ON WEAK CENTER GALOIS EXTENSIONS OF RINGS

GEORGE SZETO and LIANYONG XUE

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Abstract. Let $B$ be a ring with 1, $C$ the center of $B$, $G$ a finite automorphism group of $B$, and $B^G$ the set of elements in $B$ fixed under each element in $G$. Then, the notion of a center Galois extension of $B^G$ with Galois group $G$ (i.e., $C$ is a Galois algebra over $C^G$ with Galois group $G|_C \cong G$) is generalized to a weak center Galois extension with group $G$, where $B$ is called a weak center Galois extension with group $G$ if $BI_i = B$ for some idempotent in $C$ and $I_i = \{c - g_i(c) \mid c \in C\}$ for each $g_i \neq 1$ in $G$. It is shown that $B$ is a weak center Galois extension with group $G$ if and only if for each $g_i \neq 1$ in $G$ there exists an idempotent $e_i$ in $C$ and $\{b_k e_i g_i(c_k e_i) = \delta_{1, g_i} e_i\}$ such that $\sum_{k=1}^m b_k e_i g_i(c_k e_i) = \delta_{1, g_i} e_i$ and $g_i$ restricted to $C(1 - e_i)$ is an identity, and a structure of a weak center Galois extension with group $G$ is also given.

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1. Introduction. Galois theory for fields was generalized for rings in the sixties and seventies [3, 4, 7, 8]. Let $B$ be a ring with 1, $G = \{g_1 = 1, g_2, \ldots, g_n\}$ an automorphism group of $B$ of order $n$, $C$ the center of $B$, and $B^G$ the set of elements in $B$ fixed under each element in $G$. There are several well-known classes of noncommutative Galois extensions: (1) the DeMeyer-Kanzaki Galois extension $B$ (i.e., $B$ is an Azumaya $C$-algebra which is a Galois algebra with Galois group $G|_C \cong G$) [3, 7], (2) the $H$-separable Galois extension $B$ (i.e., $B$ is a Galois and a $H$-separable extension of $B^G$) [8], (3) the Azumaya Galois extension $B$ (i.e., $B$ is a Galois extension of $B^G$ which is an Azumaya $C^G$-algebra) [1], (4) the central Galois algebra [3, 4, 7], and (5) the center Galois extension $B$ (i.e., $C$ is a Galois algebra over $C^G$ with Galois group $G|_C \cong G$) [11]. We note that a commutative Galois extension is a DeMeyer-Kanzaki Galois extension which is a center Galois extension. It is well know that $C$ is a Galois extension of $C^G$ if and only if the ideals generated by $\{c - g(c) \mid c \in C\}$ is $C$ for each $g \neq 1$ in $G$ [2, Proposition 1.2, page 80]. This fact was generalized in [11] to a center Galois extension; that is, $B$ is a center Galois extension of $B^G$ if and only if the ideals of $B$ generated by $\{c - g(c) \mid c \in C\}$ is $B$, that is, $BI_i = B$, where $I_i = \{c - g_i(c) \mid c \in C\}$ for each $g_i \neq 1$ in $G$ (for more about center Galois extensions, see [5, 6, 9, 10, 11]). Generalizing the condition that $BI_i = B = B1$ to that $BI_i = Be_i$ for some idempotent $e_i$ in $C$ for each $g_i \neq 1$ in $G$, we obtain a broader class of rings $B$ than the class of center Galois extensions. This class of rings is called weak center Galois extensions. The purpose of the present paper is to give a characterization and a structure of a weak center Galois extension $B$ with group $G$. We shall show that $B$ is a weak center Galois extension with group $G$ if and only if for each $g_i \neq 1$ in $G$ there exists an idempotent $e_i$ in $C$ and $\{b_k e_i \in Be_i; c_k e_i \in Ce_i, k = 1, 2, \ldots, m\}$ such that $\sum_{k=1}^m b_k e_i g_i(c_k e_i) = \delta_{1, g_i} e_i$.
and $g_i$ restricted to $C(1-e_i)$ is an identity. Next, we call $B$ a $T$-Galois extension of $B^T$ if there exist elements $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{i=1}^m a_i g_i(b_i) = \delta_{1,g}$ for $g \in T \cup \{1\}$. We note that $T$ is not necessarily a subgroup of $G$. Let $B$ be a weak center Galois extension with group $G$. Then, we show that there exists a partition of $G - \{1\}$, $\{T_j \subset G, j = 1, 2, \ldots, h\}$ for some integer $h$ and some idempotents $e_j \in C, j = 1, 2, \ldots, h$ such that $B_{e_j}$ is a $T_j$-Galois extension of $(B_{e_j})^{T_j}$. Moreover, when $G$ is abelian, $e_j$ can be taken as orthogonal idempotents in $C$ so that $\sum_{j=1}^h B_{e_j}$ is a direct sum. Furthermore, a sufficient condition is given for the existence of a subgroup $H_j \subset T_j \cup \{1\}$ for $j = 1, 2, \ldots, h$. In this case, $B_{e_j}$ is a $H_j$-Galois extension of $(B_{e_j})^{H_j}$ with Galois group $H_j$.

