

## INTEGRABILITY AND $L^1$ -CONVERGENCE OF REES-STANOJEVIĆ SUMS WITH GENERALIZED SEMICONVEX COEFFICIENTS

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Integrability and  $L^1$ -convergence of modified cosine sums introduced by Rees and Stanojević (1973) under a class of generalized semiconvex null coefficients are studied, using Cesaro means of integral order.

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### 1. Introduction. Let

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad (1.1)$$

$$g_n(x) = \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \sum_{j=k}^n (\Delta a_j) \cos kx. \quad (1.2)$$

The problem of  $L^1$ -convergence of the Fourier cosine series (1.1) has been settled for various special classes of coefficients. Young [6] found that  $a_n \log n = o(1)$ ,  $n \rightarrow \infty$  is a necessary and sufficient condition for cosine series with convex ( $\Delta^2 a_n \geq 0$ ) coefficients, and Kolmogorov [5] extended this result to the cosine series with quasi-convex ( $\sum_{n=1}^{\infty} n |\Delta^2 a_{n-1}| < \infty$ ) coefficients. Later, Garrett and Stanojević [3] using modified cosine sums (1.2), proved the following theorem.

**THEOREM 1.1.** *Let  $\{a_n\}$  be a null sequence of bounded variation. Then the sequence of modified cosine sums*

$$g_n(x) = S_n(x) - a_{n+1} D_n(x), \quad (1.3)$$

where  $S_n(x)$  are the partial sums of the cosine series (1.1) and  $D_n(x)$  is the Dirichlet kernel, converges in  $L^1$ -norm to  $g(x)$ , the pointwise sum of the cosine series, if and only if for every  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$ , independent of  $n$ , such that

$$\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) \right| dx < \epsilon, \quad \text{for every } n. \quad (1.4)$$

This result contains as a special case a number of classical and neo-classical results. In particular, in [3] the following corollary to Theorem 1.1 is proved.

**THEOREM 1.2.** *Let  $\{a_n\}$  be a null sequence of bounded variation satisfying condition (1.4). Then the cosine series is the Fourier series of its sum  $g(x)$  and  $\|S_n(g) - g\| = o(1)$ ,  $n \rightarrow \infty$  is equivalent to  $a_n \log n = o(1)$ ,  $n \rightarrow \infty$ .*

In [2] Garrett and Stanojević proved the following theorem.

**THEOREM 1.3.** *If  $\{a_n\}$  is a null quasi-convex sequence, then  $g_n(x)$  converges to  $g(x)$  in the  $L^1$ -norm.*

**DEFINITION 1.4** (see [4]). A sequence  $\{a_n\}$  is said to be semiconvex if  $\{a_n\} \rightarrow 0$  as  $n \rightarrow \infty$ , and

$$\sum_{n=1}^{\infty} n |\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \quad (a_0 = 0), \tag{1.5}$$

where  $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$ ,  $\Delta a_n = a_n - a_{n+1}$ .

It may be remarked here that every quasi-convex null sequence is semi-convex. We generalize semiconvexity of null sequences in the following way: a null sequence  $\{a_n\}$  is said to be generalized semiconvex, if

$$\sum_{n=1}^{\infty} n^\alpha |\Delta^{\alpha+1} a_{n-1} + \Delta^{\alpha+1} a_n| < \infty, \quad \text{for } \alpha > 0 \ (a_0 = 0). \tag{1.6}$$

For  $\alpha = 1$ , this class reduces to the class defined in [4]. The object of this paper is to show that Theorem 1.3 of Garrett and Stanojević [2] holds good for cosine sums (1.2) with generalized semi-convex null coefficients.

**2. Notation and formulae.** In what follows, we use the following notions [7]:

$$\begin{aligned} S_n^0 &= S_n = a_0 + a_1 + \dots + a_n; \\ S_n^k &= S_0^{k-1} + S_1^{k-1} + \dots + S_n^{k-1}, \quad k = 1, 2, \dots, \ n = 0, 1, 2, \dots; \\ A_n^0 &= 1, \quad A_n^k = A_0^{k-1} + A_1^{k-1} + \dots + A_n^{k-1} \quad k = 1, 2, \dots, \ n = 0, 1, 2, \dots \end{aligned} \tag{2.1}$$

The  $A_n$ 's are called the binomial coefficients and are given by the following relation:

$$\sum_{k=0}^{\infty} A_k^\alpha x^k = (1-x)^{(-\alpha-1)}, \tag{2.2}$$

whereas  $S_n$ 's are given by

$$\sum_{k=0}^{\infty} S_k^\alpha x^k = (1-x)^{-\alpha} \sum_{k=0}^{\infty} S_k x^k, \tag{2.3}$$

and

$$\begin{aligned} A_n^\alpha &= \sum_{v=0}^n A_v^{\alpha-1}, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \\ A_n^\alpha &= \binom{n+\alpha}{n} \cong \frac{n^\alpha}{\Gamma(\alpha+1)} \quad (\alpha \neq -1, -2, \dots). \end{aligned} \tag{2.4}$$

The Cesaro means  $T_k^\alpha$  of order  $\alpha$  is denoted by  $T_k^\alpha = S_k^\alpha / A_k^\alpha$ .

