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RINGS WITH A FINITE SET OF NONNILPOTENTS

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ABSTRACT. Let R be a ring and let N denote the set of nilpotent elements of R. Let n be a nonnegative integer. The ring R is called a θ_n-ring if the number of elements in R which are not in N is at most n. The following theorem is proved: If R is a θ_n-ring, then R is nil or R is finite. Conversely, if R is a nil ring or a finite ring, then R is a θ_n-ring for some n. The proof of this theorem uses the structure theory of rings, beginning with the division ring case, followed by the primitive ring case, and then the semisimple ring case. Finally, the general case is considered.

KEY WORDS AND PHRASES. *Nil ring, Division ring, Primitive ring, Semisimple ring, Semigroup.*

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1. INTRODUCTION.

Let R be a ring and let N denote the set of nilpotent elements of R. Let n be a nonnegative integer. The ring R is called a θ_n-ring if the number of elements in R which are not in N is at most n; that is, $|R/N| \leq n$. One question which suggests itself is the following: what are necessary and sufficient conditions for a given ring R to be a θ_n-ring? The answer is given in the following.

2. MAIN RESULTS.

THEOREM 1. If R is a θ_n-ring, then R is nil or R is finite. Conversely, if R is a nil ring or a finite ring, then R is a θ_n-ring for some integer n.

In the proof of Theorem 1, we use the structure theory of rings, beginning with the division ring case, followed by the primitive ring case, and then the semisimple ring case. Finally, we consider the general case.

In preparation for the proof of Theorem 1, we first prove the following lemmas.

LEMMA 1. Any subring and any homomorphic image of a θ_n-ring is a θ_n-ring.

This follows at once from the definition of a θ_n-ring.

LEMMA 2. Let $R_1, R_2, \ldots, R_{n+1}$ be rings where each R_i has an identity. Then the direct sum $R_1 \dotplus R_2 \dotplus \ldots \dotplus R_{n+1}$ is not a θ_n-ring.

PROOF. Let $u_1 = (1,0,0,\ldots,0)$, $u_2 = (0,1,0,\ldots,0),\ldots,u_{n+1} = (0,0,\ldots,0,1)$, where each u_i is an element of the direct sum $R_1 \dotplus R_2 + \ldots \dotplus R_{n+1}$. Then $u_i \notin N$ for each $i = 1,2,\ldots,n + 1$, where N denotes the set of nilpotent elements of R. Hence

$$|\{R_1 \dotplus R_2 \dotplus \ldots \dotplus R_{n+1}\}\backslash N| > n ,$$

and thus $R_1 \dotplus R_2 \dotplus \ldots \dotplus R_{n+1}$ is not a θ_n-ring.

LEMMA 3. If D is a division ring which is also a θ_n-ring, then D is finite.

This follows at once from the difinition of a θ_n-ring.

LEMMA 4. If R is a primitive ring which is also a θ_n-ring, then R is finite. In fact, R is isomorphic to a complete matrix ring D_m over a finite division ring D.

PROOF. Suppose $R \ncong D_m$ for any division ring D and any positive integer m. (We shall show that this leads to a contradiction.) Then, by Jacobson's Density Theorem [1; p. 33], for each positive integer q, there exists a subring R_0 of R which maps homomorphically onto D_q, for some division ring D. In particular, there exists a subring R_1 of R which maps homomorphically onto D_{n+1}. Hence, by Lemma 1,

$$D_{n+1} \text{ is a } \theta_n\text{-ring.} \tag{2.1}$$

Now, define $A_1, A_2, \ldots, A_{n+1}$ to be the following $(n + 1) \times (n + 1)$ matrices in D_{n+1}:

> A_i has zero entries in all rows except row i,
> but A_i has entries of 1 throughout row i ;
> $(i = 1, 2, \ldots, n + 1)$.

It is readily verified that each A_i is idempotent and thus each A_i is not nilpotent. Hence, $|D_{n+1} \backslash N| > n$, where N denotes the set of nilpotent elements of D_{n+1}. Therefore, D_{n+1} is not a θ_n-ring, which contradicts (2.1). This contradiction shows that $R \cong D_m$ for some division ring D and some positive integer m. Now, since the division ring D is a subring of the θ_n-ring D_m, D is finite, by Lemmas 1 and 3. Thus, R is finite, and the proof is complete.

LEMMA 5. If R is a semisimple ring which is also a θ_n-ring, then R is finite.

PROOF. By contradiction. Thus, suppose that R is a semisimple ring which is also a θ_n-ring, and suppose R is not finite. Since R is semisimple, there exists ideal $I_\alpha (\alpha \in \Omega)$ of R such that [1; p. 14]

$$\bigcap_{\alpha \in \Omega} I_\alpha = (0); \quad \text{each } R/I_\alpha \text{ is primitive .} \tag{2.2}$$

By Lemma 1, R/I_α is a θ_n-ring. Hence, by (2.2) and Lemma 4,

$$R/I_\alpha \cong D_m, \ R/I_\alpha \text{ is finite, for any } \alpha \in \Omega \ . \tag{2.3}$$

