ON THE RADIUS OF UNIVALENCE OF
CONVEX COMBINATIONS OF ANALYTIC FUNCTIONS

KHALIDA I. NOOR, FATIMA M. ALOBoudi and NAEELA ALDIHAN

Mathematics Department
Science College of Education for Girls
Malaz, Sitteen Road
Riyadh, SAUDI ARABIA

(Received July 10, 1982 and in revised form February 21, 1983)

ABSTRACT. We consider for $\alpha > 0$, the convex combinations $f(z) = (1-\alpha)F(z) + \alpha zF'(z)$, where $F$ belongs to different subclasses of univalent functions and find the radius for which $f$ is in the same class.

KEY WORDS AND PHRASES. Univalent functions, alpha-quasi-convex, starlike, close-to-convex functions, convex combinations.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. Primary 30A32, Secondary 30A34.

1. INTRODUCTION.

Let $S$, $K$, $S^*$ and $C$ denote the classes of analytic functions in the unit disc $E = \{z: |z| < 1\}$ which are respectively univalent, close-to-convex, starlike, and convex. In [1,2], a new subclass $C^*$ of univalent functions was introduced and studied.

A function $f$, analytic in $E$, belongs to $C^*$ if and only if there exists a convex function $g$ such that for $z \in E$,

$$\Re \left( \frac{zf'(z)}{g'(z)} \right) > 0.$$ (1.1)

The functions in $C^*$ are called quasi-convex and $C \subset C^* \subset K \subset S$. It is shown [2] that $f \in C^*$ if and only if $zf' \in K$. Recently the functions called $\alpha$-quasi-convex have been defined and their properties studied in [3]. A function $f$, analytic in $E$, is said to be $\alpha$-quasi-convex if and only if there exists a convex function $g$ such that, for $\alpha$ real and positive

$$\Re \left( (1 - \alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \right) > 0.$$ (1.2)
It has been shown [3] that $F$ is $\alpha$-quasi-convex if and only if $f$ with
\[
f(z) = (1 - \alpha)F(z) + \alpha zF'(z)
\]
is close-to-convex. \hspace{1cm} (1.3)

All $\alpha$-quasi-convex functions are close-to-convex.

2. MAIN RESULTS.

We shall now study the mapping properties of $f$: $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$, 
$\alpha > 0$, when $F$ belongs to different subclasses of univalent functions.

THEOREM 2.1. Let $F \in S^*$ and $\alpha > 0$. The function
\[
F(z) = (1 - \alpha)F(z) + \alpha zF'(z)
\]
is starlike in $|z| < r_o$, where
\[
\frac{1}{2\alpha + \sqrt{4\alpha^2 + 1 - 2\alpha}}.
\]
(2.2)

This result is sharp.

PROOF. We can write (2.1) as
\[
f(z) = \frac{1 - \alpha}{\alpha} \left( \frac{1}{f(z)} \right)'
\]
and from this it follows that
\[
F(z) = \frac{1}{\alpha} \int_0^z \frac{1}{f(z)dz}.
\]
(2.3)

Then
\[
\frac{zF'(z)}{F(z)} = \left\{ \left( \frac{1}{\alpha} \right) \frac{1}{\alpha} \int_0^z \frac{1}{z^\alpha - 2} f(z)dz + f(z) \right\}/(z \int_0^z \frac{1}{z^\alpha - 2} f(z)dz)
\]
\[
= \left\{ \left( \frac{1}{\alpha} \right) \frac{1}{z^\alpha - 2} f(z)dz + \frac{1}{\alpha} - 1 \right\}/\left\{ \int_0^z \frac{1}{z^\alpha - 2} f(z)dz \right\} = h(z),
\]
(2.4)

where $\text{Re } h(z) > 0$, since $F \in S^*$.

