ABSTRACT. For each bounded self-adjoint operator $T$ on a Hilbert module $H$ over an $H^*$-algebra $A$ there exists a locally compact space $\mathcal{M}$ and a certain $A$-valued measure $\mu$ such that $H$ is isomorphic to $L^2(\mu)\otimes A$ and $T$ corresponds to a multiplication with a continuous function. There is a similar result for a commuting family of normal operators. A consequence for this result is a representation theorem for generalized stationary processes.


1. INTRODUCTION.

The diagonalization theorem states that for each bounded self-adjoint linear operator $T$ acting on a Hilbert space $H$ there exists a measure space $(S, \mu)$ and a real valued measurable function $h(s)$ such that $H$ is isomorphic to $L^2(S, \mu)$ and $T$ corresponds to the multiplication with $h(s)$. Furthermore, the space $(S, \mu)$ could be selected in such a way that there is a Hausdorff topology on $S$ with respect to which $h(s)$ is continuous, $S$ is locally compact and which makes $\mu$ a regular Borel measure. In this note we shall give a suitable generalization of this fact.

The situation is somewhat more complex in our case. The space $L^2(S, \mu)$ needs to be replaced by the tensor product $L^2(\mu)\otimes A$, which is less manageable. This space is properly defined below.

2. PRELIMINARIES.

Let $A$ be a proper $H^*$-algebra (Ambrose [1]) and let $\mathcal{R}_A = \{xy|x, y \in A\}$ be its trace-class (Saworotnow and Friedell [2]); let $X$ be a locally compact Hausdorff space and let $\mu$ be a positive $\mathcal{R}_A$-valued Borel measure on $X$. The last statement means that $\mu$ is defined on the class $\mathcal{B}$ of all Borel subsets $\Delta$ of $X$ having the property that $\Delta \subseteq Q$ for some compact set $Q$, and $\mu$ is such that $(\mu(\Delta)x, x) \geq 0$ for all $\Delta \in \mathcal{B}$ and each $x \in \mathcal{A}$. Members of $\mathcal{B}$ will be called bounded Borel sets (a bounded Borel set is a Borel set included in a compact set). Note that the scalar-valued function $m_A = \text{tr}_A \mu_A \Delta \in \mathcal{B}$, is an ordinary Borel measure on $X$; it coincides with the total variation $\|\cdot\|$ (Definition in III.1.4 of Dunford and Schwartz [3]) of $\nu$. defe
Let $S(X)$ and $S(X,A)$ be respectively the classes of all complex-valued and $A$-valued simple functions of $X$. One can define the integrals for members $\psi(x) = \sum a_i \phi_{\Delta_i}(x)$ and $\xi(x) = \sum a_i \phi_{\Delta_i}(x)$ ($\Delta_i \beta$, $a_i \in A$ and $\lambda_i$'s are complex numbers) of $S(X)$ and $S(X,A)$ in the usual way by setting

$$\int \psi \, d\mu = \sum \lambda_i \mu_{\Delta_i} \quad \text{and} \quad \int \xi \, d\mu = \sum a_i \mu_{\Delta_i}$$

and then extending it to larger classes using the norms

$$||\psi|| = \int |\psi| \, dm = \sum |\lambda_i| \mu_{\Delta_i}$$

and

$$||\xi|| = \sum |a_i| \mu_{\Delta_i}.$$  

(2.2)

Let $L(X)$ and $B(X,A)$ denote respectively the classes of those functions to which the integrals are extendable in this fashion. (Note that $S(X)$ is dense in $L(X)$ and $S(X,A)$ is dense in $B(X,A)$).

Then it is easy to see that

$$r(\int \psi \, d\mu) \leq ||\psi|| \quad \text{and} \quad r(\int \xi \, d\mu) \leq ||\xi||$$

hold for all $\psi \in L(X)$ and $\xi \in B(X,A)$. (For a discussion of integrals of this type we refer the reader to Bogdanowicz [4]).

**LEMMA 1.** If $a \in A$ and either $\psi \in L(X)$ or $\psi \in B(X,A)$, then $\psi \in L(X)$ and $\int \psi \, d\mu = a \int \psi \, d\mu$. If $\psi \in S(X,A)$ and $\psi \geq 0$ and almost everywhere then $\int \psi \, d\mu \geq 0$.

