ITERATIONS CONVERGING TO DISTINCT SOLUTIONS
OF SOME NONLINEAR OPERATOR EQUATIONS IN BANACH SPACE

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ABSTRACT. We examine the solvability of multilinear equations of the form
\[ M_{k}(x,x,\ldots,x) = y, \quad k = 2,3,\ldots \]
where \( M_{k} \) is a \( k \)-linear operator on a Banach space \( X \) and \( y \in X \) is fixed.

KEY WORDS AND PHRASES. Multilinear operator, contraction.

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1. INTRODUCTION.

We study the quadratic equation
\[ B(x,x) y (.) \]
in a Banach space \( X \), where \( B \) is a bounded symmetric bilinear operator on \( X \) and \( y \) is fixed in \( X \) \[2\], \[3\], \[7\], \[9\], \[10\]. We consider two cases.

CASE 1. Let \( y \neq 0 \) and set \( x = \bar{x} - h \) for some \( \bar{x} \) such that the linear operator \( 2B(\bar{x}) \) is invertible then (1.1) becomes
\[ B(h,h) = h - \bar{y} \]
where \( \bar{B} = (2B(\bar{x}))^{-1}B, \quad \bar{y} = (2B(\bar{x}))^{-1}B(\bar{x},\bar{x}) \) and \( h \in X \) is to be determined.

We introduce the iteration
\[ h_{n+1} = (\bar{B}(h_{n}))^{-1}(h_{n} - \bar{y}) \quad \text{for some } \quad h_{0} \in X \]
(1.3)
to find a solution \( h \) of (1.2) such that \( h \neq \bar{x} \).

It turns out under certain assumptions that iteration (1.3) converges to an \( h \in X \) such that \( h \neq \bar{x} \), therefore \( x = \bar{x} - h \) is a nonzero solution of (1.1).

CASE 2. Let \( y \neq 0 \), we then introduce the iteration
\[ x_{n+1} = B(x_{n})^{-1}(y) \quad \text{for some } \quad x_{0} \in X \]
(1.4)
to find solutions of (1.1).

The results obtained here can be generalized to include multilinear equations of the form
where $M_k$ is a $k$-linear operator on $X$ and $y$ is fixed in $X$ \[10\].

We now state the following lemma. The proof can be found in \[10\].

2. EXISTENCE THEORY.

**Lemma 1.** Let $L_1$ and $L_2$ be bounded linear operators in a Banach space $X$, where $L_1$ is invertible, and $\|L_1^{-1}\| \cdot \|L_2\| < 1$. Then $(L_1 + L_2)^{-1}$ exists, and

$$\|(L_1 + L_2)^{-1}\| \leq \frac{\|L_1^{-1}\|}{1 - \|L_2\| \cdot \|L_1^{-1}\|}.$$

**Lemma 2.** Let $z \neq 0$ be fixed in $X$. Assume that the linear operator $\overline{B}(z)$ is invertible then $\overline{B}(x)$ is also invertible for all $x \in U(z,r) = \{ x \in X \mid \|x-z\| < r \}$, where $r \in (0,r_0)$ and $r_0 = \left[ \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\| \right]^{-1}$.

**Proof.** We have

$$\|\overline{B}(x-z)\| \cdot \|\overline{B}(z)^{-1}\| \leq \|\overline{B}\| \cdot \|x-z\| \cdot \|\overline{B}(z)^{-1}\|$$

$$\leq \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\| \cdot r$$

$$< 1$$

for $r \in (0,r_0)$. The result now follows from Lemma 1 for $L_1 = \overline{B}(z)$, $L_2 = \overline{B}(x-z)$ and $x \in U(z,r)$.

**Definition 1.** Assume that the linear operator $\overline{B}(z)$ is invertible.

Define the operators $F, T$ on $U(z,r)$ for some $r > 0$ by

$$F(x) = \overline{B}(x,x) + \overline{y} - x, \quad T(x) = (\overline{B}(x))^{-1}(x-\overline{y})$$

and the real polynomials $f(r), g(r)$ on $R$ by

$$f(r) = a'r^2 + b'r + c', \quad g(r) = ar^2 + br + c,$$

$$a' = (\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|)^2,$$

$$b' = -2\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|,$$

$$c' = 1 - \|\overline{B}(z)^{-1}\| - \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|^2 \cdot \|z-\overline{y}\|,$$

$$a = \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|,$$

$$b = \|\overline{B}(z)^{-1}(I-\overline{B}(z))\| - 1,$$

and

$$c = \|\overline{B}(z)^{-1}F(z)\|.$$
is well defined and it converges to a unique solution $h$ of (1.2) in $\overline{U}(z,r)$ for any $h_0 \in \overline{U}(z,r)$.

PROOF. $T$ is well defined by Lemma 2.

CLAIM 1. $T$ maps $\overline{U}(z,r)$ into $\overline{U}(z,r)$.

If $x \in \overline{U}(z,r)$ then

$$T(x) - z = \overline{E}(x)^{-1}(x - y) - z$$

so

$$||T(x) - z|| \leq r$$

if

$$\frac{1}{1 - ||E|| \cdot ||E(z)^{-1}||} \left[ ||E(z)^{-1}(1 - \overline{E}(z))|| \cdot r + ||E(z)^{-1}F(z)|| \right] \leq r$$

(using Lemma 1 for $L_1 = \overline{E}(z)$ and $L_2 = \overline{E}(x-z)$) or $g(r) \leq 0$ which is true by hypothesis.

CLAIM 2. $T$ is a contraction operator on $\overline{U}(z,r)$.