From (2.4), we have
\[
\frac{1}{z^\alpha - 2} f(z) - \left( \frac{1}{\alpha} - 1 \right) \int_0^z \frac{1}{z^\alpha - 2} f(z)dz = h(z)\int_0^z \frac{1}{z^\alpha - 2} f(z)dz.
\]
(2.5)

Differentiating both sides of (2.5), we obtain
\[
\left( \frac{1}{\alpha - 1} \right) \frac{1}{z^\alpha - 2} f(z) + \frac{1}{z^\alpha - 1} f'(z) - \left( \frac{1}{\alpha - 1} \right) \frac{1}{z^\alpha - 2} f(z) = h'(z)\int_0^z \frac{1}{z^\alpha - 2} f(z)dz + h(z)\frac{1}{z^\alpha - 2} f(z).
\]
Thus
\[
\frac{zf'(z)}{f(z)} = h(z) + \left[h'(z)\int_0^z \frac{1}{z^\alpha - 2} f(z)dz\right]/(z f(z)).
\]
Now, using the well-known result [4], \(|h'(z)| \leq \{2 \text{Re } h(z)\}/(1 - r^2), |z| = r, we have

\[
\text{Re } \frac{zf'(z)}{f(z)} \geq \text{Re } h(z) \{1 - \frac{2}{1 - r^2} \left| \int_0^r \frac{z^{1/2} - 2}{z^{1/2} - 2} f(z) dz \right| \}.
\]  

(2.6)

From (2.1) and (2.3), we have

\[
\frac{1}{z^{1/2} - 2} f(z) = \frac{1}{\alpha(z^{1/2} - 2 F(z))} = \frac{1}{z^{1/2} - 2 F(z)} + \left( \frac{1}{\alpha} - 1 \right) = h(z) + \left( \frac{1}{\alpha} - 1 \right),
\]

from which it follows that

\[
\left| \frac{1}{z^{1/2} - 2} f(z)/ \int_0^r \frac{z^{1/2} - 2}{z^{1/2} - 2} f(z) dz \right| \geq \text{Re } h(z) + \left( \frac{1}{\alpha} - 1 \right) \geq \left( \frac{1}{\alpha} - 1 \right) + \frac{1 - r}{1 + r}.
\]  

(2.7)

Using (2.7), we have from (2.6)

\[
\text{Re } \frac{zf'(z)}{f(z)} \geq \text{Re } h(z) \{1 - \left( \frac{2}{1 - r^2} \right) \left( \frac{r + r^2}{1 + (\frac{1}{\alpha} - 2)r} \right) \}
\]

\[
= \text{Re } h(z) \{(\frac{1}{\alpha} - 4r - (\frac{1}{\alpha} - 2)r^2)/(1 - r)(\frac{1}{\alpha} + (\frac{1}{\alpha} - 2)r)\}.
\]  

(2.8)

The right hand side of (2.8) is positive for \(r < r_o\), where \(r_o\) is given by (2.2). This result is sharp as can be seen by

\[
f_o(z) = \{\alpha(z(\frac{1}{\alpha} - (\frac{1}{\alpha} - 2)z))\}/(1 - z)^3
\]

\[
= (1 - \alpha)F_o(z) + \alpha z F'_o(z),
\]  

(2.9)

where

\[
F_o(z) = \frac{z}{(1 - z)^2} \in S^*.
\]

REMARK 2.1. Let \(f \in C\), then \(f\), given by (2.1), is convex for \(|z| < r_o\), where \(r_o\) is given by (2.2). The proof follows on the same lines as in Theorem 2.1. See also [5] and [6].

REMARK 2.2. In [6], Nikolaeva and Repina treated the same problem, with a different notation, for the convex and starlike functions of order \(\beta\). Theorem 2.1 follows from their result when we take \(\beta = 0\) for \(0 \leq \alpha \leq 1\). On the other hand, our proof of Theorem 2.1 is much simpler and the result holds for all \(\alpha > 0\).
THEOREM 2.2. Let $F \in K$ and $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$, $\alpha > 0$. Then $f$ is close-to-convex in $|z| < r_0$, $r_0$ is given by (2.2). The function $f_0$ in (2.9) shows that this result is sharp.