**PROOF.** The first assertion is easy to verify. Let $\psi$ be a simple function such that "$\psi(x) > 0$" holds outside of some set $\Delta \beta$ with $m = \mu(\Delta) = 0$. Then $\psi$ can be represented in the form $\psi = \sum a_i \phi_{\Delta_i}$ with $\Delta_1, \Delta_2, \ldots, \Delta_n$ disjoint ($\Delta_i \beta$) and $a_i > 0$ for each $i$ for which "$m = \mu(\Delta_i) = 0"$ holds. Then

$$\int \psi \, d\mu = \sum a_i \mu_{\Delta_i} = \sum \mu_{\Delta_i} a_i \mu_{\Delta_i} \

\quad \geq \sum \mu_{\Delta_i} a_i \mu_{\Delta_i} = \sum \mu_{\Delta_i} a_i \mu_{\Delta_i} \geq 0.$$

Let $L^2(\mu) = \{f : X \rightarrow C | f \text{ is } m\text{-measurable} \text{ and } \int |f|^2 \, dm < \infty\}$ (m = tr$\mu$) be the set of all square $m$-measurable complex-valued functions. Then there is a $RA$-valued inner product

$$[\psi_1, \psi_2] = \int \bar{\psi}_1 \psi_2 \, d\mu$$

defined on $L^2(\mu)$ such that $(\psi_1, \psi_2) = \int \bar{\psi}_1 \psi_2 \, dm$ is an ordinary scalar product on $L^2(\mu)$ making $L^2(\mu)$ a Hilbert space.

**LEMMA 2.** Let $\psi_1, \psi_2, \ldots, \psi_n \in L^2(\mu)$ and let $a_1, a_2, \ldots, a_n \in A$. Then

$$\text{tr} \left( \sum_{i,j} a_i^* a_j \int \psi_i \psi_j \, d\mu \right) > 0$$

(2.6)

**PROOF.** Let $n(\psi)$ denote the norm on $L^2(\mu)$: $n(\psi)^2 = \int |\psi|^2 \, dm$. Let $\epsilon > 0$ be arbitrary; let $\eta_1, \eta_2, \ldots, \eta_n \in S(X)$ be such that $n(\psi_i - \eta_i) < \epsilon$ for $i = 1, 2, \ldots, n$. Then

$$\left| \text{tr} \left( \sum_{i,j} a_i^* \int \psi_i \psi_j \bar{\eta}_i \bar{\eta}_j \, d\mu \right) \right| \leq \epsilon r(a_i^* a_j) \int \left| \psi_i \psi_j \bar{\eta}_i \bar{\eta}_j \right| \, d\mu \leq 0$$

and the last sum can be made arbitrarily small by selecting $\epsilon$ small enough. On the other hand one can see that

$$\text{tr} \left( \sum_{i,j} a_i^* \int \psi_i \psi_j \, d\mu \right) = \text{tr} \left( \sum_{i,j} a_i^* \eta_i \eta_j \right) \geq 0$$

(2.7)
since \((\sum_{j=1}^n a_j^*\eta_j)(\sum_{j=1}^n a_j^*\eta_j)^*\) is positive and simple. Hence \(\text{tr}\sum a_j^*\int \psi_j^* d\mu a_j \geq 0\).

COROLLARY. The expression \(z = \sum_{j=1}^n (a_j^*\int \psi_j^* d\mu)\) is a positive member of \(\mathfrak{A}\).

PROOF. Note that the expression \((za,a) = \text{tr}(a^*za)\) is of the same form as \(\text{tr}z\).

Hence \((za,a) \geq 0\) for each \(a \in \mathfrak{A}\).

Now consider the space \(K\) of all tensors \(f = \sum_{i=1}^n \psi_i \otimes a_i\) with \(\psi_1, \psi_2, \ldots, \psi_n \in L^2(\mu)\) and \(a_1, a_2, \ldots, a_n \in \mathfrak{A}\). Define the positive form \([f,g]\) on \(K\) by setting

\[ [f,g] = \sum_{i,j}^n a_i^* \int \psi_i \eta_j^* d\mu b_j \quad (2.8) \]

(here \(g = \sum \eta_j \otimes b_j\)). Let \(\mathcal{N} = \{f \in K : [f,f] = 0\}\), \(K' = K''\); we define \(L^2(\mu) \otimes \mathfrak{A}\) to be the completion of \(K'\) with respect to the norm \(\|f\| = \sqrt{\text{tr}[f,f]}\) (modulo the set \(\mathcal{N}\)).

