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GENERALIZED RAMSEY NUMBERS FOR PATHS IN 2-CHROMATIC GRAPHS

R. MEENAKSHI

Mathematics Department
The University of Toledo
Toledo, Ohio 43606

and

P.S. SUNDARARAGHAVAN

Computer Systems Department
The University of Toledo
Toledo, Ohio 43606

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ABSTRACT. Chung and Liu have defined the d-chromatic Ramsey number as follows. Let $1\le d\le c$ and let $t = \binom{c}{d}$. Let $1,2,\ldots,t$ be the ordered subsets of d colors chosen from c distinct colors. Let $G_1,G_2,\ldots,G_t$ be graphs. The d-chromatic Ramsey number denoted by $r_d^c(G_1,G_2,\ldots G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i, the subgraph whose edges are colored in the ith subset of colors contains a G_i. In this paper it is shown that $r_2^3(P_i,P_j,P_k) = [(4k+2j+i-2)/6]$ where $i\le j\le k<r(P_i,P_j)$, r_2^3 stands for a generalized Ramsey number on a 2-colored graph and P_i is a path of order i.

KEY WORDS AND PHRASES. Ramsey Number, Generalized Ramsey Number, d-Chromatic Ramsey number, Colored Graph.

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1. INTRODUCTION AND NOTATION.

Chung and Liu [1] have defined the d-chromatic Ramsey number as follows. Let $1\le d\le c$ and let $t = \binom{c}{d}$. Let $1,2,\ldots,t$ be the ordered subsets of d colors chosen from c distinct colors. Let $G_1,G_2,\ldots,G_t$ be graphs. The d-chromatic Ramsey number denoted by $r_d^c(G_1,G_2,\ldots,G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i, the subgraph whose edges are colored in the ith subset of colors contains a G_i. In this paper the value of $r_2^3(P_i,P_j,P_k)$ is found. Let $P_{i(r,s)}$ and $C_{i(r,s)}$ respectively denote a path or a cycle connecting i nodes whose edges are colored in color r or color s. Let $d_i(x)$ denote the degree of node x in color i. Let $|N_i(x)UN_i(y)|$ denote the number of vertices adjacent to x or y in color i, and $r(P_i,P_j)$ be the least number p such that when the edges of the full graph K_p are colored in colors 1 and 2 contains a $P_{i(1)}$ or $P_{j(2)}$. It is assumed throughout that $2\le i\le j\le k$. Let [i] and {i} respectively denote the largest integer less than or equal to i and the smallest integer greater than or equal to i. A colored graph G is a complete graph whose edges are colored in colors 1, 2, or 3. V(G) and E(G) denote the set of nodes and edges of G and E_i is the set of edges in color i.

2. MAIN RESULT.

First a series of Lemmas are presented which is followed by a bounding theorem for $r_2^3(P_i,P_j,P_k)$ and finally, an example shows that the bound is tight.

Lemma 1:

$$r_2^3(P_i,P_j,P_k) = r(P_i,P_j) \text{ when } i\leq 3. \tag{2.1}$$

Proof: It is well known that

$$r(P_i,P_j) = j+[i/2] \text{ when } j\geq i\geq 2. \tag{2.2}$$

In [1] it is shown that

$$r_2^3(G_i,G_j,G_k)\leq r(G_i,G_j). \text{ for } i\leq j\leq k, \tag{2.3}$$

and the equality holds if $k\geq r(G_i,G_j)$.

Lemma 2:

$$r_2^3(P_i,P_j,P_k)\leq[(4k+2j+i-2)/6] \tag{2.4}$$

when $i=4$ and $k<r(P_i,P_j)$.

Proof: From (2.2) and (2.3), $k=j$ and $[(4k+2j+i-2)/6]=j+[(i-2)/6]$. Let $j=4$ and G be a colored K_4 with no $P_{4(1,2)}$ as a subgraph which implies that $\exists$ a $x\varepsilon V(G)\ni d_3(x)\geq 2$. If y and z are adjacent to x is color 3 then G has a $P_{4(1,3)}$ or $P_{4(2,3)}$. Assume that

$$r_2^3(P_4,P_{j-1},P_{j-1}) = j-1 \qquad \text{for all } j>4. \tag{2.5}$$

Let G be a colored K_j with no $P_{4(1,2)}$ as a subgraph. If $\exists$ a $x\varepsilon V(G)\ni d_3(x)\geq j-3$, then by (2.5) $G-x$ contains a $P_{[j-1](1,3)}$ or $P_{[j-1](2,3)}$ and so G contains a $P_{j(1,3)}$ or $P_{j(2,3)}$. Hence, let $d_{(1,2)}(x) \geq 3$ $\forall$ $x\varepsilon N(G)$, which implies that G contains a $P_{4(1,2)}$, a contradiction.