If $w, v \in \overline{U}(z,r)$ then

$$||T(w) - T(v)||$$

$$= ||\overline{E}(w)^{-1}(w - \overline{E}(v)^{-1}(v - y))||$$

$$= ||\overline{E}(w)^{-1}(1 - \overline{E}(\overline{E}(v)^{-1}(v - y))(w - v))||$$

$$= \frac{1}{1 - ||E|| \cdot ||E(z)^{-1}||} \left[ ||E(z)^{-1}|| + ||E|| \cdot ||E(z)^{-1}||^2 \cdot r + \frac{||E|| \cdot ||E(z)^{-1}||^2 \cdot ||y - z||}{||E|| \cdot ||E(z)^{-1}|| \cdot r} \right] \cdot ||w - v||$$

$$= q \cdot ||w - v||$$

So $T$ is a contraction on $\overline{U}(z,r)$ if $0 < q < 1$, where

$$q = \frac{1}{1 - ||E|| \cdot ||E(z)^{-1}|| \cdot r} \left[ ||E(z)^{-1}|| + ||E|| \cdot ||E(z)^{-1}||^2 \cdot r + \frac{||E|| \cdot ||E(z)^{-1}||^2 \cdot ||y - z||}{||E|| \cdot ||E(z)^{-1}|| \cdot r} \right]$$

which is true since $f(r) > 0$.

THEOREM 2. Assume that there exist $r > 0$, $z, \overline{x} \in \overline{X}$ satisfying the hypotheses of Theorem 1 and

(a) $0 < ||\overline{x}|| < -1 + \sqrt{1 + 4 ||E|| ||y||}$;

(b) $r + ||z|| < \frac{||y||}{1 + ||E|| \cdot ||\overline{x}||}$

then if $||\overline{x}|| < h_0 = r + ||z||$, the solution $h$ if (1.2) is such that

$$||\overline{x}|| < ||h|| \leq r + ||z||$$

Moreover, $x = \overline{x} - h$ is a nonzero solution of (1.1).

PROOF. By Theorem 1 $h \in \overline{U}(z,r)$ therefore

$$||h|| \leq r + ||z||.$$
Assume that \( \| x_k \| > \| x \| \) for \( k = 0,1,2,\ldots,n \). By iteration (1.3) we have
\[
B(h_{n+1}, h_n) = h_n - y
\]
or
\[
\| B \| \| x_{n+1} \| \cdot \| h_n \| \geq \| x_n - y \| \geq \| y \| - \| h_n \|
\]
so
\[
\| h_{n+1} \| \geq \frac{\| y \| - \| h_n \|}{\| B \| \cdot \| h_n \|},
\]
to show that
\[
\| h_{n+1} \| > \| x \|,
\]
it suffices to show
\[
\frac{\| y \| - \| h_n \|}{\| B \| \cdot \| h_n \|} > \| x \|
\]
which is true by (b). For consistency we must have
\[
\| x \| < \frac{\| y \|}{1 + \| B \| \cdot \| x \|}
\]
which is true by (a). The result now follows by taking the limit as \( n \to \infty \) in (2.1).

Finally note that since \( \| h \| > \| x \| \), \( x - h \neq 0 \) therefore \( x = x - h \) is a non-zero solution of (1.1).

**DEFINITION 2.** Assume that the linear operator \( B(z) \) is invertible for some \( z \in X \). Define the operator \( \overline{P} \) on \( U(z,r) \) for some \( r > 0 \) by
\[
\overline{P}(x) = B(x,x) - y, \quad y \neq 0
\]
and the real polynomials \( \overline{f}(r), \overline{g}(r) \) on \( \mathbb{R} \) by
\[
\overline{f}(r) = s_1 r^2 + s_2 r + s_3, \quad \overline{g}(r) = s_1 r^2 + s_2 r + s_3,
\]
where
\[
\begin{align*}
s_1' &= (\| B \| \cdot \| B(z)^{-1} \|)^2 \\
s_2' &= -2\| B \| \cdot \| B(z)^{-1} \| \\
s_3' &= 1 - \| B \| \cdot \| B(z)^{-1} \| \\
s_1 &= \| B \| \cdot \| B(z)^{-1} \| \\
s_2 &= \| B \| \\
s_3 &= \| B(z)^{-1} \|.
\end{align*}
\]

The proofs of the following theorems are omitted as similar to Theorems 1 and 2.

**THEOREM 3.** Let \( z \in X \) be such that the linear operator \( B(z) \) is invertible and that the following are true:

\( a) \) \( s_3' > 0; \)
\( b) \) \( s_2 > 0, \ s_2 - 4s_1 s_2 > 0, \) and
\( c) \) there exists \( r > 0 \) such that \( \overline{f}(r) > 0 \) and \( \overline{g}(r) \leq 0 \)
then the iteration
\[ x_{n+1} = B(x_n)^{-1}(y) \]
for some \( x_0 \in X \) is well defined and it converges to a solution \( x \) of (1.1) which is unique in \( \overline{U}(z,r) \) for any \( x_0 \in \overline{U}(z,r) \).

**THEOREM 4.** Let \( z,r \) be such that the hypotheses of Theorem 3 are satisfied. Let \( p < q \) be positive numbers such that

\[ a) \quad p \|B\| \leq \|y\| ; \]
\[ b) \quad \frac{\|B(z)^{-1}\|}{1 - \|B\| \cdot \|B(z)^{-1}\| r} \leq q \leq r + \|z\| \]

then if \( p \leq \|x_0\| \leq q \) then the solution \( x \) of (1.1) is such that
\[ p \leq \|x\| \leq q. \]

**REFERENCES**

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