STRONG UNIQUE CONTINUATION OF EIGENFUNCTIONS FOR $p$-LAPLACIAN OPERATOR

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Abstract. We show the strong unique continuation property of the eigenfunctions for $p$-Laplacian operator in the case $p < N$.

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1. Introduction. This paper is primarily concerned with the problem:

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + V|u|^{p-2}u = 0 \text{ in } \Omega,$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and the weight functions $V$ is assumed to be not equivalent to zero and to lie in $L^{N/p}(\Omega)$.

Also, we know that the unique continuation property is defined by a different form. In this work, we are interested to study a family of functions which enjoys the strong unique continuation property (SUCP), that is, functions besides possibly the zero functions has a zero of infinite order.

Definition 1.1. A function $u \in L^p(\Omega)$ has a zero of infinite order in $p$-mean at $x_0 \in \Omega$, if for each $n \in \mathbb{N}$,

$$\int_{|x-x_0| \leq R} |u|^p = 0(\mathbb{R}^n) \text{ as } R \to 0.$$

(1.2)

There is an extensive literature on unique continuation. We refer to the work of Jerison-Kenig on the unique continuation for Shrödinger operators (cf. [3]). The same work is done by Gossez and Figueiredo, but for linear elliptic operator in the case $V \in L^{N/2}$, where $N > 2$, (cf. [1]). Also, Loulit extended this property to $N = 2$ by introducing Orlicz’s space, (cf. [2, 5]). In this work, we generalize this property for the $p$-Laplacian in the case $V \in L^{N/p}(\Omega)$ and $p < N$.

2. Strong unique continuation theorem. In this section, we proceed to establish the strong unique continuation property of the eigenfunctions for the $p$-Laplacian operator in the case $V \in L^{N/p}(\Omega)$ and $p < N$.

Theorem 2.1. Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ solution of (1.1). If $u = 0$ on a set $E$ of positive measure, then $u$ has a zero of infinite order in $p$-mean.
To prove this theorem we need the following lemmas.

**Lemma 2.2.** Let \( g \in W^{1,p}_0(\Omega) \) and \( V \in L^{N/p} \). Then for each \( \epsilon > 0 \) there exists a positive constant \( k_\epsilon \) such that

\[
\int_\Omega V|g|^p \leq \epsilon \int_\Omega |\nabla g|^p + k_\epsilon \int_\Omega |g|^p. \tag{2.1}
\]

**Proof.** Set \( G = \{ x \in \Omega : V(x) \geq c \} \).

So

\[
\int_\Omega V|g|^p \leq \int_G V|g|^p + k \int_\Omega |g|^p. \tag{2.3}
\]

By using the Hölder and Poincaré’s inequalities, we get

\[
\int_\Omega V|g|^p \leq c \| \chi_G V \|_{L^\infty} \int_\Omega |\nabla g|^p + k \int_\Omega |g|^p. \tag{2.4}
\]

But \( \| \cdot \| \) is absolutely continuous. So, given \( \epsilon > 0 \), there exists \( k \) such that \( c \| \chi_G V \| \leq \epsilon \).

Which gives immediately the inequality (2.1). \( \square \)

**Lemma 2.3.** Let \( B_r \) and \( B_{2r} \) be two concentric balls contained in \( \Omega \). Then

\[
\int_{B_r} |\nabla u|^p \leq \frac{c}{r^p} \int_{B_{2r}} |u|^p, \tag{2.5}
\]

where the constant \( c \) does not depend on \( r \).

**Proof.** Take \( \varphi \in C_0^\infty(\Omega) \), with \( \text{supp} \varphi \subset B_{2r}, \varphi(x) = 1 \) for \( x \in B_r \) and \( |\nabla \varphi| \leq c/r \).

Using \( \varphi^p u \) as a test function in (1.1), we get

\[
\int_{B_{2r}} -\text{div}(|\nabla u|^{p-2} \nabla u) \varphi^p u + \int_{B_{2r}} V|u|^{p-2} u \varphi^p u = 0. \tag{2.6}
\]

So

\[
\int_{B_{2r}} |\nabla u|^p \varphi^p = -p \int_{B_{2r}} |\nabla u|^{p-2} \varphi^p \nabla \varphi \cdot \nabla (\varphi u) - \int_{B_{2r}} V|\varphi u|^p. \tag{2.7}
\]

Using Young’s inequalities for \((((p-1)/p) + 1/p = 1)\), we can estimate the first integral in the right-hand side of (2.7) by

\[
(p-1)\epsilon^{p/(p-1)} \int_{B_{2r}} |\nabla u|^p \varphi + \epsilon^{-p} \int_{B_{2r}} |\nabla \varphi|^p |u|^p. \tag{2.8}
\]

Also by the result of Lemma 2.2, we can estimate the second integral in the right-hand side of (2.7) by

\[
\epsilon \int_{B_{2r}} |\nabla (\varphi u)|^p + c \epsilon \int_{B_{2r}} |\varphi u|^p. \tag{2.9}
\]

