ON A CLASS OF EVEN-DIMENSIONAL MANIFOLDS STRUCTURED BY AN AFFINE CONNECTION

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We deal with a 2m-dimensional Riemannian manifold \((M,g)\) structured by an affine connection and a vector field \(\mathcal{F}\), defining a \(\mathcal{F}\)-parallel connection. It is proved that \(\mathcal{F}\) is both a torse forming vector field and an exterior concurrent vector field. Properties of the curvature 2-forms are established. It is shown that \(M\) is endowed with a conformal symplectic structure \(\Omega\) and \(\mathcal{F}\) defines a relative conformal transformation of \(\Omega\).

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1. Introduction. In [5], a class of odd-dimensional manifolds endowed with a \(\mathcal{F}\)-parallel connection was investigated.

In the present paper, we consider a 2m-dimensional Riemannian manifold \((M,g)\), structured by an affine connection defined by the torsion 2-forms \(S^A, A \in \{1,2,\ldots,2m\}\). If \(\{e_A\}\) and \(\{\omega^A\}\) are a vector and a covector basis, respectively, and \(\mathcal{F}(T^A)\) a vector field (called the structure vector field of \(M\)), we assume that \(\mathcal{F}\) defines a \(\mathcal{F}\)-parallel connection, in the sense of [9] (see also [2, 4]), that is, the connection forms associated with \(\{e_A\}\) and \(\{\omega^A\}\) satisfy

\[ \theta^A_B = \langle \mathcal{F}, e_B \wedge e_A \rangle = T^B \omega^A - T^A \omega^B, \tag{1.1} \]

where \(\wedge\) means the wedge product of vector fields, which implies \(\nabla_{\mathcal{F}} e_A = 0\).

Next, we assume that the torsion forms \(S^A\) are exterior recurrent (abbreviated ER) [1] with \(\alpha = \mathcal{F}^\perp\) as recurrence form, that is, \(dS^A = \alpha \wedge S^A\).

Assuming that \(T^A\) are also ER with a certain Pfaffian \(u\) as recurrence form, that is, \(dT^A = T^A u\), and denoting \(2t = \|\mathcal{F}\|^2\), we have

\[ \nabla \mathcal{F} = 2t dp + (u - \alpha) \otimes \mathcal{F}, \tag{1.2} \]

where \(dp\) is the soldering form of \(M\) [3], which says that \(\mathcal{F}\) is a torse forming vector field [8, 11, 12].

We derive

\[ \nabla^2 \mathcal{F} = 2t (u + \alpha) \wedge dp, \tag{1.3} \]

that is, \(\mathcal{F}\) is an exterior concurrent vector field [10] (see also [4]).

Setting \(S = S^1 \wedge S^2 \wedge \cdots \wedge S^{2m}\), we find that the \(4m\)-form \(S\) associated with \(M\) is ER with \(4m\alpha\) as recurrence form.
It is shown that the curvature 2-forms $\Theta^A_B$ are ER having the closed 1-form $2(u + \alpha)$ as recurrence form. We agree to define such a manifold as an exterior recurrent curvature 2-form manifold.

Finally, assuming that $M$ carries an almost symplectic form $\Omega$, that is, a nondegenerate differential 2-form, we prove that $\Omega$ is a conformal symplectic form.

It is shown that $\mathcal{T}$ defines a relative conformal transformation of the conformal symplectic form $\Omega$ (see [5]).

The above results are stated in Theorem 3.1.

2. Preliminaries. Let $(M,g)$ be a $2m$-dimensional oriented Riemannian manifold structured by an affine differential operator $\nabla$.

Let $\Gamma(TM)$ be the set of sections of the tangent bundle and $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$ the classical musical isomorphisms defined by $g$ (i.e., $\flat$ is the index lowering operator and $\sharp$ is the index raising operator).

Following [7], we denote by

$$A^q(M,TM) = \Gamma \text{Hom}(\wedge^q TM, TM)$$

the set of vector-valued $q$-forms ($q \leq \dim M$) and we write for the affine operator $\nabla$

$$d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM).$$

If $dp \in A^1(M,TM)$ is the canonical vector-valued 1-form of $M$, then as an extension of the Levi-Civita operator and by [3], we agree to call $dp$ the soldering form of $M$.