PROOF. Since $F \in K$, there exists a $G \in S^*$ such that, for $z \in E$, $\Re \frac{zF'(z)}{G(z)} > 0$. Now let $g(z) = (1 - \alpha)G(z) + \alpha zG'(z)$. Then by Theorem 2.1, $g$ is starlike for $|z| < r_0$, $r_0$ is defined by (2.2). Using the same technique of Theorem 2.1, we can easily show that $\Re \frac{zf'(z)}{g(z)} > 0$ for $|z| < r_0$.

REMARK 2.3. For $\alpha = \frac{1}{2}$, this result has been proved in [7].

As an easy consequence of (1.3) and Theorem 2.2, we have the following.

COROLLARY 2.1. Let $F \in K$ and $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$, $\alpha > 0$. Then $F$ is $\alpha$-quasi-convex in $|z| < r_0$. This means that the radius of $\alpha$-quasi-convexity for close-to-convex functions is given by (2.2).

THEOREM 2.3. Let $F \in C^*$ and $\alpha > 0$. Let $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$. Then $f$ is in $C^*$, for $|z| < r_0$, $r_0$ is given by (2.2).

PROOF. Since $F \in C^*$, there exists a $G \in C$ such that for $z \in E$, $\Re \frac{zF'(z)}{G(z)} > 0$. Now let $g(z) = (1 - \alpha)G(z) + \alpha zG'(z)$, then $g$ is convex in $|z| < r_0$. We can write

\[ f(z) = (1 - \alpha)F(z) + \alpha zF'(z) = \frac{2 - \frac{1}{\alpha}}{\alpha - 1} \left( \frac{\alpha}{z} \right) F(z) \]

and

\[ g(z) = (1 - \alpha)G(z) + \alpha zG'(z) = \frac{2 - \frac{1}{\alpha}}{\alpha - 1} \left( \frac{\alpha}{z} \right) G(z) \]

Thus

\[ \frac{(zf'(z))'}{g'(z)} = \left( \frac{2 - \frac{1}{\alpha}}{(z^{-1} F(z))'} \right) / \left( \frac{2 - \frac{1}{\alpha}}{(z^{-1} G(z))'} \right) \]

Now

\[ (z - \frac{1}{\alpha} (z^{-1} F(z))')' = (z(\frac{1}{\alpha} - 1)F(z) + zF'(z))' = (\frac{1}{\alpha} zF'(z) + z^2 F''(z))' \]

\[ = \frac{2 - \frac{1}{\alpha}}{\alpha} \left( \frac{1}{\alpha} F'(z) + z^2 F''(z) \right) = \frac{2 - \frac{1}{\alpha}}{\alpha} \left( zF'(z) \right)' \]

Let $zF'(z) = H(z)$, then from (2.10), we have

\[ \frac{(zf'(z))'}{g'(z)} = \left( z - \frac{1}{\alpha} (z^{-1} H(z))' \right) / \left( \frac{2 - \frac{1}{\alpha}}{\alpha} (z^{-1} G(z))' \right) \]

Since from Theorem 2.2, the function $(1 - \alpha)H(z) + zH'(z) = z \frac{2 - \frac{1}{\alpha}}{\alpha} (z^{-1} H(z))'$ belongs to $K$ with respect to a convex function $g$: $g(z) = (1 - \alpha)G(z) + \alpha zG'(z)$ in
\[ |z| < r_0, \text{ so } f \text{ is in } C^* \text{ for } |z| < r_0, \text{ where } r_0 \text{ is given by (2.2).} \]

**Remark 2.4.** For \( F \in C^* \) and \( \alpha = \frac{1}{2} \), Theorem 2.3 has been proved in [1].

We now deal with a generalized form of (1.1) by taking \( g \) to be starlike and prove the following.