2. Definitions and notation. Throughout this paper, $B$ represents a ring with 1, $G = \{g_1 = 1, g_2, \ldots, g_n\}$ an automorphism group of $B$ of order $n$ for some integer $n$, $C$ the center of $B$, and $B^G$ the set of elements in $B$ fixed under each element in $G$. We denote $I_i = \{c - g_1(c) \mid c \in C\}$ and $B I_i$ the ideal of $B$ generated by $I_i$ for $g_i \in G$.

$B$ is called a $G$-Galois extension of $B^G$ if there exist elements $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{i=1}^m a_i g_i(b_i) = \delta_{1,g}$ for $g \in T \cup \{1\}$. Such a set $\{a_i, b_i\}$ is called a $G$-Galois system for $B$. $B$ is called a weak center Galois extension of $B^G$ with group $G$ if $B I_i = B_{e_i}$ for some idempotent in $C$ for each $g_i \neq 1$ in $G$. For a subset $T$ (not necessarily a subgroup) of $G$, $B$ is called a $T$-Galois extension of $B^T$ if there exist elements $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{i=1}^m a_i g_i(b_i) = \delta_{1,g}$ for $g \in T \cup \{1\}$. Such a set $\{a_i, b_i\}$ is called a $T$-Galois system for $B$. For a $B$-module $M$, we denote $\text{Ann}_B(M) = \{b \in B \mid bm = 0 \text{ for all } m \in M\}$.

3. Weak center Galois extensions. In [11], the present authors showed that a center Galois extension $B$ is equivalent to each of the following statements: (i) $B I_i = B$ for each $g_i \neq 1$ in $G$ and (ii) $B$ is a Galois extension of $B^G$ with a Galois system $\{b_i \in B, c_i \in C, i = 1, 2, \ldots, m\}$ for some integer $m$. In this section, we generalize this characterization to a weak center Galois extension $B$ with group $G$. We begin with the following lemma.

**Lemma 3.1.** If $B$ is a weak center Galois extension with group $G$, then

1. $g_i$ restricted to $B_{e_i}$ is an automorphism of $B_{e_i}$.
2. $B_{e_i}$ is a $(g_i)$-Galois extension of $(B_{e_i})^{[g_i]}$.

**Proof.** (1) For any $b = \sum_{k=1}^m b_k (c_k - g_i(c_k)) \in B I_i = B_{e_i}$, where $b_k \in B$ and $c_k \in C$, $k = 1, 2, \ldots, m$ for some integer $m$, we have $g_i(b) = g_i(\sum_{k=1}^m b_k (c_k - g_i(c_k))) = \sum_{k=1}^m g_i(b_k)(g_i(c_k) - g_i(g_i(c_k))) \in B I_i = B_{e_i}$. Hence, $g_i(B_{e_i}) \subset B_{e_i}$. Thus, $g_i$ restricted to $B_{e_i}$ is an automorphism of $B_{e_i}$ since $g_i$ is an automorphism of $B$.

