TOPOLOGICAL CONJUGACIES OF PIECEWISE MONOTONE INTERVAL MAPS

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Abstract. Our aim is to establish the topological conjugacy between piecewise monotone expansive interval maps and piecewise linear maps. First, we are concerned with maps satisfying a Markov condition and next with those admitting a certain countable partition. Finally, we compute the topological entropy in the Markov case.

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1. Introduction and preliminaries. Let I be a closed interval in \( \mathbb{R} \), which is usually taken to be the interval \([0, 1]\), and \( f : I \to I \) a mapping. The iterates of \( f \) are the maps \( f^n \) defined inductively by \( f^0 = \text{id}_I, f^1 = f, f^{n+1} = f^n \circ f \). The (forward or positive) orbit of a point \( x \in I \) is the set \( O(x) = \{ f^n(x) : n \in \mathbb{N} \} \). The \( \omega \)-limit set of \( x \) is the set of the limit points of \( O(x) \) and is denoted by \( \omega(x) \). Two maps \( f : I \to I \) and \( g : J \to J \) are called topologically conjugate if there exists a homeomorphism \( h : I \to J \) such that \( h \circ f = g \circ h \).

The study of topological conjugacies has commenced with Poincaré in the 1880s. He considered homeomorphisms \( f : S^1 \to S^1 \) of the unit circle \( S^1 = \mathbb{R}/\mathbb{Z} \) with no periodic points and showed that there exist a rotation \( R : S^1 \to S^1 \) and a continuous, surjective and monotone map \( h : S^1 \to S^1 \) such that \( h \circ f = R \circ h \), that is, \( f \) and \( R \) are topologically semiconjugate. Similar results for piecewise monotone interval maps \( f \) were proved later by Parry [10] and Milnor and Thurston [9]. According to them, if \( f : I \to I \) is continuous, piecewise monotone with positive topological entropy \( h(f) \), then there exists a piecewise linear map \( T : [0, 1] \to [0, 1] \) with slope \( \pm \exp(h(f)) \) such that \( f, T \) are topologically semiconjugate. \( f \) and \( T \) become topologically conjugate, if there are no attracting periodic points and no wandering intervals for \( f \). The nonexistence of wandering intervals has been proved for a large class of functions satisfying some mild smoothness conditions (see [3, 6, 7, 8]).

In this paper, we consider the family \( \mathcal{M} \) of functions which are piecewise monotone (but not necessarily continuous) and expansive. Particularly, \( f : [0, 1] \to [0, 1] \) belongs to the family \( \mathcal{M} \) if there exists a partition \( 0 = a_0 < a_1 < \cdots < a_r = 1 \) (\( r \geq 2 \)) of \([0, 1]\) such that \( f \mid [a_{i-1}, a_i] \) \( (i = 1, 2, \ldots, r) \) is a monotone \( C^1 \) function and satisfy the following Markov condition: for every \( i = 1, 2, \ldots, r \), there exist \( p(i), q(i) \in \{0, 1, \ldots, r\} \) with \( p(i) < q(i) \) such that \( f(a_{i-1}, a_i) = (a_{p(i)}, a_{q(i)}) \). Furthermore, we assume that there is \( \lambda > 1 \) such that \( |f'(x)| \geq \lambda \), for almost every \( x \in [0, 1] \), in which case, \( f \) is called expansive. Our aim is to show that every \( f \in \mathcal{M} \) is topologically conjugate to a map \( T \) which is linear on each interval \([(i-1)/r, i/r] \) \( (i = 1, 2, \ldots, r) \). Next, we
consider the class $\mathcal{M}_c$ where $[0,1]$ accepts a countable partition accumulating to $1$. Finally, in the last section, we compute the topological entropy for continuous maps in $\mathcal{M}$.