It is not difficult to see that \(L^2(\mu) \otimes \mathfrak{A}\) is a Hilbert module.

Let \(h\) be a bounded continuous real valued function on \(X\). Define the operator \(T_h\) on \(L^2(\mu) \otimes \mathfrak{A}\) by setting

\[ T_h(f) = \sum_{i} \psi_i \otimes a_i \quad (2.9) \]

Then \(T_h\) is a bounded self-adjoint (in the sense that \([T_h(f),g] = [f,T_h(g)]\) holds).

Also \(T_h\) is \(\mathfrak{A}\)-linear (additive and \(\mathfrak{A}\)-homogeneous in the sense that \(T_h(fa) = T_h(f)a\) for all \(f \in L^2(\mu) \otimes \mathfrak{A}\), \(a \in \mathfrak{A}\)).

The fact that \(T_h\) is bounded (in the sense that \(\|T_h(f)\| \leq M\|f\|\) holds for some \(M\)) can be verified directly, using \(\S 10\) of Naimark [5]. Let \(f = \sum_{i} \psi_i \otimes a_i\) be a fixed member of \(K\). Consider the positive linear functional

\[ p(y) = \text{tr}[f,Ty(f)] = \text{tr} \sum_{i} a_i^* \int \psi_i \eta_j^* d\mu a_j \quad (2.10) \]

on the space \(BC(\mathfrak{X})\) of all bounded continuous (complex) functions on \(X\). It follows from the proposition I in subsection 4 of \(\S 10\) in Naimark [5] that \(p(h^*h) \leq \|h^*h\|p(e) = \|h\|^2p(e)\). Thus:

\[ \|T_h(f)\|^2 = \text{tr}[T_h(f),T_h(f)] = \text{tr}[f,T_h^*h(f)] = p(h^*h) \leq \|h\|^2p(e) = \|h\|^2\|f\|^2. \quad (2.11) \]

We also see that \(\|T_h\| \leq \|h\|\|f\|\). It turns out that each bounded self-adjoint \(\mathfrak{A}\)-linear operator is of the form \(T_h\) described above.

3. MAIN RESULTS.

Definition. An \(\mathfrak{A}\)-linear operator \(T\) on a Hilbert module \(H\) is said to be cyclic if there exists \(f \in H\) such that the set \(\{\sum_{k=0}^{n} \lambda_k T_k(f) : \lambda_k \in \mathfrak{A}, \lambda_k \text{ complex}\}\) is dense in \(H\) (we assume that \(T_0(f) = 0\)).

THEOREM 1. For each bounded \(\mathfrak{A}\)-linear self-adjoint operator \(T\) on a Hilbert module \(H\) there exists a locally compact Hausdorff space \(X\), a \(\mathfrak{A}\)-valued positive regular measure \(\mu\) defined on the class \(\beta\) of bounded (dominated by compact sets) Borel subsets of \(X\) and a bounded continuous real valued function \(h\) on \(X\) such that \(H\) is isometrically isomorphic to \(L^2(\mu) \otimes \mathfrak{A}\) and \(T\) corresponds to the operator \(T_h\) (described above) acting on \(L^2(\mu) \otimes \mathfrak{A}\). If \(T\) is cyclic, then \(X\) is homeomorphic to the compact subset of the real line.

PROOF. Let \(B\) be the commutative \(B^*\)-algebra generated by \(T\) and the identity operator \(I\) (note that each member of \(B\) is \(\mathfrak{A}\)-linear). Let \(\mathfrak{M}\) be the set of maximal ideals of \(B\), let \(\tau\) be the standard Gelfand topology on \(\mathfrak{M}\) and let \(S \to S(\mathfrak{M})\) be the Gelfand map of \(B\) into the continuous complex functions on \(\mathfrak{M}\). Note that \(\mathfrak{M}\) is homeomorphic to the spectrum of \(T\), which is a compact subset of the real line. We consider 2 cases.
CASE I. First assume that there exists \( f \in H \) such that the set

\[
H^1 = \left\{ \sum_{i=1}^{n} s_i (f)_a S_i \right\}_{S_i \in B, a_i \in A}
\]

is dense in \( H \) (this is equivalent to the statement that \( T \) is cyclic).