(2) Since $B I_i = B_{e_i}$, there exist $\{b_k \in B, c_k \in C, k = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{k=1}^m b_k (c_k - g_i(c_k)) = e_i$. Therefore, $\sum_{k=1}^m b_k c_k = e_i + \sum_{k=1}^m b_k g_i(c_k)$. Let $b_{m+1} = -\sum_{k=1}^m b_k g_i(c_k)$ and $c_{m+1} = 1$. Then $\sum_{k=1}^{m+1} b_k c_k = e_i$ and $\sum_{k=1}^{m+1} b_k g_i(c_k) = 0$. Noting that $e_i$ is the identity of $B_{e_i}$ and $g_i$ restricted to $B_{e_i}$ is an automorphism
of $B_{e_i}$, we have $g_i(e_i) = e_i$. Hence, $\sum_{k=1}^{m+1} b_k e_i g_i(c_k e_i) = \delta_1 g_i e_i$, that is, $\{b_k e_i; c_k e_i, k = 1, 2, \ldots, m + 1\}$ is a $\{g_i\}$-Galois system for $B_{e_i}$.

The following is an equivalent condition for a weak center Galois extension with group $G$.

**Theorem 3.2.** $B$ is a weak center Galois extension with group $G$ (i.e., $B_{l_i} = B_{e_i}$ for some idempotent $e_i$ in $C$ for each $g_i \neq 1$ in $G$) if and only if for each $g_i \neq 1$ in $G$ there exists an idempotent $e_i$ in $C$ and $\{b_k e_i; c_k e_i \in C_{e_i}, k = 1, 2, \ldots, m\}$ such that

$\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_1 g_i e_i$ and $g_i$ restricted to $C(1-e_i)$ is an identity.

**Proof.** $(\Rightarrow)$ By Lemma 3.1(2), $B_{l_i} (= B_{e_i})$ contains a $\{g_i\}$-Galois system $\{b_k e_i \in B_{e_i}; c_k e_i \in C_{e_i}, k = 1, 2, \ldots, m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_1 g_i e_i$. Next, we show that $g_i$ restricted to $(1-e_i)$ is an identity. In fact, by Lemma 3.1(1), $g_i(e_i) = e_i$. Hence, for any $c \in C$, $(c - g_i(c(1-e_i))) = (c - g_i(c)) \in C_{e_i} \cap C(1-e_i) = \{0\}$. Thus, $g_i(c(1-e_i)) = c(1-e_i)$ for all $c \in C$. This proves that $g_i$ restricted to $C(1-e_i)$ is an identity.

$(\Leftarrow)$ By hypothesis, for each $g_i \neq 1$ in $G$ there exists an idempotent $e_i$ in $C$ and $\{b_k e_i \in B_{e_i}; c_k e_i \in C_{e_i}, k = 1, 2, \ldots, m\}$ such that $\sum_{k=1}^{m} b_k e_i g_i(c_k e_i) = \delta_1 g_i e_i$. Hence, $e_i = \sum_{k=1}^{m} b_k e_i c_k e_i g_i(c_k e_i) \in B_{l_i}$. Hence, $B_{e_i} \subset B_{l_i}$. But $e_i$ is an idempotent, so $B_{l_i} = B_{e_i} e_i \subset B_{l_i} e_i \subset B_{e_i}$. Thus, $B_{l_i} = B_{e_i} e_i$. Since $g_i$ restricted to $C(1-e_i)$ is an identity, $g_i(c(1-e_i)) = c(1-e_i)$ for all $c \in C$ (in particular, $g_i(e_i) = e_i$). Hence, $c - g_i(c) = c e_i - g_i(c e_i) = (c - g_i(c)) e_i$ for all $c \in C$. This implies that $B_{e_i} = B_{l_i} e_i = B_{l_i}$.

Recall that $B$ is called a $T$-Galois extension of $B^T$ for a subset $T$ (not necessary a subgroup) of $G$ if $B$ contains a $T$-Galois system. Next, we give a structure of a weak center Galois extension with group $G$.

**Lemma 3.3.** Assume $B$ is a weak center Galois extension with group $G$. Let $T_j = \{g_i \in G \mid B_{l_i} = B_{e_i}, \text{i.e., } e_i = e_j\}$. Then $B_{e_j}$ is a $T_j$-Galois extension of $(B_{e_j})^{(1)}$ for each $j \neq 1$.