**Notation.** If $J \subset [0,1]$ is an interval, we denote $|J|$ its length.

2. **Topological conjugacies for maps in $\mathcal{M}$.** In this section, we study the topological conjugacies for maps $f \in \mathcal{M}$. If $0 = a_0 < a_1 < \cdots < a_r = 1$ is the partition corresponding to $f$, we say that $f$ is of order $r$. The points of the partition are called critical points of $f$. We denote by $I_1, \ldots, I_r$ the intervals of the partition, that is, $I_j = (a_{j-1}, a_j)$.

We assume that these intervals are maximal in the sense that if $I$ is an interval which strictly contains one of them, then $f \mid I$ is neither continuous nor monotone. Also, we denote by $f_j$ the restriction of $f$ to $I_j$. Finally, we denote by $F_{j_1, j_2, \ldots, j_k}$ the composition $f_{j_k}^{-1} \circ f_{j_{k-1}}^{-1} \cdots \circ f_{j_1}^{-1}$. Note that $F_{j_1, j_2, \ldots, j_k}$ is not necessarily defined for every (finite) sequence $j_1, j_2, \ldots, j_k$. Moreover, $F_{j_1, j_2, \ldots, j_k}(x)$ is the unique point $y \in I_{j_1}$ such that $f(y) = I_{j_2}, \ldots, f^{k-1}(y) \in I_{j_k}$ and $f^k(y) = x$.

An open interval $J \subset [0,1]$ is called a branch of $f^n$ if $f^n \mid J$ is continuous, monotone and $J$ is maximal with these properties. The set of branches of $f^n$ is denoted by $B_n(f)$.

Moreover, we define the sets

$$\mathcal{C}_n(f) = \bigcup_{j=0}^{r-1} \bigcup_{i=0}^{n-1} f^{-i}(a_j), \quad n = 1, 2, \ldots,$$

$$\mathcal{C}(f) = \bigcup_{j=0}^{r-1} \bigcup_{i=0}^{\infty} f^{-i}(a_j).$$

Frequently, we write $\mathcal{C}_n$ and $\mathcal{C}$ instead of $\mathcal{C}_n(f)$ and $\mathcal{C}(f)$.

In what follows, we introduce some notions from symbolic dynamics. To each point $x$ of $\mathcal{C}$, there corresponds a sequence of symbols which is related with the order of the points of $O(x)$.

**Definition 2.1.** The itinerary of $x \in \mathcal{C}$ with respect to $f \in \mathcal{M}$ is a sequence $i_f(x) = \{i_n(x)\}_{n=0}^{\infty}$, where

$$i_n(x) = \begin{cases} j, & \text{if } f^n(x) \in I_j, \\ 2j + 1, & \text{if } f^n(x) = a_j. \end{cases}$$

An interesting notion in symbolic dynamics is the shift map $\sigma$: if $x = \{x_n\}_{n=0}^{\infty}$, then $\sigma(x) = y$, where $y = \{x_n\}_{n=1}^{\infty}$. Inductively, we have $\sigma^k(x) = \{x_n\}_{n=k}^{\infty}$. To each $f \in \mathcal{M}$ of order $r$, we associate a subset of $\{1/2, 1, 3/2, \ldots, r, (2r + 1)/2\}^\mathbb{N}$. We describe this set in the following definition.

**Definition 2.2.** Let $f \in \mathcal{M}$ with partition $0 = a_0 < a_1 < \cdots < a_r = 1$. We define the set of sequences $\Sigma(f) = \{a : a = \{x_n\}_{n=0}^{\infty} \}$ with entries from the set $\{1/2, 1, 3/2, \ldots, r, (2r + 1)/2\}$, which satisfy the following conditions:

(i) Let $a = \{x_n\} \in \Sigma(f)$. Then there exists an entry $x_k$ of $a$ of the form $(2k + 1)/2$, where $k = 0, 1, \ldots, r$. Furthermore, if $x_N$ is the first entry of $a$ with this property, then $\sigma^N(a) = i_f(a_k)$. 


(ii) If \( n < N - 1 \) and \( x_n = j \), then \( p(j) + 1 \leq x_{n+1} \leq q(j) \).