Let \( \beta \) be the class of all Borel subsets of \( M \) (each \( \Delta \in \beta \) is bounded since \( M \) is compact) and let \( \Delta \longrightarrow \mu \Delta \) be a spectral measure on \( \beta \) (\( \beta \) is the class of compact subsets of \( \mathbb{R} \)).

Let \( \mathcal{L}_1 \) be the class of all Borel subsets of \( M \), and let \( \Delta \longrightarrow \mu \Delta \) be a spectral measure on \( \mathcal{L}_1 \) such that \( S = \int_{M} S(M)d\mu \). Note that each \( P \) is \( \sigma \)-finite since it commutes with linear maps \( f \longrightarrow f(a) \) (which commute with all \( S \in B \)).

Then \( \mu \Delta \rightarrow \Delta \rightarrow \mathcal{L}_1 \) is a \( \mathcal{L}_1 \)-valued positive measure on \( \mathcal{L}_1 \), and for each \( S(B) \) we have

\[
\int_{M} S(M)d\mu = \int_{M} S(M)d\mu = [f_o, \int_{M} S(M)d\mu] = [f_o, S(f)]
\]

(here, as above, \( [\cdot, \cdot] \) denotes the generalized inner product on \( H \)). In this case we can take \( X = M \). The correspondence

\[
Sf_o \longleftrightarrow S(M)
\]

is a \( \mathcal{L}_1 \)-valued positive measure on \( \mathcal{L}_1 \). For each \( S \in B \) we have

\[
\sum_{i=1}^{n} S_i (f)_a S_i \longleftrightarrow \sum_{i=1}^{n} S_i (M)S_i (a)
\]

It is easy to check that \( T \) corresponds to the operator \( T \) of multiplication with function \( h(M) T(M) \):

\[
T(\sum_{i=1}^{n} S_i (f)_a S_i \longleftrightarrow \sum_{i=1}^{n} S_i (M)S_i (a)
\]

The function \( h \) is real valued since \( T^* = T \) and \( \left\| T \right\| = \left\| h \right\| \). Note also that in this case \( M \) is homeomorphic to the spectrum of \( T \), which is a compact subset of the real line. This implies the last assertion of the theorem.

CASE II. Now let us consider the general case. For any \( f \in H \) let \( H(f) \) be the closure of the set \( \{ \sum_{i=1}^{n} S_i (f)_a S_i \} \). Then it follows from Lemma 2 in Saworotnow [6] that \( fH(f) \). Also both \( H(f) \) and its orthogonal complement \( H(f) \) (which coincides with the set \( H(f)^{\perp} = \{ \left\langle g, h \right\rangle = 0 \text{ for all } h \in H(f) \} \) (Lemma 3 of Saworotnow [6]) are invariant under \( T \).

It follows from this fact and Zorn's Principle that there exists a set \( \{ f \gamma \}_{\gamma \in \Gamma} \) of mutually orthogonal members of \( H \) such that \( H = \sum_{\gamma \in \Gamma} H(f) \) and each \( H(f) \) is invariant under \( T \).

For each \( \gamma \in \Gamma \) and \( S \in B \) let \( S \gamma \) be the restriction of \( S \) to \( H(f) \), and let \( B \gamma = \{ S \gamma : S \in B \} \). It follows from part I (case I) of this proof that for each \( \gamma \in \Gamma \) there exists a compact Hausdorff space \( (\mathcal{M}_\gamma, \tau_\gamma) \), a \( \mathcal{L}_1 \)-valued positive Borel measure \( \mu \) and
a continuous real valued function $h_\gamma(\cdot)$ on $\mathcal{M}_\gamma$ such that $H(f_\gamma)$ is isomorphic to $L^2(\mu_\gamma)\otimes A$ and the action of the operator $T_\gamma$ (the restriction of $T$) corresponds to the multiplication with $h_\gamma$ on $L^2(\mu_\gamma)$. Note also that $h_\gamma(M) \leq \|T\|$ for each $M \in \mathcal{M}_\gamma$.