Lemma 3: Let $k\geq 3$ and $j = k-2[(k+4)/6]$. Then

$$r_2^3(P_j,P_j,P_k)\leq k-1.$$

Proof: Let s be the least non-negative integer $\ni$ $s \equiv k(\text{mod } 6)$.
It is easily shown that

$$r(P_j,P_j) = j+[j/2]-1 = K-1-[s/2]. \tag{2.6}$$

From (2.3) and (2.6) the lemma follows.

Lemma 4: Let $k\geq 3$ and $\ell = k+[(k-2)/6]$. Let G be a colored K_ℓ. If G contains a $C_{[k-1](1,2)}$, $C_{[k-1](1,3)}$, or $C_{[k-1](2,3)}$, then G contains a $P_{k(1,2)}$, $P_{k(1,3)}$, or $P_{k(2,3)}$, respectively.

Proof: Without loss of generality assume that G contains a $C_{[k-1](2,3)}$ but not $P_{k(2,3)}$ which implies that the $[(k+4)/6]$ vertices of G not in $C_{[k-1](2,3)}$ are adjacent in color 1 to C_{k-1}. By Lemma 3, the subgraph generated by nodes of C_{k-1} contains a $P_{j(1,2)}$ or $P_{j(1,3)}$ where $j = k-2[(k+4)/6]$. Without loss of generality assume that $P_{j(1,2)}$ is present and let x be one of its end vertices. Consider the remaining $2[(k+4)/6]-1$ vertices of C_{k-1}. Since there exists $[(k+4)/6]$ vertices not in C_{k-1}, but adjacent to every vertex of C_{k-1} in color 1, there exists a path P with $2[(k+4)/6]$ vertices in color 1, vertex disjoint from $P_{j(1,2)}$ referred above.

This path P has an end vertex adjacent to x in color 1 and hence G contains $P_{k(1,2)}$ as a subgraph.

Theorem 1:

$$r_2^3(P_i,P_j,P_k) \leq [(4k+2j+i-2)/6] \tag{2.7}$$

when $\quad k<j+[i/2]-1 = r(P_i,P_j)$.

Proof: If $i\leq 4$, the theorem follows from Lemma 2. The rest of the proof is divided into three main cases. Assume that the theorem holds when $i'<i, j'<j$ or $k'<k$ where $i\geq 5$. Define $\ell = [(4k+2j+i-2)/6]$. Let G be a colored K_ℓ.

Case 1: Let $i = j = k$. Without loss of generality let $x_1\varepsilon V(G)$ be $\ni$

$$n = d_1(x_1)\geq d_i(x) \tag{2.8}$$

For $i = 2,3$ and $x\varepsilon V(G)$ and $2 \leq n$.

Consider $G-x_1$. By the induction hypothesis,

$$r_2^3(P_{i-2},P_{i-2},P_i)\leq i-2+[(i+4)/6]$$

which implies that $G-x_1$ contains a $P_{[i-2](1,2)}$, $P_{[i-2](1,3)}$ or $P_{i(2,3)}$.

Without loss of generality assume that $G-x_1$ has $P_{[i-2](1,2)}$ and denote this path by $P = (y,\dots,z)$.

Case 1.1: Let $(x_1,y)\varepsilon E_1$ and $(x_1,z)\varepsilon E_3$, since otherwise the proof follows from Lemma 4. If $(x_1,u)\varepsilon E-E(P)$ and $(x_1,u)\varepsilon E_1$ then G has a $P_{i(1,2)}$. Thus X_1 is adjacent to n vertices of V(P) in color 1. Let $v \neq y$ be $\ni$ $(x_1,v)\varepsilon E_1$ and $v\varepsilon V(P)$. Let u be adjacent to v in P on the segment from v to y. Let $(y,u)\not\varepsilon E_3$. Then the existence of cycle $(x_1,y,\dots,u,z,\dots,v,x_1)$, by Lemma 4 implies the existence of $P_{i(1,2)}$ completing the proof.

Suppose we let $(z,u)\varepsilon E_3$. Let $f\varepsilon$ V(G)-V(P). If $(z,f)\varepsilon E_1UE_2$, then G has a $P_{i(1,2)}$ and hence let $(z,f)\varepsilon E_3$ for all f. Since $V(G)-V(P) = [(i+4)/6]+1$, $d_3(z)\geq[(i+4)/6]+1+n-1$. Since $d_3(z)\leq n$, $[(i+4)/6] = 0$ contradicting $i\geq 5$.