It is possible to define an order on the set \( \mathcal{I}_f(\mathcal{E}) \) which is consistent with the natural order of real numbers. Two sequences of symbols \( x = \{x_n\}_{n=0}^{\infty} \) and \( y = \{y_n\}_{n=0}^{\infty} \) belonging to \( (1/2, 1/3, 2, \ldots, r, (2r + 1)/2)^N \) are called to have discrepancy \( n \) if \( x_i = y_i \), for \( i = 0, 1, \ldots, n - 1 \), and \( x_n \neq y_n \). If the itineraries of two points of \( \mathcal{E} \) have discrepancy \( n \), then the first \( n \) points of their orbits are visiting simultaneously the same intervals of \( B_1(f) \). Moreover, we define \( 1/2 < 1 < 3/2 < \cdots < r < (2r + 1)/2 \).

**Definition 2.3.** Let \( f \in \mathcal{M} \) and \( x, y \in \mathcal{E} \) with \( x \neq y \). We assume that itineraries \( \hat{i}_f(x) \) and \( \hat{i}_f(y) \) have discrepancy \( n \) and that \( f \) is decreasing in \( k \) common intervals.

(i) When \( k \) is even, then \( \hat{i}_f(x) < \hat{i}_f(y) \) if and only if \( i_n(x) < i_n(y) \).

(ii) When \( k \) is odd, then \( \hat{i}_f(x) < \hat{i}_f(y) \) if and only if \( i_n(x) < i_n(y) \).

**Lemma 2.4.** Let \( f \in \mathcal{M} \) be of order \( r \) and let \( x, y \in \mathcal{E} \) with \( x \neq y \). Then \( \hat{i}_f(x) < \hat{i}_f(y) \) if and only if \( x < y \).

**Proof.** We assume that itineraries \( \hat{i}_f(x) \) and \( \hat{i}_f(y) \) have discrepancy \( n \). That is, \( i_k(x) = i_k(y) = j_k \), for \( k = 0, 1, \ldots, n - 1 \), and \( i_n(x) \neq i_n(y) \). We claim that \( j_0, j_1, \ldots, j_{n-1} \) are not of the form \((2s + 1)/2\). To prove this, we assume the contrary, whence \( \hat{i}_f(x) = \hat{i}_f(y) \), which is a contradiction, since \( i_n(x) \neq i_n(y) \). From Definition 2.1, \( x, y \) belong to \( I_{j_0} \) and successively visit the intervals \( I_{j_1}, \ldots, I_{j_{n-1}} \). So, we can write \( x = F_{j_0 j_1 \cdots j_{n-1}}(f^n(x)) \) and \( y = F_{j_0 j_1 \cdots j_{n-1}}(f^n(y)) \). We assume that \( f \) is decreasing in \( k \) intervals among \( I_{j_0}, I_{j_1}, \ldots, I_{j_{n-1}} \). There are two cases.

(i) When \( k \) is even, then \( F_{j_0 j_1 \cdots j_{n-1}} \) is increasing. Assume that \( \hat{i}_f(x) < \hat{i}_f(y) \), then from Definition 2.3 we have \( i_n(x) < i_n(y) \). This means that \( f^n(x) < f^n(y) \) and, hence, \( x = F_{j_0 j_1 \cdots j_{n-1}}(f^n(x)) < y = F_{j_0 j_1 \cdots j_{n-1}}(f^n(y)) \).

(ii) When \( k \) is odd, then \( F_{j_0 j_1 \cdots j_{n-1}} \) is decreasing. Assume that \( \hat{i}_f(x) < \hat{i}_f(y) \), then from Definition 2.3 we have \( i_n(x) < i_n(y) \). This means that \( f^n(x) > f^n(y) \) and, hence, \( x = F_{j_0 j_1 \cdots j_{n-1}}(f^n(x)) < y = F_{j_0 j_1 \cdots j_{n-1}}(f^n(y)) \).