**Theorem 2.4.** Let \( F \) be analytic in \( E \) and let for \( z \in E, \text{Re } G'(z) > 0, G \in S^* \).

Let \( f(z) = (1 - \alpha)F(z) + azF'(z) \) and \( g(z) = (1 - \alpha)G(z) + azG'(z), \) with \( \alpha > 0 \). Then \( \text{Re } \frac{(zF'(z))'}{G'(z)} > 0 \) for \( |z| < r_1 \), where

\[ r_1 = \frac{1}{3\alpha + \sqrt{9\alpha^2 + 1 - 2\alpha}} \]

For \( \alpha = \frac{1}{2} \), the problem has been solved in [8].

**Proof.** From (2.3), we can write

\[ F(z) = \frac{1}{\alpha} z \left[ 1 - \frac{1}{\alpha} \right] \int_0^z \frac{1}{z^{\frac{1}{\alpha} - 2}} f(z) \, dz \]

\[ zF'(z) = \frac{1}{\alpha} z \left[ 1 - \frac{1}{\alpha} \right] \left( 1 - \frac{1}{\alpha} \right) \int_0^z \frac{1}{z^{\frac{1}{\alpha} - 2}} f(z) \, dz + \frac{1}{\alpha} f(z) \]

Thus

\[ \frac{(zF'(z))'}{G'(z)} = \frac{1}{\left( \frac{1}{\alpha} - 1 \right)} \int_0^z \frac{1}{z^{\frac{1}{\alpha} - 1}} f'(z) \, dz \]

Thus

\[ \frac{(zF'(z))'}{G'(z)} = \frac{h(z)}{g'(z)} \]

where \( \text{Re } h(z) > 0, z \in E \).

From (2.11), we write

\[ \frac{1}{z^{\frac{1}{\alpha} - 1}} f'(z) \, dz = h(z) \int_0^z \frac{1}{z^{\frac{1}{\alpha} - 1}} g'(z) \, dz. \]

Differentiating both sides, and simplifying, we obtain

\[ \frac{(zF'(z))'}{G'(z)} = h(z) + \frac{h'(z) \left( \int_0^z \frac{1}{z^{\frac{1}{\alpha} - 1}} g'(z) \, dz \right)}{\frac{1}{\alpha - 1} g'(z)} . \]

Using \( |h'(z)| \leq \frac{2\text{Re } h(z)}{1 - r^2} \), (2.12) gives
Now
\[
\frac{1}{g'(z)} \int_0^z \frac{1}{\alpha} g'(z)dz = \frac{1}{\alpha} G(z) = \frac{1}{\alpha - 1} + \frac{(zG(z))^\prime}{G(z)}. \tag{2.14}
\]

Since \( G \in S^* \), so
\[
\frac{(zG'(z))^\prime}{G'(z)} \geq \frac{1 - 4r + r^2}{1 - r^2}. \tag{2.15}
\]

From (2.13), (2.14) and (2.15), we obtain
\[
\Re \left( \frac{zf'(z)}{g'(z)} \right) \geq \Re h(z) \left[ 1 - \frac{2}{1 - r^2} \frac{r(1 - r^2)}{\frac{1}{\alpha} - 4r - (\frac{1}{\alpha} - 2)r^2} \right]
= \Re h(z) \frac{1 - 6ar - (1 - 2a)r^2}{1 - 4ar - (1 - 2a)r^2},
\]

and this positive for \(|z| < r_1\), where
\[
r_1 = \frac{1}{3a + \sqrt{9a^2 + 1 - 2a}}.
\]

ACKNOWLEDGEMENT. The authors are grateful for the referee's helpful comments and suggestions on the earlier version of this paper. In particular, the reference to Nikolaeva and Repnina was kindly supplied by him.

REFERENCES

Special Issue on
Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/jamds/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>June 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>September 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>December 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be