Also for  $0 < x \leq \pi$ , let

$$\begin{aligned}
 \bar{D}_0(x) &= -\frac{1}{2} \cot \frac{x}{2}, \\
 \bar{S}_n(x) &= \bar{D}_0(x) + \sin x + \sin 2x + \dots + \sin nx, \\
 \bar{S}_n^1(x) &= \bar{S}_0(x) + \bar{S}_1(x) + \bar{S}_2(x) + \dots + \bar{S}_n(x), \\
 \bar{S}_n^2(x) &= \bar{S}_0^1(x) + \bar{S}_1^1(x) + \bar{S}_2^1(x) + \dots + \bar{S}_n^1(x), \\
 &\vdots \\
 \bar{S}_n^k(x) &= \bar{S}_0^{k-1}(x) + \bar{S}_1^{k-1}(x) + \bar{S}_2^{k-1}(x) + \dots + \bar{S}_n^{k-1}(x).
 \end{aligned}
 \tag{2.5}$$

The conjugate Cesaro means  $\bar{T}_k^\alpha$  of order  $\alpha$  is denoted by  $\bar{T}_k^\alpha = \bar{S}_k^\alpha / A_k^\alpha$ . We use the following lemma for the proof of our result.

**LEMMA 2.1** (see [1]). *If  $\alpha \geq 0, p \geq 0$ ,*

$$\begin{aligned}
 \epsilon_n &= o(n^{-p}), \\
 \sum_{n=0}^{\infty} A_n^{\alpha+p} |\Delta^{\alpha+1} \epsilon_n| &< \infty,
 \end{aligned}
 \tag{2.6}$$

then

$$\sum_{n=0}^{\infty} A_n^{\lambda+p} |\Delta^{\lambda+1} \epsilon_n| < \infty \quad \text{for } -1 \leq \lambda \leq \alpha,
 \tag{2.7}$$

$A_n^{\lambda+p} \Delta^\lambda \epsilon_n$  is of bounded variation for  $0 \leq \lambda \leq \alpha$  and tends to zero as  $n \rightarrow \infty$ .

**3. Main result.** The main result of this paper is the following theorem.

**THEOREM 3.1.** *If  $\{a_n\}$  is a generalized semiconvex null sequence, then  $g_n(x)$  converges to  $g(x)$  in  $L^1$ -metric if and only if  $\lim_{n \rightarrow \infty} \Delta a_n \log n = o(1)$ , as  $n \rightarrow \infty$ .*

**PROOF.** We have

$$\begin{aligned}
 g_n(x) &= \frac{1}{2} \sum_{k=0}^n \Delta a_k + \sum_{k=1}^n \sum_{j=k}^n \Delta a_j \cos kx \\
 &= \frac{a_0}{2} + \sum_{k=1}^n a_k \cos kx - a_{n+1} D_n(x) \\
 &= \sum_{k=1}^n a_k \cos kx - a_{n+1} D_n(x) \quad (a_0 = 0) \\
 &= \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2 \sin x} + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} - a_{n+1} D_n(x),
 \end{aligned}
 \tag{3.1}$$

where

$$\begin{aligned}
 D_n(x) &= \frac{\sin nx + \sin(n+1)x}{2 \sin x}, \\
 g_n(x) &= \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \frac{\sin kx}{2 \sin x} + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \\
 &\quad - a_{n+1} \frac{\sin nx}{2 \sin x} - a_{n+1} \frac{\sin(n+1)x}{2 \sin x} \\
 &= \frac{1}{2 \sin x} \sum_{k=1}^n (\Delta a_{k-1} + \Delta a_k) \sin kx + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}.
 \end{aligned} \tag{3.2}$$

Applying Abel's transformation, we have

$$\begin{aligned}
 g_n(x) &= \frac{1}{2 \sin x} \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \sum_{v=1}^k \sin vx + (\Delta a_{n-1} + \Delta a_n) \sum_{v=1}^n \sin vx \\
 &\quad + \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \\
 &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) (\bar{S}_k^0(x) - \bar{S}_0(x)) + (\Delta a_{n-1} + \Delta a_n) (\bar{S}_n^0(x) - \bar{S}_0(x)) \right] \\
 &\quad + \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \\
 &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \bar{S}_k^0(x) - \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) \bar{S}_0(x) \right] \\
 &\quad + \frac{1}{2 \sin x} [(\Delta a_{n-1} + \Delta a_n) \bar{S}_n^0(x) - (\Delta a_{n-1} + \Delta a_n) \bar{S}_0(x)] + \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \\
 &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-1} (\Delta^2 a_{k-1} + \Delta^2 a_k) (\bar{S}_k^0(x)) - (\Delta a_{n-1} + \Delta a_n) \bar{S}_n^0(x) + a_2 \bar{S}_0(x) \right] \\
 &\quad + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}.
 \end{aligned} \tag{3.3}$$