**Lemma 2.5.** Let \( f \in \mathcal{M} \) be of order \( r \). The map \( \hat{i}_f : \mathcal{E} \to \Sigma(f) \) is a bijection.

**Proof.** Let \( x, y \in \mathcal{E} \) with \( \hat{i}_f(x) = \hat{i}_f(y) \). Let \( k, m \) be the minimal integers for which \( f^k(x) \), \( f^m(y) \) are critical points of \( f \). Assume that \( k + m \) (let \( k < m \)). Since \( f^k(x) \) is a critical point, then \( f^{k+1}(x) = 0 \) or 1, and, so, \( i_{k+1}(x) = 1/2 \) or \( (2r + 1)/2 \). On the other hand, \( i_k(y) = 1, 2, \ldots, r \), and, hence, \( i_{k+1}(y) \neq 1/2 \) and \( i_{k+1}(y) \neq (2r + 1)/2 \), which is a contradiction, since \( i_{k+1}(x) = i_{k+1}(y) \). So, \( k = m \). Furthermore, we observe that \( f^k(x) = f^k(y) \), since \( i_k(x) = i_k(y) \) and it is of the form \((2j + 1)/2\). Consequently, \( f^k(x) = f^k(y) = a_j \).

Assume that \( i_n(x) = i_n(y) = j_n \in \mathbb{N} \), for \( n = 0, 1, \ldots, k - 1 \). From Definition 2.1, \( x, y \) belong to \( I_{j_0} \) and successively visit the intervals \( I_{j_1}, \ldots, I_{j_{k-1}} \). So, we can write \( x = F_{j_0 j_1 \cdots j_{k-1}}(f^k(x)) \) and \( y = F_{j_0 j_1 \cdots j_{k-1}}(f^k(y)) \). Since \( f^k(x) = f^k(y) \), we have \( x = y \). Thus, \( \hat{i}_f \) is injective.

Let \( \alpha = \{x_n\} \in \Sigma(f) \). We shall show that there exists an \( x \in \mathcal{E} \) such that \( \hat{i}_f(x) = \alpha \).

From Definition 2.2, an entry of the sequence \( \alpha \) is of the form \((2k + 1)/2\). Let \( x_n \) be the first entry with this property. Then \( x = F_{x_0 x_1 \cdots x_{n-1}}(a_k) \) satisfies the desired property.
Proposition 2.6. Let \( f \in \mathcal{M} \) be of order \( r \). Then \( \mathcal{C} \) is dense in \([0, 1]\).

Proof. Let \( \tilde{J} \subset [0, 1] \) be an open interval such that \( \tilde{J} \cap \mathcal{C} = \emptyset \). First, we show that \( f^n(\tilde{J}) \cap \mathcal{C} = \emptyset \), for \( n \in \mathbb{N} \). We assume, in the contrary, that there exists \( x \in f^n(\tilde{J}) \cap \mathcal{C} \), then there is \( y \in \tilde{J} \) such that \( x = f^n(y) \). But, \( f^m(x) = a_k \), for some \( m \in \mathbb{N} \) and \( k = 0, 1, 2, \ldots, r \), since \( x \in \mathcal{C} \). So, \( f^{m+n}(y) = f^m(x) = a_k \), that is, \( y \in \mathcal{C} \), which is a contradiction, since \( \tilde{J} \cap \mathcal{C} = \emptyset \).

As \( f^n(\tilde{J}) \cap \mathcal{C} = \emptyset \), for \( n \in \mathbb{N} \), it turns out that \( f \) is monotone and \( C^1 \) on each interval \( \tilde{J}, f(\tilde{J}), f^2(\tilde{J}), \ldots \).