Using these estimates in (2.7), we have

\[
\int_{B_{2r}} |\nabla u|^p \varphi \leq ((p-1)\epsilon^{p/(p-1)} + \epsilon) \int_{B_{2r}} |\nabla u|^p |\varphi|^p + (\epsilon^{-p} + \epsilon) \int_{B_{2r}} |u|^p |\nabla \varphi|^p + c \epsilon \int_{B_{2r}} |u|^p |\varphi|^p. \tag{2.10}
\]

Using the fact that \( |\nabla \varphi| \leq c/r, |\varphi| \leq c/r, \text{ and } \varphi = 1 \) in \( B_r \), we have immediately inequality (2.5). \( \square \)
**Lemma 2.4.** Let \( u \in W^{1,1}(B_r) \), where \( B_r \) is the ball of radius \( r \) in \( \mathbb{R}^N \) and let \( E = \{ x \in B_r : u(x) = 0 \} \). Then there exists a constant \( \beta \) depending only on \( N \) such that

\[
\int_A |u| \leq \beta \frac{r^N}{|E|^{1/N}} \int_{B_r} |\nabla u|
\]  

(2.11)

for all ball \( B_r \), \( u \) as above and all measurable sets \( A \subset B_r \).

To prove this lemma see [4].

**Proof of Theorem 2.1.** We know that almost every point of \( E \) is a point of density of \( E \). Let \( x_0 \in E \) be such a point. This means that

\[
\lim_{r \to 0} \frac{|E \cap B_r|}{|B_r|} = 1,
\]

(2.12)

where \( B_r \) denotes the ball of radius \( r \) centered at \( x_0 \) and \( |S| \) denotes the Lebesgue measure of a set \( S \). So, given \( \epsilon > 0 \) there is an \( r_0 = r_0(\epsilon) \) such that

\[
\frac{|E^c \cap B_r|}{|B_r|} < \epsilon, \quad \frac{|E \cap B_r|}{|B_r|} > 1 - \epsilon \quad \text{for} \quad r \leq r_0,
\]

(2.13)

where \( E^c \) denotes the complement of the set \( E \). Taking \( r_0 \) smaller, if necessary, we can assume \( B_{r_0} \subset \Omega \). Since \( u = 0 \) on \( E \), by Lemma 2.4 and (2.13) we have

\[
\int_{B_r} |u|^p = \int_{B_r \cap E^c} |u|^p \leq \beta \frac{r^N}{|E \cap B_r|} |E^c \cap B_r|^{1/N} \int_{B_r} |\nabla (u)^p| \leq p \beta \frac{r^N}{|B_r|^{(1-1/N)}} \frac{e^{1/N}}{1-\epsilon} \int_{B_r} |u|^{p-1} |\nabla u|.
\]

(2.14)

By Hölder’s inequality

\[
\int_{B_r} |u|^p \leq c \frac{e^{1/N}}{1-\epsilon} r \left( \int_{B_r} |\nabla u|^p \right)^{1/p} \left( \int_{B_r} |u|^p \right)^{(p-1)/p},
\]

(2.15)

and by using the Young’s inequality, we get

\[
\int_{B_r} |u|^p \leq c \frac{e^{1/N}}{1-\epsilon} r \left( r^{p-1} \int_{B_r} |\nabla u|^p + \frac{p-1}{r} \int_{B_r} |u|^p \right).
\]

(2.16)

Finally, by Lemma 2.3, we have

\[
\int_{B_r} |u|^p \leq c \frac{e^{1/N}}{1-\epsilon} \int_{B_{2r}} |u|^p,
\]

(2.17)

where \( c \) is independent of \( \epsilon \) and of \( r \) as \( r \to 0 \).
Now let us introduce the following functions:
\[ f(r) = \int_{B_r} |u|^p. \quad (2.18) \]

And let us fix \( n \in \mathbb{N} \), choose \( \epsilon > 0 \) such that \( (c\epsilon^{1/N})/(1-\epsilon) \leq 2^{-n} \). Observe that consequently \( r_0 \) depends on \( n \). Then (2.17) can be written as
\[ f(r) \leq 2^{-n} f(2r) \quad \text{for} \quad r \leq r_0. \quad (2.19) \]

Iterating (2.19), we get
\[ f(\rho) \leq 2^{-kn} f(2^k \rho), \quad \text{if} \quad 2^{k-1} \rho \leq r_0. \quad (2.20) \]

Now given \( 0 < r < r_0(n) \) and choose \( k \in \mathbb{N} \) such that
\[ 2^{-k} r_0 \leq r \leq 2^{-k+1} r_0. \quad (2.21) \]

From (2.20), we obtain
\[ f(r) \leq 2^{-kn} f(2^k r) \leq 2^{-kn} f(2r_0). \quad (2.22) \]

Since \( 2^{-k} \leq r/r_0 \), we finally obtain
\[ f(r) \leq \left( \frac{r}{r_0} \right)^n f(2r_0), \quad (2.23) \]

which shows that \( x_0 \) is a zero infinite order in \( p \)-mean. \( \Box \)

REFERENCES


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