from the series $G = K_1 \supset K_2 \supset \dots \supset K_n \supset \langle 0\rangle$ the coset representatives have been chosen as in Lemma 5 so that $A = H_1 \oplus H_3 \oplus \dots$ and $B = H_2 \oplus H_4 \oplus \dots$, where $0 \in H_i$, $1 \le i \le n$.

THEOREM 4. If $G = A \oplus B$ is a Z-factorization of G arising from the series $G = K_1 \supset K_2 \supset \dots \supset K_n \supset \langle 0\rangle$ then K_n is pure in K_{n-1}.

PROOF. We prove the result for n odd; the proof for n even is similar.

We may assume

$$A = H_1 \oplus H_3 \oplus \dots \oplus H_n ,$$

$$B = H_2 \oplus H_4 \oplus \dots \oplus H_{n-1} ,$$

where $0 \in H_i$, $1 \le i \le n$.

Since $K_{n-1} = H_{n-1} \oplus K_n$ we have that $H_{n-1} = B \cap K_{n-1}$ is a Z-set. The result follows from Proposition 1.

LEMMA 6. Let $G = A \oplus B$ be a Z-factorization of G arising from the series $G = K_1 \supset K_2 \supset \dots \supset K_n \supset \langle 0\rangle$. For $3 \le i \le n$ let ψ_i be the natural epimorphism with kernel K_i. Then $\psi_i(G) = \psi_i(A) \oplus \psi_i(B)$ is a Z-factorization of $\psi_i(G)$ arising from the series $\psi_i(G) = \psi_i(K_1) \supset \psi_i(K_2) \supset \dots \supset \psi_i(K_{i-1}) \supset \psi_i(K_i)$.

PROOF. The result follows from the homomorphic properties of the epimorphisms ψ_i.

THEOREM 5. If $G = A \oplus B$ is a Z-factorization of G arising from the series $G = K_1 \supset K_2 \supset \dots \supset K_n \supset \langle 0\rangle$ then K_i/K_{i+1} is pure in K_{i-1}/K_{i+1}, $2 \le i \le n-1$.

PROOF. By Lemma 6 a Z-factorization of G/K_{i+1}, $2 \le i \le n-1$, arises from the series $G/K_{i+1} = K_1/K_{i+1} \supset K_2/K_{i+1} \supset \dots \supset K_{i-1}/K_{i+1} \supset K_i/K_{i+1} \supset K_{i+1}/K_{i+1}$. Application of Theorem 4 completes the proof.

5. EXAMPLES.

We now show that the converses of theorems 4 and 5 are false.

EXAMPLE 2. Let G be a group of type $(2^2, 2, 2)$ and let a, b, and c of orders 2^2, 2, and 2 respectively generate G. Consider the series $G = K_1 \supset K_2 \supset K_3 \supset K_4 \supset K_5 = \langle 0\rangle$, where $K_4 = \langle 2a\rangle$, $K_3 = \langle b\rangle \oplus \langle 2a\rangle$, $K_2 = \langle c\rangle \oplus \langle b\rangle \oplus \langle 2a\rangle$. Then K_i/K_{i+1} is pure in K_{i-1}/K_{i+1}, $2 \le i \le 4$. Suppose $G = A \oplus B$ is a Z-factorization arising from the above series. We may assume $A = H_1 \oplus H_3$, $B = H_2 \oplus H_4$, $0 \in H_i$, $1 \le i \le 4$. The only possible choices for H_3 are $\langle b\rangle$ and $\langle 2a+b\rangle$, and H_1 must have the form $H_1 = \{0, \gamma\}$, $\gamma \ne 0$, $\gamma \in K_2$. Since K_2 contains all the non-zero elements of order 2, γ must be of order 2^2. Thus γ has the form $\gamma = a + k_2$ for some $k_2 \in K_2$. We have that $\gamma \in H_1 \oplus H_3$. Therefore $2\gamma \in H_1 \oplus H_3$. But $2\gamma \in K_2$. Hence $2\gamma \in (H_1 \oplus H_3) \cap K_2 = H_3$. Depending on the choice for H_3, we have that $2\gamma = b$ or $2\gamma = 2a+b$. Clearly both cases are impossible and we conclude that for every choice of H_3 we cannot choose H_1 such that $A = H_1 \oplus H_3$ is a Z-set.

Example 3 answers the following questions negatively:

If G is a "bad" group, are all its "good factorizations" (i.e., the factorizations in which at least one factor is periodic) obtained from the variation of Sands' method?

If G is a "Z-good" group, are all its Z-factorizations obtained from the variation of Sands' method?

EXAMPLE 3. Let G be a group of type (p,p,p,p,2), p an odd prime, and let a_1, a_2, b_1, b_2, and c of orders p, p, p, p, and 2 respectively generate G. Let $T = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle b_1 \rangle \oplus \langle b_2 \rangle$,

$$A' = (\langle a_1, a_2 \rangle \setminus \langle a_2 \rangle) \cup \langle a_2 + b_2 \rangle ,$$

$$B = (\langle b_1, b_2 \rangle \setminus (\bigcup_{i=1}^{p-1} \langle b_1 + ib_2 \rangle)) \cup (\bigcup_{i=1}^{p-1} \langle b_1 + ib_2 + 2a_2 \rangle).$$

By Lemma 4 we have that A' and B are non-periodic Z-sets and $T = A' \oplus B$. Thus T is "Z-bad" and therefore "bad." Consequently, G itself is "bad" [6]. However, in view of Lemma 2, the Sylow 2-subgroup of G, $\langle c \rangle$, is "Z-good" so that G is "Z-good" by Lemma 1.

Let $A = A' \oplus \langle c \rangle$. Clearly A is a periodic Z-set and $\langle c \rangle \subseteq S$, the subgroup of periods of A. Let $s \in S$ so that for all $a \in A$, $a+s \in A$. Then for all $a' \in A'$, $a'+s \in A$. Thus $a' + s = \tilde{a} = \tilde{a}' + x$, $\tilde{a} \in A$, $\tilde{a}' \in A'$, $x \in \langle c \rangle$. Hence for all $a' \in A'$, $a' + s - x \in A'$ and $s - x$ is a period of A'. Since A' is non-periodic we must have that $s - x = 0$, i.e., $s = x \in \langle c \rangle$. Therefore $S = \langle c \rangle$.

We have that $G = T \oplus \langle c \rangle = A \oplus B$. Suppose this factorization arises from the series $G = K_1 \supset K_2 \supset \cdots \supset K_n \supset \langle 0 \rangle$. Since B is non-periodic, $H_n = K_n$ is not a factor of B. Thus there exist transversals H_i such that $0 \in H_i$, $1 \leq i \leq n$, and

$$A = H_n + H_{n-2} + \cdots ,$$

$$B = H_{n-1} + H_{n-3} + \cdots .$$

H_n is contained in the subgroup of periods of A so that $H_n = \langle c \rangle$.

Note that $B \oplus \langle c \rangle$ is not a subgroup. Thus $K_{n-1} \neq B \oplus \langle c \rangle$ and consequently $H_{n-1} \neq B$. But $|H_{n-1}|$ divides $|B| = p^2$. Hence $|H_{n-1}| = p$. $H_{n-1} = B \cap K_{n-1}$ implies that H_{n-1} is a Z-set. Thus H_{n-1} is a subgroup and we conclude that B is periodic, a contradiction.

Let G be a finite abelian group such that all Sylow subgroups of G are "Z-good." It remains an open question as to whether all "Z-factorizations" of G can be obtained from the variation of Sands' method.

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