We prove by induction that \( |f^n(\tilde{J})| \geq \lambda^n |\tilde{J}| \), for \( n \geq 1 \). From the mean value theorem and since \( f|\tilde{J} \) is monotone, we have \( |f(\tilde{J})|/|\tilde{J}| = |f'(a)| \), for some \( a \in \tilde{J} \). But, \( |f'(a)| \geq \lambda \) and, hence \( |f(\tilde{J})| \geq \lambda |\tilde{J}| \). We assume that the claim is true for \( k < n \). From the mean value theorem and since \( f|f^{n-1}(\tilde{J}) \) is monotone, we have \( |f^n(\tilde{J})|/|f^{n-1}(\tilde{J})| = |f'(a_1)| \geq \lambda \), for some \( a_1 \in f^{n-1}(\tilde{J}) \). From the induction assumption, we have \( |f^{n-1}(\tilde{J})| \geq \lambda^{n-1} |\tilde{J}| \). Combining the last two inequalities, we have \( |f^n(\tilde{J})| \geq \lambda^n |\tilde{J}| \).

Thus, for some \( n \in \mathbb{N} \), \( \lambda^n |\tilde{J}| > 1 \), which is a contradiction, since \( |f^n(\tilde{J})| \leq 1 \).

Theorem 2.7. Let \( f \in \mathcal{M} \) be of order \( r \) with partition \( 0 = a_0 < a_1 < \cdots < a_r = 1 \). We consider the map \( T \in \mathcal{M} \) with partition \( 0 < 1/r < 2/r < \cdots < (r-1)/r < 1 \) which is linear in each interval \([i/i, i/r)\) and \( T([i-1/i, i/r) = (p(i)/r, q(i)/r) \). Furthermore, \( T|[i-1/i, i/r) \) is of the same monotonicity type with \( f|[a_{i-1}, a_i) \) and it is continuous, from the right or from the left at \( i/r \), when \( f \) is continuous, from the right or from the left at \( a_i \), respectively. Then \( f \) and \( T \) are topologically conjugate. (Figure 2.1)

Proof. From Definition 2.2, we have \( \Sigma(f) = \Sigma(T) \). With this observation and since \( i_f \) and \( i_T \) are bijections (Lemma 2.5), we can define a correspondence \( h : \mathcal{C}(f) \rightarrow \mathcal{C}(T) \).
\( \mathcal{C}(T) \), which is an order preserving bijection and such that \( h \circ f = T \circ h \). For \( x \in \mathcal{C}(f) \), we define \( h(x) \) to be the unique element of \( \mathcal{C}(T) \), for which \( \dot{i}_f(h(x)) = \dot{i}_T(f(x)) \).

Equivalently, \( h = \dot{i}_T^{-1} \circ \dot{i}_f \). But since \( \dot{i}_f \) and \( \dot{i}_T \) are bijections, we have that \( h \) is also a bijection. From Lemma 2.4, \( \dot{i}_f \) and \( \dot{i}_T \) are order preserving maps and, so, the same holds for \( h \).

Let \( x \in \mathcal{C}(f) \). We shall show that \( h \circ f(x) \) and \( T \circ h(x) \) have the same itinerary with respect to \( T \). Indeed,

\[
\dot{i}_T(h(f(x))) = \dot{i}_f(f(x)) = \sigma(\dot{i}_f(x)).
\]

On the other hand,

\[
\dot{i}_T(T(h(x))) = \sigma(\dot{i}_T(h(x))) = \sigma(\dot{i}_f(x)).
\]

Since \( \dot{i}_T \) is an injection, we have that \( h \circ f(x) = T \circ h(x) \).

Since \( \mathcal{C}(f) \) and \( \mathcal{C}(T) \) are dense in \([0,1]\) (Proposition 2.6), \( h \) can extend to a homeomorphism \( \tilde{h} : [0,1] \to [0,1] \) such that \( \tilde{h} \circ f = T \circ \tilde{h} \).

3. Topological conjugacies for maps in \( \mathcal{M}_\infty \). In the previous sections, we had studied functions with a finite partition. Here we study a special class of functions with countable partition. Some modifications are necessary.