Let the unit vector fields $\{e_A\}$ be an orthonormal vector basis and $\{\omega^A\}$ its corresponding cobasis on $M$, $A = 1, \ldots, 2m$. Then, if $\theta^A_B$, $S^A$, and $\Theta^A_B$ denote the connection forms, the torsion 2-forms and the curvature 2-forms, respectively, Cartan’s structure equations are expressed by

$$\nabla e_A = \theta^A_B \otimes e_B,$$
$$d\omega^A = \omega^B \wedge \theta^A_B + S^A,$$
$$d\theta^A_B = \Theta^A_C \wedge \theta^C_B + \Theta^A_B.$$

We recall the following definitions (cf. [4]).

A vector field $\mathcal{T}$ is said to be a torse forming vector field [12] if it satisfies

$$\nabla \mathcal{T} = f \mathcal{T} + v \otimes \mathcal{T}, \quad f \in C^\infty M, \quad v \in \wedge^1 M.$$

Also, the vector field $\mathcal{T}$ is called exterior concurrent [10] if

$$\nabla^2 \mathcal{T} = \pi \wedge dp, \quad \pi \in \wedge^1 M.$$

If $Z, Z' \in \Gamma(TM)$, we also have the following formula:

$$d\omega(Z, Z') = \mathcal{L}_{Z'} \omega(Z) - \mathcal{L}_Z \omega(Z') + \omega([Z, Z']),$$

where $\mathcal{L}$ is the Lie derivative.
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Since \( dp = \omega^A \wedge e_A \), then it follows that

\[
d^\nabla (dp) = S^A \otimes e_A. \tag{2.9}
\]

3. Manifolds with affine connection. In the present paper, we assume first that the \( 2m \)-dimensional Riemannian manifold \((M,g)\) carries a structure vector field \( \mathcal{T}(T^A) \) which defines a \( \mathcal{T} \)-parallel connection, in the sense of [9] (see also [2, 4]). Such a connection is expressed by

\[
\theta^A_B = \langle \mathcal{T}, e_B \wedge e_A \rangle = T^B \omega^A - T^A \omega^B. \tag{3.1}
\]

Since we quickly find from (3.1) that

\[
\nabla_{\mathcal{T}} e_A = 0, \tag{3.2}
\]

this agrees with the definition of \( \mathcal{T} \)-parallel connection.

Setting \( 2t = \|T\|^2 \), we derive

\[
\nabla \mathcal{T} = 2t \ dp - \alpha \otimes \mathcal{T} + \sum_A dT^A \otimes e_A, \tag{3.3}
\]

where \( \alpha = \mathcal{T}^\ast \) is the dual 1-form of \( \mathcal{T} \). Also, we find by (3.1) and (2.4) that

\[
d \omega^A = \alpha \wedge \omega^A + S^A. \tag{3.4}
\]

Second, we assume that the torsion forms \( S^A \) are exterior recurrent [1] having \( \alpha \) as recurrence form, that is,

\[
d S^A = \alpha \wedge S^A, \tag{3.5}
\]

and \( T^A \) are ER with the Pfaffian \( u \) as recurrence form, that is,

\[
d T^A = T^A u. \tag{3.6}
\]

We obtain \( d \alpha = 0 \), that is, \( \alpha^\ast = \mathcal{T} \) is a closed vector field.

Under these conditions, it follows from (3.3) and (3.6) that

\[
\nabla \mathcal{T} = 2t \ dp + (u - \alpha) \otimes \mathcal{T}; \tag{3.7}
\]

this proves that \( \mathcal{T} \) is a torse forming vector field [4, 8, 11, 12]. Since the operator \( \nabla \) acts inductively and clearly by (3.6), then

\[
d t = 2t u, \tag{3.8}
\]

we infer

\[
d^\nabla (\nabla \mathcal{T}) = \nabla^2 \mathcal{T} = 2t (u + \alpha) \wedge dp. \tag{3.9}
\]

This means that the vector field \( \mathcal{T} \) is an exterior concurrent vector field [6, 10].
By [6], (3.9) implies that
\[ R(\mathcal{F}, Z) = -(2m - 1)2tg(\mathcal{F}, Z), \quad Z \in \Gamma(TM), \] (3.10)
where \( R \) denotes the Ricci tensor field on \( M \).