Case 1.2: Let (x_1,y) and (x_1,z) be in E_3. Let x_1 be adjacent to n_1 vertices of V(P) and n_2 vertices of V(G)-V(P) in color 1 where $n_1,n_2\geq 0$. Let $v\varepsilon V(P)$ be such that $(x_1,v)\varepsilon E_1$. Let u be a vertex adjacent to v on the segment $(y,\dots,v)$. By an argument similar to that used in Case 1.1, it can be shown that if $(z,u)\not\varepsilon E_3$ the proof follows from Lemma 3. For the other case, let z be adjacent to at least n_1 vertices of V(P) in color 3. For $w\varepsilon V(G)-V(P)$ if $(x,w)\varepsilon E_1$ and $(z,w)\varepsilon E_1UE_2$ the theorem follows. So z is adjacent to at least n_2 vertices of V(G)-V(P) in color 3. So $d_3(z)\geq n_1+n_2+1$, contradicting (2.8).

Case 2: Let $i = j < k$. If x_1 is $\ni$ $d_1(x_1) \geq d_i(x)$ for $i = 1,2,3$ and $x\varepsilon V(G)$ then $r_2^3(P_{i-2},P_{i-2},P_k) \leq \ell-1$ and Case 1 applies. Hence, without loss of generality assume that $d_2(x_2) \geq d_i(x)$ for $i = 1,2,3$ and $x\varepsilon V(G)$. Consider $G-x_2$. By induction hypothesis $r_2^3(P_{i-2},P_j,P_{k-1}) \leq \ell-1$ and hence $G-x_2$ contains $P_{[i-2](1,2)}$, $P_{j(1,3)}$ or $P_{[k-1](2,3)}$. If $P_{[i-2](1,2)}$ is present the proof is similar to Case 1. Let $G-x_2$ contain $P_{[k-1](2,3)}$. If z is an end vertex of this path then by arguments similar to Case 1 a contradiction, $d_1(z)$ $d_2(x_2)$ is derived thus proving the theorem. The theorem again follows if $P_{j(1,3)}$ is a subgraph of $G-x_2$.

Case 3: Let $i < j \le k < j+[i/2]-1$. Let $x \in V(G)$ be such that $d_1(x)+d_2(x) \le d_1(y)+d_2(y)$ for $y \in V(G)$. By induction hypothesis $r_2^3(P_i, P_{j-1}, P_{k-1}) \le \ell-1$, $G-x$ contains a $P_{i(1,2)}$, $P_{[j-1](1,3)}$ or $P_{[k-1](2,3)}$. The case is not obvious, if one of the latter two paths is present. If $d_1(x)+d_2(x) \le [j/2]$, then $d_3(x) \ge \{k/2\}$ so that x is adjacent to more than half the vertices of the graph and hence of the path under consideration in color 3. Therefore, G contains a $P_{j(1,3)}$ or $P_{k(2,3)}$. If $d_1(x)+d_2(x) \ge \{j/2\}$ and if $\langle E_1 U E_2 \rangle$ is connected, it is a standard result that G contains a $P_{\ell(1,2)}$, $\ell \ge 2\{j/2\}$ and hence G has a $P_i(1,2)$. However, if $\langle E_1 U E_2 \rangle$ is disconnected, it contains at least two components, each of which is of order $\{j/2\}$ or greater and hence G contains a $P_{j(3)}$.

Theorem 2:

$$r_2^3(P_i, P_j, P_k) > [(4k+2j+i-2)/6]-1.$$

Proof: Let $G = k_{\ell-1}$, where $\ell = [(4k+2j+i-2)/6]$. Let X, Y, Z be pairwise disjoint subgraphs of G such that $|X| = \{(2k+j-i-1)/3\}$, $|Y| = [(2j-2K+i-2)/6]$ and $|Z| = [(k-j+i-2)/3]$. It can be verified that $V(G) = |X| + |Y| + |Z|$. Color the edges of G as follows. Color the edges of X using color 3, edges of Y using color 1, edges of Z using color 2, edges between X and Y using color 1, edges between X and Z using color 2, and edges between Y and Z using color 1. It can be shown that $|X| + |Z| = k-1$ which rules out the existence of $P_{k(2,3)}$. Similarly $2|Y| + |X| \le j-1$ ruling out $P_{j(1,3)}$ and $2|Y|+2|Z|+1 \le i-1$ ruling out $P_{i(1,2)}$.

Theorem 3:

$$r_2^3(P_i, P_j, P_k) = [(4k+2j+i-2)/6] \text{ when } k < r(P_i, P_j) = j + [\tfrac{i}{2}] - 1$$

and

$$r_2^3(P_i, P_j, P_k) = r(P_i, P_j) \text{ when } k \ge r(P_i, P_j) = j + [\tfrac{i}{2}] - 1.$$

Proof: Follows from (2.3) and Theorems 1 and 2.

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