**Definition 3.1.** A map \( f : [0,1] \to [0,1] \) belongs to the class of functions \( \mathcal{M}_\infty \) if there exists a sequence of real numbers \( \{a_n\}_{n=0}^\infty \) with \( 0 = a_0 < a_1 < a_2 < \cdots \) and \( \lim_{n \to \infty} a_n = 1 \) such that:

(i) \( f \) is \( C^1 \) and monotone on each interval \([a_{i-1},a_i]\) of the partition.

(ii) For every \( i \in \mathbb{N}^* \), there exist unique \( p(i), q(i) \in \mathbb{N} \) such that \( f(a_{i-1},a_i) = (a_{p(i)},a_{q(i)}) \).

(iii) There exists \( \lambda > 1 \) such that \( |f'(x)| \geq \lambda \), for every \( x \neq a_i \).

In this case, \( \mathcal{C}(f) = \bigcup_{j=0}^\infty \bigcup_{i=0}^\infty f^{-i}(a_j) \).

**Definition 3.2.** Let \( f \in \mathcal{M}_\infty \) with partition \( 0 = a_0 < a_1 < a_2 < \cdots < 1 \). We define the set of sequences \( \Sigma = \{a : a = \{x_n\}_{n=0}^\infty \} \) with entries from \( \{1/2,1,3/2,\ldots\} \), which satisfy the following conditions:

(i) Let \( a = \{x_n\} \in \Sigma \). Then there exists an entry \( x_n \) of \( a \), of the form \( (2k+1)/2 \), where \( k = 0,1,\ldots \). Furthermore, if \( x_N \) is the first entry of \( a \) with this property, then \( \sigma^N(a) = \dot{i}_f(a_k) \).

(ii) If \( n < N - 1 \) and \( x_n = j \), then \( p(j) + 1 \leq x_{n+1} \leq q(j) \).

**Theorem 3.3.** Let \( f \in \mathcal{M}_\infty \) with partition \( 0 = a_0 < a_1 < a_2 < \cdots < 1 \). We consider the map \( T \in \mathcal{M}_\infty \) with partition \( 0 < 1/2 < 2/3 < 3/4 < \cdots < 1 \) which is linear in each interval \([i-1/i,i/i+1)\) and \( T((i-1)/i,i/(i+1)) = (p(i)/(p(i)+1),q(i)/(q(i)+1)) \). Furthermore, \( T \mid [a_{i-1},a_i] \) is of the same monotonicity type with \( f \mid [a_{i-1},a_i] \) and it is continuous, from the right or from the left at \( i/(i+1) \), when \( f \) is continuous, from the right or from the left at \( a_i \), respectively. Then \( f \) and \( T \) are topologically conjugate.

**Proof.** The proof of this theorem is the same as the proof of Theorem 2.7. \( \square \)
4. Computation of topological entropy for continuous Markov maps. Topological entropy is a measure of the dynamical complexity of a map and it is a topological invariant. There is an important theorem connecting topological entropy with the number $c_n$ of maximal intervals of monotonicity of the iterate $f^n$ (see [1, 4]).

**Theorem 4.1** (Misiurewicz-Szlenk). Let $f : I \to I$ be a continuous, piecewise monotone map. Then the topological entropy of $f$ is equal to the number

$$\lim_{n \to \infty} \frac{1}{n} \ln c_n. \quad (4.1)$$

As a corollary of the above theorem, if $f$ is a piecewise linear map with slope $\pm s$, then the topological entropy of $f$ is equal to $\max\{0, \ln s\}$.

Let $f$ be a continuous map in $\mathcal{M}$ and $T$ as in Theorem 2.7. The slope of $T$ is not necessarily constant. Observe that Theorem 2.7 still holds if we change the partition $0 < b_0 < 1/2 < b_1 < \cdots < (r-1)/r < 1$ with any other partition $0 = b_0 < b_1 < \cdots < b_r = 1$ of $[0,1]$. So, it is natural to ask the following question. Can we find a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ of $[0,1]$ such that $|b_{q(i)} - b_{p(i)}|/(b_i - b_{i-1})$ is constant?