**Proof.** By the proof of Lemma 3.1(2), for each $g_i \in T_j$, there is a $\{g_i\}$-Galois system $\{b^{(i)}_{k} e_j; c^{(i)}_{k} e_j, k = 1, 2, \ldots, m_i\}$ for $B_{e_j}$, where $b^{(i)}_{k} \in B$ and $c^{(i)}_{k} \in C$, $k = 1, 2, \ldots, m_i$, for some integer $m_i$. Denote the elements in $T_j$ by $\{g_{i_1}, g_{i_2}, \ldots, g_{i_t}\}$ for some integer $t$. Let

$\sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} b^{(i_{k_1})}_{k_1} b^{(i_{k_2})}_{k_2} \cdots b^{(i_{k_t})}_{k_t} e_j$ and $c^{(i_{k_1})}_{k_1} c^{(i_{k_2})}_{k_2} \cdots c^{(i_{k_t})}_{k_t} e_j$ for $k_l = 1, 2, \ldots, m_{i_l}$ and $l = 1, 2, \ldots, t$. Noting that $c^{(i_l)}_{k_l} \in C$, $l = 1, 2, \ldots, t$, we have

$$
\begin{align*}
\sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} b^{(i)}_{k_1} b^{(i)}_{k_2} \cdots b^{(i)}_{k_t} e_j & \left( c^{(i)}_{k_1} c^{(i)}_{k_2} \cdots c^{(i)}_{k_t} e_j \right) \\
= \sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \sum_{k_1=1}^{m_{i_t}} \left( b^{(i_{k_1})}_{k_1} b^{(i_{k_2})}_{k_2} \cdots b^{(i_{k_t})}_{k_t} e_j \right) \\
= \sum_{k_1=1}^{m_{i_1}} \left( b^{(i)}_{k_1} e_j \right) \\
= e_j,
\end{align*}
$$

(3.1)
and, for each \( g_i \in T_j, \)

\[
\sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} b_{k_1,k_2,\ldots,k_t} g_i(c_{k_1,k_2,\ldots,k_t}) \\
= \sum_{k_1=1}^{m_{i_1}} \sum_{k_2=1}^{m_{i_2}} \cdots \sum_{k_t=1}^{m_{i_t}} (b_{k_1}^{(i_1)} b_{k_2}^{(i_2)} \cdots b_{k_t}^{(i_t)}) g_i(c_{k_1}^{(i_1)} c_{k_2}^{(i_2)} \cdots c_{k_t}^{(i_t)}) e_j \\
= \sum_{k_1=1}^{m_{i_1}} (b_{k_1}^{(i_1)}) g_i(c_{k_1}^{(i_1)}) e_j \\
\sum_{k_2=1}^{m_{i_2}} (b_{k_2}^{(i_2)}) g_i(c_{k_2}^{(i_2)}) e_j \\
\cdots \\
\sum_{k_t=1}^{m_{i_t}} (b_{k_t}^{(i_t)}) g_i(c_{k_t}^{(i_t)}) e_j \\
= 0.
\]

Thus, \( \{b_{k_1,k_2,\ldots,k_t} c_{k_1,k_2,\ldots,k_t}, k_t = 1,2,\ldots, m_{i_t} \text{ and } l = 1,2,\ldots,t \} \) is a \( T_j \)-Galois system for \( B e_j \). This completes the proof. \( \square \)

**Theorem 3.4.** If \( B \) is a weak center Galois extension with group \( G \), then there exists a partition \( \{T_j \subset G, j = 1,2,\ldots,m \} \) of \( G - \{1\} \) and a finite set of central idempotents \( \{e'_i \mid i = 1,2,\ldots,m \} \) such that (1) \( B e'_j \) is a \( T_j \)-Galois extension of \( (Be'_j)^T_j \), (2) \( B = \sum_{j=1}^{m} Be'_j \oplus B(1 - \vee_{j=1}^{m} e'_j) \), where \( \vee_{j=1}^{m} e'_j \) is the sum of \( e'_1, e'_2,\ldots,e'_m \) in the Boolean algebra of all idempotents in \( C \), and (3) \( G|_{C(1-\vee_{j=1}^{m} e'_j)} = \{1\} \).