Let $X = \mathcal{M}_\gamma$ and let $r$ be the topology on $X$ defined by the requirement that a set $O \in r$ is open if and only if $O \cap \mathcal{M}_\gamma$ belongs to $r_\gamma$ for each $\gamma \in \Gamma$. Let $\beta$ be the class of all bounded Borel subsets of $X$. For each index $\beta$ there are indices (we use a simplified notation here) $1,2,\ldots,n \in \Gamma$ such that $\Delta_{\gamma} = \bigcup_{i=1}^n \mathcal{M}_{i\gamma}$. We set

$$\mu(\Delta) = \sum_{i=1}^n \mu_i(\Delta_{\gamma} \cap \mathcal{M}_{i\gamma}) \tag{3.9}$$

Then $\beta$ is a ring and $\mu$ is a positive $rA$-valued measure on $\beta$. We define the function $h$ on $X$ by setting $h(M) = h_\gamma(M)$ where $\gamma \in \Gamma$ is such that $M \in \mathcal{M}_{\gamma}$. Then it is easy to see that $h$ has the required properties.

To complete the proof it is now sufficient to show that $L^2(\mu) \otimes A = \sum_{i=1}^n L^2(\mu_\gamma) \otimes A$. First note that each $L^2(\mu_\gamma)$ is included in $L^2(\mu)$ and that $L^2(\mu) = \sum_{i=1}^n L^2(\mu_\gamma)$ (easy to verify). Now let $f \in L^2(\mu)\otimes A$. For each $\epsilon > 0$ one can find $g = \sum_{i=1}^n \psi_i \otimes a_i$ such that $\|f - g\| < \epsilon$ with $\psi_i \in L^2(\mu_\gamma)$. But each $\psi_i$ can be approximated in $L^2(\mu)$ by expressions of the form $\sum_{j=1}^n \phi_j \otimes b_j$ with $\phi_j \in L^2(\mu_\gamma)$ for some $\gamma_j \in \Gamma$. Thus $f$ can be approximated (as close as we please) by members $\sum_{i=1}^n (\psi_i \otimes a_i \otimes b_i)$ of $\sum_{i=1}^n L^2(\mu_\gamma) \otimes A$, i.e., $g$ is a member of $\sum_{i=1}^n L^2(\mu_\gamma) \otimes A$.

Conversely, let $f \in \sum_{i=1}^n L^2(\mu_\gamma) \otimes A$; then $f$ can be approximated by finite sums of expressions of the type $\sum_{i=1}^n \psi_i \otimes a_i \otimes b_i$ with $a_i \in A$ and $\psi_1,\psi_2,\ldots,\psi_n$ belonging to some $L^2(\mu_\gamma)$ with $\beta \in \Gamma$. We may conclude that $f \in L^2(\mu)\otimes A$ since $L^2(\mu_\gamma) \subset L^2(\mu)$ for each $\gamma$. The reader should be able to give a precise argument here.

**THEOREM 2.** Let $Z$ be a family of bounded $A$-linear operators on a Hilbert module $H$ (over an $A^*$-algebra $A$) such that each member of $Z$ and its adjoint (with respect to the generalized inner product) commute with any other member of $Z$. In particular, $Z$ could be a commutative $*$-algebra of $A$-linear operators on $H$. Then there exists a locally compact Hausdorff space $X$, a $rA$-valued positive Borel measure $\mu$ on $X$ and a map $\mathcal{T} \mapsto h_\mathcal{T}$ of $Z$ into complex valued functions on $X$ such that $H$ is isomorphic to $L^2(\mu)\otimes A$ and each $\mathcal{T}$ corresponds to multiplication with some function $h_\mathcal{T}$. Moreover $\|h_\mathcal{T}\| \leq \|\mathcal{T}\|$ for each $\mathcal{T} \in Z$.

**PROOF.** The proof is essentially the same as the proof of Theorem 1 above. We use the $*$-algebra of operators generated by $Z$ (and the identity operator $I$) instead of the algebra generated by the operator $T$ (and $I$).

**COROLLARY 1.** Each $*$-representation of a commutative $*$-algebra by bounded $A$-linear operators is of the form $x \mapsto T_h$, where $T_h$ is an operator of multiplication with a complex valued function $h = h_x$ described before Theorem 1.

This corollary could be considered as a generalization of Theorem 65 in Mackey [7] if we disregard the fact that Mackey considers more general (self-adjoint) algebras and we do not specify the space $X$ on which the functions $h = h_x$ act (also our Hilbert module does not have to be separable (as a Hilbert space)).