To answer this question, to each $f \in \mathcal{M}$, we associate an $r \times r$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 0, & \text{if } (b_{i-1}, b_i) \cap f^{-1}(b_{j-1}, b_j) = \emptyset, \\ 1, & \text{if } (b_{i-1}, b_i) \cap f^{-1}(b_{j-1}, b_j) \neq \emptyset. \end{cases} \quad (4.2)$$

Observe that $A$ is nonnegative. According to the Perron-Frobenius theorem, there exists a unique nonnegative eigenvalue $s \geq 0$, which is maximal in absolute value among all the other eigenvalues and corresponding to a nonnegative eigenvector (see Gantmacher [5]).

**Proposition 4.2.** Assume that $f \in \mathcal{M}$ is a continuous map of order $r$, $A$ is the corresponding matrix, and $s$ is the "maximal" eigenvalue of $A$.

(a) If $s > 1$ and the corresponding eigenvector is positive, then the topological entropy of $f$ is $\ln s$.

(b) If $s \leq 1$ or at least one component of the corresponding eigenvector is zero, then the topological entropy of $f$ is zero.

**Proof.** (a) Assume that there exist a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ and a constant $s > 1$ such that $|T(b_{i-1}, b_i)| = s|(b_{i-1}, b_i)|$, for $i = 1,2,\ldots,r$. If we let $x_i = b_i - b_{i-1} > 0$, the above relation gives

$$x_{p(i)+1} + x_{p(i)+2} + \cdots + x_{q(i)} = s x_i, \quad i = 1,2,\ldots,r, \quad (4.3)$$

or, equivalently,

$$A x = s x, \quad \text{where } x = (x_1, \ldots, x_r)^T. \quad (4.4)$$

Thus, there exist a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ and a constant $s > 1$ such that $|T(b_{i-1}, b_i)| = s|(b_{i-1}, b_i)|$, for $i = 1,2,\ldots,r$, if and only if (a) holds.

(b) Assume on the contrary that $h(f) > 0$. Then $f$ is conjugate to a piecewise linear
map with constant slope [9]. It follows that there exist a partition $0 = b_0 < b_1 < \cdots < b_r = 1$ and a constant $s > 1$ such that $|T(b_{i-1}, b_i)| = s|((b_{i-1}, b_i)|$, for $i = 1, 2, \ldots, r$. This is equivalent to (a), which contradicts (b).

**Remark 4.3.** There is a similar result in [2]. The proof we give here is more simple and is based heavily on Theorem 2.7.

The above proposition gives a method to construct the partition $0 = b_0 < b_1 < \cdots < b_r = 1$, when we are in case (a). Assume that $(u_1, u_2, \ldots, u_r)^\tau$ is an eigenvector corresponding to the maximal eigenvalue. Then $b_0 = 0$ and

$$b_k = \frac{\sum_{i=1}^{k} u_i}{\sum_{i=1}^{r} u_i} \quad \text{for } k = 1, 2, \ldots, r. \tag{4.5}$$

Consider the map $f \in \mathcal{M}$ whose graph is shown in Figure 2.1. According to Theorem 2.7, $f$ is topologically conjugate with $T$ which is piecewise linear (the graph of $T$ is shown in Figure 4.1). The associated matrix to $f$ is

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}. \tag{4.6}$$

The maximal eigenvalue is $s = 2.8393$ and an eigenvector is

$$(0.6478, 0.4196, 0.7718, 1)^\tau. \tag{4.7}$$
Then from (4.5) we have $b_0 = 0, b_1 = 0.2282, b_2 = 0.3759, b_3 = 0.6478, b_4 = 1$. $f$ is topologically conjugate to $T'$ whose graph is shown in Figure 4.2. Since the slope of $T'$ is constant in absolute value we have that $h(f) = \ln s = 1.0435$.

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**REFERENCES**


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