**Proof.** (1) Since \( BI_i = Be_i \) for some idempotent \( e_i \) in \( C \) for each \( g_i \neq 1 \) in \( G \), we have a set of central idempotents \( \{e_i \mid g_i \neq 1 \} \) in \( G \). Let \( E = \{e'_j \mid j = 1,2,\ldots,m \} \) be the set of all distinct idempotents in \( \{e_i \mid g_i \neq 1 \} \) and let \( T_j = \{g_i \in G \mid BI_i = Be'_j \} \), i.e., \( e_i = e'_j \). Then \( Be'_j \) is a \( T_j \)-Galois extension of \( (Be'_j)^T_j \) for each \( j = 1,2,\ldots,m \) by Lemma 3.3. Moreover, since \( E = \{e'_j \mid j = 1,2,\ldots,m \} \) is the set of all distinct idempotents in \( \{e_i \mid BI_i = Be_i \} \) for \( g_i \neq 1 \) in \( G \), it is easy to see that \( T_i \cap T_j = \emptyset \), the empty set for \( i \neq j \) and that \( \bigcup_{j=1}^{m} T_j = G - \{1\} \), that is, \( \{T_j \subset G, j = 1,2,\ldots,m \} \) is a partition of \( G - \{1\} \).

Part (2) is an immediate consequence of part (1), and Theorem 3.2 implies part (3).

We remark that the partition of \( G - \{1\} \), \( \{T_j \subset G, j = 1,2,\ldots,m \} \) is determined by the set of all distinct idempotents in \( \{e_i \mid BI_i = Be_i \} \) for \( g_i \neq 1 \) in \( G \). \( \square \)

When \( G \) is abelian, we obtain a stronger structure of a weak center Galois extension with group \( G \).

**Lemma 3.5.** Assume that \( B \) is a weak center Galois extension with group \( G \). If \( G \) is abelian, then \( g_j(e_i) = e_i \) for all \( i,j = 2,3,\ldots,n \).

**Proof.** For any \( c - g_i(c) \in I_i, g_j(c - g_i(c)) = g_j(c) - g_i(g_j(c)) \in I_i \). Hence, \( g_j(BI_i) \subset BI_i \). Thus, \( g_j \) restricted to \( BI_i (= Be_i) \) is an automorphism of \( Be_i \) since \( g_j \) is an automorphism of \( B \). Therefore, \( g_j(e_i) = e_i \). \( \square \)

**Theorem 3.6.** Assume that \( B \) is a weak center Galois extension with group \( G \). If \( G \) is abelian, then there exist orthogonal idempotents \( \{f_i \mid i = 1,2,\ldots,p \} \) for some integer \( p \) and some subset \( T^{(i)} \) of \( G, i = 1,2,\ldots,p \) such that \( B = \oplus_{i=1}^{p} B f_i \oplus B(1 - \vee_{i=1}^{p} f_i) \), where \( \vee_{i=1}^{p} f_i \) is the sum of \( f_1,f_2,\ldots,f_p \) in the Boolean algebra of all idempotents in \( C \) and \( B f_i \) is a \( T^{(i)} \)-Galois extension of \( (B f_i)^{T^{(i)}} \) for \( i = 1,2,\ldots,p \).
**Proof.** By Theorem 3.4, there exists a set of distinct idempotents $E = \{e'_j \mid j = 1, 2, \ldots, m\}$ in $C$ and a partition $\{T_j \mid j = 1, 2, \ldots, m\}$ of $G - \{1\}$ such that $B e'_j$ is a $T_j$-Galois extension of $(B e'_j)^{T_j}$ for $j = 1, 2, \ldots, m$. Now, let $S$ be the Boolean subalgebra generated by $E$ with all nonzero minimal elements $f_1, f_2, \ldots, f_p$ in $S$. Then, it is easy to see that $f_i f_j = 0$ for $i \neq j$, and so $f_1, f_2, \ldots, f_p$ are orthogonal idempotents in $C$. For each $f_i$, $i = 1, 2, \ldots, p$, $f_i = e'_j_1 e'_j_2 \cdots e'_j_p$. By Theorem 3.4, $B e'_j$ is a $T_j$-Galois extension of $(B e'_j)^{T_j}$ for each $l = 1, 2, \ldots, p_i$ with a $T_j$-Galois system $\{b_{l1}^{(i)} e_{j_1}' \cdots e_{j_p}' \mid b_{l1}^{(i)} \in C, j_1, \ldots, j_p \}$ of $\cup_{i=1}^p B e'_i$. Hence, by using the same patching method as given in Lemma 3.3, $\{b_{l1}^{(i)} b_{l2}^{(i)} \cdots b_{lp_i}^{(i)} f_i \mid c_{l1}, \ldots, c_{lp_i} \in C\}$ is a $T_j$-Galois system for $B f_i$, where $T_j = \cup_{i=1}^p T_{jl}$, and $T_{jl}$ is a $T_j$-Galois extension of $(B f_i)^{T_j}$ for $i = 1, 2, \ldots, p$ and $\{f_1, f_2, \ldots, f_p\}$ is a set of orthogonal idempotents in $C$. 