**COROLLARY 2.** Let $G$ be a commutative locally compact group with composition $+$ and let $\mathcal{T} \mapsto U_\mathcal{T}$ be a $*$-representation of $G$ by $A$-linear unitary operators acting on a Hilbert module $H$. Assume that there exists a vector $f_0 \in H$ such that the submodule $H_0$, generated by the vectors of the form $U_\mathcal{T}(f_0)$, is dense in $H$. Then there exists a compact Hausdorff space $\mathcal{M}$, a positive $rA$-valued Borel measure $\mu$ on $\mathcal{M}$ and a map...
t \rightarrow g_t of G into the continuous functions on $\mathcal{M}_G$ such that H is (isometrically) isomorphic to $L^2(\mu)\otimes A$ and each $U_t$ corresponds to multiplication members of $L^2(\mu)$ with $g_t$.

The map $t \rightarrow g_t$ has the following properties (for each $t \in G$ and all $\mathcal{M}_G$):

$$g_0(M) = 1 \quad \text{(here 0 is the identity of G)} \quad (3.10)$$

$$|g_t(M)| = 1 \quad (3.11)$$

$$g_{-t}(M) = g_t(M) \quad (3.12)$$

$$g_{t+s}(M) = g_t(M)g_s(M) \quad (3.13)$$

It is appropriate at this point to mention a certain application of the last corollary. Let $G$, $A$ and $H$ be as above, and let $\xi : G \rightarrow H$ be a generalized stationary process (Saworotnow [8]), i.e., $\xi$ is an $H$-valued function on $G$ such that $((t+r),(s+t)) = (\xi(t),\xi(s))$ for all $t, r, s \in G$. Let $H_{\xi}$ be the submodule generated by the vectors of the form $\xi(t)$, $t \in G$ ($H_{\xi}$ is the closure of $\sum_{k=1}^{n} \xi(t_k)a_k : t_k \in G$).

For each $t \in G$ consider the operator $U_t$ on $H_{\xi}$ defined by

$$U_t(\sum_{k=1}^{n} \xi(t_k)a_k) = \sum_{k=1}^{n} \xi(t_k+t)a_k \quad \text{and let } f_0 = \xi(0). \quad (1.14)$$

Then the map $t \rightarrow U_t$ is a representation of $G$ by $A$-linear unitary operators and it is easy to see that the assumptions of Corollary 2 are fulfilled. Let $M, \mu$ and $g_t$ be as in Corollary 2 and let $f(M)$ be the member of $\mathcal{M}$ corresponding to $f_0 = \xi(0)$.

Then the space $H_{\xi}$ is isomorphic to $L^2(\mu)\otimes A$ and each $U_t$ corresponds to multiplication of members of $L^2(\mu)$ with $g_t$. For each $t \in G$ let $h_t(M) = g_t(M)f(M)$. In this fashion we arrived at a concrete representation of the abstract stationary process $\xi$ by the complex valued continuous function $h_t$ defined on $M$. Note that the scalar product $(\xi(t),\xi(s))$ corresponds to the expression

$$\int h_t(M)h_s(M)d\mu(M) = \int g_t(M)g_s(M)f(M)f(M)d\mu(M) = \int g_t(M)h_{-s}(M)|f(M)|^2d\mu(M) = \int g_{t+s}(M)|f(M)|^2d\mu(M) \quad (3.15)$$

and this expression depends on $t-s$ only and is independent of a particular choice of $t$ and $s$.

4. CONCLUDING REMARK.

To conclude the paper we make the following remark about the operator $T_h$ discussed above. It is easy to see that we do not need at all to assume existence of a (locally compact) topology on the space $X$ (discussed at the beginning of this paper). Let $\mu$ be a positive $\sigma$-finite measure defined on some $\sigma$-ring of subsets of $X$. If $h$ is any $\mathcal{M}$-measurable essentially bounded real valued function on $X$ then the corresponding operator $T_h$ on $L^2(\mu)\otimes A$,

$$T_h(\sum \psi_i \otimes a_i) = \sum (\psi_i h) \otimes a_i \quad (3.16)$$

is also self-adjoint, $A$-linear and bounded. The fact that $T_h$ is bounded can be verified in the same way as above using the algebra $B$ of all essentially bounded $\mathcal{M}$-measurable complex-valued functions on $X$. 

REFERENCES


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