Now, choose $\alpha_1 \in \Omega$, and having chosen $\alpha_1, \ldots, \alpha_k$ so that

$$\sum_{i=1}^{k}{}^{\bullet}\, R/I_{\alpha_i} \cong R/\bigcap_{i=1}^{k} I_{\alpha_i} \quad , \tag{2.4}$$

choose $\alpha_{k+1} \in \Omega$ such that $\bigcap_{i=1}^{k} I_{\alpha_i} \not\subseteq I_{\alpha_{k+1}}$. That such α_{k+1} can always be so chosen is proved as follows: suppose no such α_{k+1} exists. Then $(0) = \bigcap_{\alpha\in\Omega} I_\alpha = \bigcap_{i=1}^{k} I_{\alpha_i}$, and hence (see (2.4))

$$R \cong R/\bigcap_{i=1}^{k} I_{\alpha_i} \cong \sum_{i=1}^{k}{}^{\bullet}\, R/I_{\alpha_i} \ .$$

Thus, using (2.3), we see that R is finite, a contradiction. This contradiction shows that there exists $\alpha_{k+1} \in \Omega$ such that $\bigcap_{i=1}^{k} I_{\alpha_i} \not\subseteq I_{\alpha_{k+1}}$. Now, as we can see from (2.3), $R/I_{\alpha_{k+1}}$ is simple. Since, moreover, $\bigcap_{i=1}^{k} I_{\alpha_i} \not\subseteq I_{\alpha_{k+1}}$, we have $\bigcap_{i=1}^{k} I_{\alpha_i} + I_{\alpha_{k+1}} = R$. Hence, by applying the second isomorphism theorem, we readily verify that

$$R/\bigcap_{i=1}^{k+1} I_{\alpha_i} \cong R/\bigcap_{i=1}^{k} I_{\alpha_i} \dotplus R/I_{\alpha_{k+1}} \cong \sum_{i=1}^{k+1}{}^{\bullet}\, R/I_{\alpha_i} \ ,$$

by (2.4). In particular, we have

$$\sum_{i=1}^{n+1}{}^{\bullet}\, R/I_{\alpha_i} \cong R/\bigcap_{i=1}^{n+1} I_{\alpha_i} \ .$$

Hence, using Lemma 1, $\sum_{i=1}^{n+1}{}^{\bullet}\, R/I_{\alpha_i}$ is a θ_n-ring. This, however, contradicts Lemma 2 (see (2.3)). This contradiction shows that R is finite, and the lemma is proved.

We are now in a position to prove our main result, stated at the beginning.

PROOF of Theorem 1. Let R be a θ_n-ring, and let J denote the Jacobson radical of R. We claim that J is nil. We prove this by contradiction. Thus, suppose J is not nil. Then, for some a, $a \in J$, $a \notin N$, where N denotes the set of nilpotent elements of R. Let

$$S = \{a, a^2, a^3, \ldots, a^{n+1}\}, \quad (a \in J,\ a \notin N) \ . \tag{2.5}$$

Since $a \notin N$, $a^i \notin N$ for each $i = 1,2,\ldots,n + 1$, and hence $S \subseteq R\backslash N$. Therefore,

$$|S| \leq |R\backslash N| \leq n \ , \tag{2.6}$$

since R is a θ_n-ring. In view of (2.5) and (2.6), we see that $a^i = a^j$ for some distinct positive integers i, j, and hence some power of a is idempotent, say a^k. Thus, a^k is an idempotent element of J, and hence, as is well known, $a^k = 0$. Therefore, $a \in N$, a contradiction. This contradiction proves that J is nil.

Now, if $J = R$, then R is nil. So suppose $J \neq R$. Then R/J is a semisimple ring with more than one element. Moreover, by Lemma 1, R/J is a θ_n-ring, and hence by Lemma 5, R/J is finite. We have thus shown that R/J is a finite semisimple ring with more than one element, and hence, as is well known, R/J has an idenitity. Let $\bar{u}$ be the identity element of R/J, and let

$$N + u = \{n + u \mid n \in N\}; \text{ N is the set of nilpotents of R.} \tag{2.7}$$

We claim that

$$(N + u) \cap N = \emptyset \ . \tag{2.8}$$

For, if $a \in (N + u) \cap N$, then $a \in N$ and $a = n + u$ for some $n \in N$. Hence $\bar{a} = \bar{n} + \bar{u}$ = an invertible element of R/J. Thus, $\bar{a}$ is an invertible element of R/J, which contradicts the hypothesis that $a \in N$ (and hence $\bar{a}$ is nilpotent). This contradiction proves (2.8). In view of (2.8), we have $N + u \subseteq R\backslash N$, and hence

$$|N + u| \leq |R\backslash N| \leq n \ , \tag{2.9}$$

since R is a θ_n-ring. But $|N| = |N + u|$, and hence by (2.9), N is finite. The net result, then, is that both N and $R\backslash N$ are finite, and hence R is finite. We have thus shown that, in any case, R is nil (if $J = R$) or R is finite (if $J \neq R$). The converse part of Theorem 1 is trivial.

We conclude with the following

REMARK. The analogue of Theorem 1 is not true for semigroups. To see this, let N be an infinite nil semigroup, and let $1 \notin N$ be an identity element; that is,

$$1 \cdot x = x \cdot 1 = x \quad \text{for all} \quad x \quad \text{in} \quad N, \quad \text{and} \quad 1 \cdot 1 = 1 .$$

Let $R = N \cup \{1\}$. Then R is a semigroup. Also, $|R \backslash N| = 1$. But R is neither nil nor finite.

REFERENCE

1. Jacobson, N. Structure of Rings, rev. ed., Amer. Math. Soc. Colloq. Publ., Vol. 37, Providence, R.I., 1964.