4. **Special cases.** We note that the $T_i$’s in Theorem 3.4 and $T^{(i)}$’s in Theorem 3.6 may not be subgroups of $G$. Next, we give a sufficient condition for each $T_i \cup \{1\} \subset G$ containing a subgroup $H_i$ so that $B e_i$ is a $H_i$-Galois extension of $(B e_i)^{H_i}$ with Galois group $H_i$. Consequently, $B e_i$ becomes a center Galois extension of $(B e_i)^{H_i}$ with Galois group $H_i$, and $B$ is a center Galois extension of $G$ with Galois group $G$ if $e_i = 1$ for all $g_i \neq 1$. We first show a relation between $B(1 - e_p)$, $B(1 - e_q)$, and $B(1 - e_t)$, where $g_p g_q = g_t \in G$.

**Lemma 4.1.** Let $J_l = \{b \in B \mid bc = g_t(c)b \text{ for all } c \in C\}$ for each $g_t \in G$. Then, $J_p J_q \subset J_t$ if $g_p g_q = g_t \in G$.

**Proof.** Let $a \in J_p$ and $b \in J_q$. Then, for any $c \in C$, $(a b)c = a g_q(c)b = g_p(g_q(c))a b = g_t(c)(a b)$, where $g_p g_q = g_t$. Hence, $ab \in J_t$. Thus, $J_p J_q \subset J_t$. 

**Corollary 4.2.** If $B$ is a weak center Galois extension with group $G$, then $B(1 - e_p)B(1 - e_q) \subset B(1 - e_t)$, where $g_p g_q = g_t \in G$.

**Proof.** Since $B$ is a weak center Galois extension with group $G$, $B I_l = B e_i$ for some idempotent $e_i$ in $C$ for each $g_i \neq 1$ in $G$. But $I_l = \{c - g_t(c) \mid c \in C\}$, so $J_l = \{b \in B \mid b (c - g_t(c)) = 0 \text{ for all } c \in C\}$. Hence, $J_l = \text{Ann}_B(I_l) = \text{Ann}_B(B I_l) = \text{Ann}_B(B e_i) = B(1 - e_t)$. Thus, by Lemma 4.1, we have $B(1 - e_p)B(1 - e_q) \subset B(1 - e_t)$, where $g_p g_q = g_t \in G$.

**Theorem 4.3.** Assume that $B$ is a weak center Galois extension with group $G$. Let $T_i$, for each $i = 2, 3, \ldots, n$, be the subset of $G$ as given in Theorem 3.4 such that $B e_i$ is a $T_i$-Galois extension of $(B e_i)^{T_i}$, the Boolean subalgebra generated by $\{e_i \mid g_i \neq 1 \text{ in } G\}$ with all nonzero minimal elements $\{f_1, f_2, \ldots, f_k\}$ in $S$, and $H_j = \{1\} \cup \{g_i \in G \mid e_i f_j = f_j \text{ and } e_i f_l = 0 \text{ for all } l \neq j\}$. Then, $H_j$ is a subgroup of $G$ for each $j = 1, 2, \ldots, k$ such that $g_i(f_j) = f_j$ for each $g_i \in H_j$.