By (3.9) and by standard calculation, we derive
\[ \nabla^4 \mathcal{F} = 0 \] (3.11)
and therefore we may say that the vector field \( \mathcal{F} \) is an element of
\[ \Gamma \text{Hom}\left(\wedge^4 TM, TM\right). \] (3.12)

On the other hand, recall that the Bianchi forms in the sense of Tachibana are defined by
\[ \Omega^{(p)}_{\alpha_1, \ldots, \alpha_{2p}} = \Omega_{\alpha_2}^{\alpha_1} \wedge \Omega_{\alpha_3}^{\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2p}}^{\alpha_{2p-1}}, \] (3.13)
where \( \Omega_{\alpha_{2p+1}}^{\alpha_{2p+1}} \) are 2-forms. Thus, setting
\[ S = S^1 \wedge S^2 \wedge \cdots \wedge S^{2m}, \] (3.14)
we find that
\[ dS = 4m\alpha \wedge S. \] (3.15)

Therefore, we may say that the \( 4m \)-form \( S \) associated with \( M \) is ER with \( 4m\alpha \) as recurrence form.

By (3.4) we may set
\[ S^A = u \wedge \omega^A \] (3.16)
and by (3.1) and the structure equations (2.5) we get after some calculations
\[ \Theta^3_B = 2(u + \alpha) \wedge \omega^3_B + 2t \omega^B \wedge \omega^A. \] (3.17)

Next, performing the exterior differentiation of \( \Theta^3_B \), we derive, taking account of (3.8)
\[ d\Theta^3_B = 2(u + \alpha) \wedge \Theta^3_B. \] (3.18)

This shows that all curvature forms \( \Theta^3_B \) are ER and have the closed 1-form \( 2(u + \alpha) \) as recurrence form.

We agree to define such an even-dimensional manifold \( M \) as an exterior recurrent curvature 2-form manifold.

Finally, assume that \( M \) carries an almost symplectic form \( \Omega \). Then, we may express \( \Omega \) as
\[ \Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a*}, \quad a^* = a + m. \] (3.19)
Taking the exterior differentiation of $\Omega$, we find by (3.4) and (3.16) that
\[ d\Omega = 2(\alpha + u) \wedge \Omega. \] (3.20)

This shows that the manifold under consideration is endowed with a conformal symplectic structure having $\alpha + u$ as covector of Lee.

Moreover, taking the Lie differentiation of $\Omega$ with respect to the structure vector field $\mathcal{T}$, we infer
\[ \mathcal{L}_\mathcal{T} \Omega = ut \Omega + 2(u + \alpha) \wedge \sum_{a=1}^{m} (T^a \omega^a - T^a \omega^b). \] (3.21)

Using (3.8) and (3.6), the exterior differentiation of (3.21) gives
\[ d\mathcal{L}_\mathcal{T} \Omega = 8t u \wedge \Omega. \] (3.22)

Hence, by [4], the above equation says that $\mathcal{T}$ defines a relative conformal transformation of the conformal symplectic form $\Omega$.

Summing up, we state the following theorem.

**Theorem 3.1.** Let $(M, g)$ be a $2m$-dimensional Riemannian manifold structured by an affine connection defined by the torsion 2-forms $S^A$, $A = 1, \ldots, 2m$. Let $\mathcal{F}(T^A)$ be a structure vector field, which defines a $\mathcal{F}$-parallel connection and assume that $S^A$ are exterior recurrent, having $\mathcal{T}^b = \alpha^b$ as recurrence form ($\mathcal{T}^b = \alpha^b$ is a closed Pfaffian).

Then the following properties hold:
(i) $\mathcal{T}$ is both a torse forming and an exterior concurrent vector field;
(ii) the structure curvature 2-forms $\Theta_A^B$ are exterior recurrent with the closed Pfaffian $2(u + \alpha)$ as recurrence form;
(iii) the manifold $M$ is endowed with a conformal symplectic structure $\Omega$ having $u + \alpha$ as covector of Lee;
(iv) the vector field $\mathcal{T}$ defines a relative conformal transformation of $\Omega$, that is, $d\mathcal{L}_\mathcal{T} \Omega = 8t u \wedge \Omega$, where $2t = \|\mathcal{T}\|^2$.

**References**


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