□

If we use Abel's transformation  $\alpha$  times, we have similarly,

$$\begin{aligned}
 g_n(x) &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{n-\alpha} (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \bar{S}_k^{\alpha-1}(x) + \sum_{k=1}^{\alpha} \Delta^k a_{n-k} \bar{S}_{n-k+1}^{k-1}(x) \right] \\
 &\quad + \frac{1}{2 \sin x} \left[ \sum_{k=1}^{\alpha} \Delta^k a_{n-k+1} \bar{S}_{n-k+1}^{k-1}(x) + a_2 \bar{S}_0(x) \right] + \Delta a_n \frac{\sin(n+1)x}{2 \sin x}.
 \end{aligned} \tag{3.4}$$

Since  $\bar{S}_n(x)$  and  $\bar{T}_n(x)$  are uniformly bounded on every segment  $[\epsilon, \pi - \epsilon]$ ,  $\epsilon > 0$ .

$$\begin{aligned}
 g(x) &= \lim_{n \rightarrow \infty} g_n(x) \\
 &= \frac{1}{2 \sin x} \left[ \sum_{k=1}^{\infty} (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \bar{S}_k^{\alpha-1}(x) + a_2 \bar{S}_0(x) \right].
 \end{aligned} \tag{3.5}$$

Thus

$$\begin{aligned}
 g(x) - g_n(x) &= \frac{1}{2 \sin x} \left[ \sum_{k=n-\alpha+1}^{\infty} (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \bar{S}_k^{\alpha-1}(x) - \sum_{k=1}^{\alpha} \Delta^k a_{n-k} \bar{S}_{n-k+1}^{k-1}(x) \right] \\
 &\quad - \frac{1}{2 \sin x} \left[ \sum_{k=1}^{\alpha} \Delta^k a_{n-k+1} \bar{S}_{n-k+1}^{k-1}(x) \right] - \Delta a_n \frac{\sin(n+1)x}{2 \sin x}, \\
 \|g(x) - g_n(x)\| &\leq C \left[ \int_0^{\pi} \left| \sum_{k=n-\alpha+1}^{\infty} (\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k) \bar{S}_k^{\alpha-1}(x) \right| dx \right] \\
 &\quad + C \left[ \int_0^{\pi} \left| \sum_{k=1}^{\alpha} \Delta^k a_{n-k} \bar{S}_{n-k+1}^{k-1}(x) \right| dx + \int_0^{\pi} \left| \sum_{k=1}^{\alpha} \Delta^k a_{n-k+1} \bar{S}_{n-k+1}^{k-1}(x) \right| dx \right] \\
 &\quad + \int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx, \\
 \|g(x) - g_n(x)\| &\leq C \left[ \sum_{k=n-\alpha+1}^{\infty} |(\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k)| \int_0^{\pi} |\bar{S}_k^{\alpha-1}(x)| dx \right] \\
 &\quad + C \left[ \sum_{k=1}^{\alpha} |\Delta^k a_{n-k}| \int_0^{\pi} |\bar{S}_{n-k+1}^{k-1}(x)| dx + \sum_{k=1}^{\alpha} |\Delta^k a_{n-k+1}| \int_0^{\pi} |\bar{S}_{n-k+1}^{k-1}(x)| dx \right] \\
 &\quad + \int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \\
 &\leq C \left[ \sum_{k=n-\alpha+1}^{\infty} A_k^{\alpha} |(\Delta^{\alpha+1} a_{k-1} + \Delta^{\alpha+1} a_k)| \int_0^{\pi} |\bar{T}_k^{\alpha}(x)| dx \right] \\
 &\quad + C \left[ \sum_{k=1}^{\alpha} A_{n-k+1}^k |\Delta^k a_{n-k}| \int_0^{\pi} |\bar{T}_{n-k+1}^k(x)| dx \right] \\
 &\quad + C \left[ \sum_{k=1}^{\alpha} A_{n-k+1}^k |\Delta^k a_{n-k+1}| \int_0^{\pi} |\bar{T}_{n-k+1}^k(x)| dx \right] \\
 &\quad + \int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx.
 \end{aligned} \tag{3.6}$$

The first three terms of the above inequality are of  $o(1)$  by Lemma 2.1 and the hypothesis of Theorem 3.1.

Moreover, since

$$\int_0^{\pi} \left| \frac{\sin(n+1)x}{2 \sin x} \right| dx \leq C \log n, \quad n \geq 2, \tag{3.7}$$

therefore

$$\int_0^{\pi} \left| \Delta a_n \frac{\sin(n+1)x}{2 \sin x} \right| dx \sim \Delta a_n \log n. \tag{3.8}$$

It follows that  $\int_0^\pi |g(x) - g_n(x)| dx \rightarrow 0$ , if and only if  $\Delta a_n \log n \rightarrow o(1)$  as  $n \rightarrow \infty$ . This completes the proof of the theorem.  $\square$

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