**Proof.** (1) For any $g_p$ and $g_q$ in $H_j$, let $g_p g_q = g_t$ for some $g_t \in G$. We claim that $g_t \in H_j$ if $g_t \neq 1$. Since $g_t \neq 1$, $B I_l = B e_i$ for some idempotent $e_i \neq 0$ in $C$. By Corollary 4.2, $B(1 - e_p)B(1 - e_q) \subset B(1 - e_t)$. Therefore, in the Boolean algebra of all
idempotents in $C$ with operations $\land$, $\lor$, complement, and the relation $<, (1 - e_p)(1 - e_q) < (1 - e_t)$. So $e_t < e_p \lor e_q = e_p + e_q - e_pe_q$. Thus, $e_t = e_t(e_p + e_q - e_pe_q)$. Since $g_p, g_q \in H_j, e_p, f_i = 0$ and $e_q, f_i = 0$ for all $l \neq j$. Hence, $e_t, f_i = e_t(e_p + e_q - e_pe_q) f_i = 0$ for all $l \neq j$. Moreover, since $S$ is the Boolean subalgebra generated by $\{e_t | g_i \neq 1 \in G\}$, there is at least one nonzero minimal element in $S$ less than $e_t$. But $e_t, f_i = 0$ for all $l \neq j$, so $f_j$ must be less than $e_t$. Hence, $e_t, f_j = f_j$. Thus, $g_t(=g_p, g_q) \in H_j$, and so $H_j$ is a subgroup of $G$. Moreover, suppose $g_i \in H_j$. Then $e_t, f_j = f_j$ and $e_t, f_i = 0$ for all $l \neq j$. Hence, $e_t$ is greater than $f_j$, but not greater than $f_i$ for all $l \neq j$. Since $g_i(e_t) = e_t, g_i(f_j)$ is a nonzero minimal element in $S$ less than $e_t$. Thus, $g_i(f_j) = f_j$. □

**Corollary 4.4.** Keeping the notation in Theorem 4.3, if $H_j \neq \{1\}$ for $j = 1, 2, \ldots, p$, then $B = \sum_{j=1}^p B(f_j) \oplus B(1 - \lor_{j=1}^p f_j)$, where $\lor_{j=1}^p f_j$ is the sum of $f_1, f_2, \ldots, f_p$ in the Boolean algebra of all idempotents in $C$, such that $B(f_j)$ is a $H_j$-Galois extension of $(B f_j)^{H_j}$ with Galois group $H_j$ for $j = 1, 2, \ldots, p$.

**Corollary 4.5.** If $B_{1j} = B$ for each $g_j \neq 1$ in $G$, then $B$ is a center Galois extension of $B^G$ with Galois group $G$.

**Proof.** Since $e_0 = e_1 = \ldots = e_n$, $T_1 = T_2 = \ldots = T_n = G - \{1\}$, so $T_i \cup \{1\} = G$. Thus, $B$ is a Galois extension of $B^G$ with a Galois system $\{b_i \in B; c_i \in C, i = 1, 2, \ldots, m\}$ for some integer $m$, that is, $B$ is a center Galois extension of $B^G$ with Galois group $G$. □

If the order of each nonidentity element in $G$ has order 2 (hence, $G$ is abelian), the following theorem shows that $T_i \cup \{1\}$ contains a subgroup of $G$ for each $g_j \neq 1$ in $T_i$.

**Theorem 4.6.** Assume that $B$ is a weak center Galois extension with group $G$. If each nonidentity element $g_i$ in $G$ has order 2, then $T_i$ contains a subgroup of $H_i$ of order 2 for each $g_j \neq 1$ in $G$ such that $B e_i$ is a $H_i$-Galois extension of $(B e_i)^{H_i}$ with Galois group $H_i$.

**Proof.** Let $B_{1i} = B e_i$ for $g_i \neq 1$ in $G$. Then $H_i = \{1, g_i\}$ is a subgroup contained in $T_i \cup \{1\}$, where $T_i = \{g_k \in G | B_{1k} = B e_i\}$ as defined in Theorem 3.4. Since $B e_i$ is a $T_i$-Galois extension of $(B e_i)^{T_i}$, $B e_i$ is a $H_i$-Galois extension of $(B e_i)^{H_i}$ with Galois group $H_i$. □

Theorem 3.4 shows that a weak center Galois extension is a sum of $T_i$-Galois extensions for some $T_i \subset G$ and Theorem 4.6 states a weak center Galois extension as a direct sum of center Galois extensions. The following is an example of a weak center Galois extension with group $G$ as given in Theorem 4.6, but not a Galois extension.

**Example 4.7.** Let $\mathbb{Q}$ be the rational field, $B = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$, and $G = \{g_1 = 1, g_2, g_3, g_4 = g_2 g_3\}$ such that $g_2(a_1, a_2, a_3, a_4, a_5) = (a_1, a_3, a_4, a_5)$ and $g_3(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_4, a_3, a_5)$ for all $(a_1, a_2, a_3, a_4, a_5) \in B$. Then,

1. $B_{1i} = B e_i$ for each $g_i \neq 1$ in $G$, where $e_2 = (1, 1, 0, 0, 0), e_3 = (0, 0, 1, 1, 0), \text{ and } e_4 = (1, 1, 1, 1, 0)$. Hence, $B$ is a weak center Galois extension with group $G$.

2. $B$ is not a Galois extension since $G$ restricted to $\{(0, 0, 0, 0, a) | a \in \mathbb{Q}\}$ is identity.

3. Let $H_i = \{1, g_i\}, i = 2, 3, 4$. Then $H_i$ is a subgroup of $G$ of order 2. Moreover, $B_{12} = B e_2$ is a center $H_2$-Galois extension of $(B e_2)^{H_2}$ with Galois system $\{b_1 = (1, 0, 0, 0, 0), b_2 = (0, 1, 0, 0, 0); c_1 = (1, 0, 0, 0, 0), c_2 = (0, 1, 0, 0, 0)\}, B_{13} = B e_3$ is a center $H_3$-Galois extension of $(B e_3)^{H_3}$ with Galois system $\{b_1 = (0, 0, 1, 0, 0), b_2 = (0, 0, 0, 1, 0); c_1 = (0, 0, 1, 0, 0), c_2 = (0, 1, 0, 0, 0)\}$,
1,0,0), c_2 = (0,0,0,1,0)), and Bf_4 = B_{e_4} is a center $H_4$-Galois extension of $(B_{e_4})^{H_4}$ with Galois system $\{b_1 = (1,0,0,0,0), b_2 = (0,1,0,0,0), b_3 = (0,0,1,0,0), b_4 = (0,0,0,1,0); c_1 = (1,0,0,0,0), c_2 = (0,1,0,0,0), c_3 = (0,0,1,0,0), c_4 = (0,0,0,1,0)\}.

(4) $S = \{0 = (0,0,0,0,0), e_2, e_3, e_4, 1 = (1,1,1,1,1)\}$ is the Boolean subalgebra generated by $E = \{e_2, e_3, e_4\}$ in the Boolean algebra of all idempotents in the center of $B$. The minimal elements in $S$ are $f_1 = e_2$ and $f_2 = e_3$, and $f_1 \vee f_2 = e_4$. We have that $Bf_1 = \{(a_1, a_2, 0, 0, 0) | a_1, a_2 \in \mathbb{Q}\}$, $Bf_2 = \{(0, 0, 0, 0, 0) | a_3, a_4 \in \mathbb{Q}\}$, and $B(1 - f_1 \vee f_2) = \{(0, 0, 0, 0, 0) | a_5 \in \mathbb{Q}\}$. So $B = Bf_1 \oplus Bf_2 \oplus B(1 - f_1 \vee f_2)$ and $Bf_j$ is a $H_j$-Galois extension of $(Bf_j)^{H_j}$ for $j = 1, 2$.

(5) Since $e_2 = (1,1,0,0,0), e_3 = (0,0,1,1,0), and e_4 = (1,1,1,1,0)$, we have $C(1 - e_2) = \{(0,0,a_3,a_4,a_5) | a_3,a_4,a_5 \in \mathbb{Q}\}, C(1 - e_3) = \{(a_1,a_2,0,0,a_5) | a_1,a_2,a_5 \in \mathbb{Q}\}, and C(1 - e_4) = \{(0,0,0,0,a_5) | a_5 \in \mathbb{Q}\}$. So $g_i$ restricted to $C(1 - e_i)$ is an identity for each $g_i \neq 1$ in $G$.

REFERENCES


GEORGE SZETO: MATHEMATICS DEPARTMENT, BRADLEY UNIVERSITY, PEORIA, IL 61625, USA E-mail address: szeto@hilltop.bradley.edu
LIANYONG XUE: MATHEMATICS DEPARTMENT, BRADLEY UNIVERSITY, PEORIA, IL 61625, USA E-mail address: lxue@hilltop.bradley.edu
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**Guest Editors**

*José Roberto Castilho Piqueira*, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

*Elbert E. Neher Macau*, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

*Celso Grebogi*, Department of Physics, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk