

CERTAIN REMARKS ON A CLASS OF EVOLUTION QUASI-VARIATIONAL INEQUALITIES

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ABSTRACT. We prove two existence theorems, one for evolution quasi-variational inequalities and the other for a time-dependent quasi-variational inequality modeling the quasi-static problem of elastoplasticity with combined kinetic-isotropic hardening.

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1. Introduction. A comprehensive updated account of the elliptic quasi- and evolution quasi-variational inequalities is presented by Baiocchi and Capelo [1]. Evolution variational inequalities have been investigated by Lions and Stampacchia [9], Brézis [2, 3], Duvaut and Lions [4], and Lions [8]. A related general problem of Moreau's sweeping process can be found in [10]. Elliptic variational and quasi-variational inequalities are discussed in [1] along with the references on evolution variational inequalities. In 1994, Mosco [11] formulated the implicit Signorini problem in terms of the elliptic quasi-variational inequality problem and indicated many open problems including the numerical methods and the probable outlines of the method. Recently, Kočvara and Outrata [6] and Outrata and Zowe [12] have studied numerical solutions of quasi-variational inequalities using techniques of nonsmooth optimization. The evolution quasi-variational inequality modeling the evolving shape of a growing pile has been studied by Prigozhin [13]. Han, Reddy, and Schroeder [5] have investigated a new class of evolution inequalities where rate quantities occur in all of its terms. An inequality of this type represents, for example, the quasi-static evolution of an elastoplastic body where the stress law is of the linear kinematic or isotropic hardening type. Such a variational inequality may be useful in the modeling of financial derivatives and option pricing, see Wilmott et al. [14]. In this paper, we discuss the existence of solutions of the implicit evolution quasi-variational inequalities. In Section 2, we present an existence theorem which is closely related to an open problem mentioned in [13]. The evolution quasi-variational inequality problem of the Han-Reddy-Schroeder type is considered in Section 3.

2. Evolution quasi-variational inequalities of Bensoussan-Lions-Mosco type. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \| = (\langle \cdot, \cdot \rangle)^{1/2}$. Let $L_2(0, T; H)$ denote the space of all measurable functions $u : (0, T) \rightarrow H$, then it is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{L_2(0,T,H)} = \int_0^T \langle u(t), v(t) \rangle_H dt. \tag{2.1}$$

If H^* is the topological dual of H , then the dual of $L_2(0, T, H) = L_2(0, T, H^*)$ in case of real H .

Let $H^{1,p}(0, T, H)$ or, very often, $H^p(0, T, H)$ denote the space of functions $f \in L_p(0, T, H)$ such that their distributional derivatives Df also belong to $L_p(0, T, H)$. $H^p(0, T, H)$ equipped with the norm

$$\|f\|_{H^p(0,T,H)}^2 = \|f\|_{L_p(0,T,H)}^2 + \|Df\|_{L_p(0,T,H)}^2 \tag{2.2}$$

is a Banach space and $H^2(0, T, H)$ is a Hilbert space. Let $F \in H^2(0, T, H^*)$ and let C be a nonempty, closed, and convex subset of $H^2(0, T, H)$. Further, let $a(\cdot, \cdot)$ be a symmetric bounded and coercive bilinear form. We consider here the following problem.

BENSOUSSAN-LIONS-MOSCO PROBLEM. Find $u \in K(u) \cap C$ such that

$$\begin{aligned} \left\langle \frac{\partial u}{\partial t}, v - u(t) \right\rangle + a(u, v - u(t)) &\geq \langle F(t), v - u(t) \rangle \\ \forall v \in K(u), u(t) \in K, u(0) = u^0 &\in K(u^0). \end{aligned} \tag{2.3}$$

Here, $K(\cdot) : C \rightarrow 2^H$ is a set-valued mapping defined on C which associates a nonempty closed convex subset $K(z)$ of H with every $z \in C$; and $F(t) \in H^*$.

We also assume the following conditions.

The set-valued map $z \rightarrow K(z)$ is weakly continuous on C , in the sense that for every sequence $z_h \in C$ converging to some $z \in C$, weakly in H , the sequence of subsets $K(z_h)$ converges to $K(z)$ in H , equivalently to the following conditions:

(i) if $w_h \in K(z_h)$ and w_h converges weakly to some w in H , then

$$w \in K(z); \tag{2.4}$$

(ii) for every $v \in K(z)$, there exists $v_h \in K(z_h)$ such that v_h converges strongly to v in H .

We further assume that sets $K(z)$ for all $z \in C$ have a nonempty intersection, that is, $u^0 \in \bigcap_{h \in H} K(h)$ and S maps C into itself.

THEOREM 2.1. *Let $a(\cdot, \cdot)$ be a coercive continuous bilinear form. Then, under the above conditions, (2.3) has at least one solution.*

PROOF OF THEOREM 2.1. We can associate with problem (2.3) the following family of variational inequalities indexed by the fixed $z \in C$, namely $u \in K(z)$,

$$\begin{aligned} \left\langle \frac{\partial u}{\partial t}, v - u(t) \right\rangle + a(u, v - u(t)) &\geq \langle F(t), v - u(t) \rangle \\ \forall v \in K(z), u(t) \in K(z), u(0) = u^0 &\in K(z). \end{aligned} \tag{2.5}$$

Equation (2.5) can be written in the form:

for some $u \in K(z)$

$$\left\langle \frac{\partial u}{\partial t} + Au(t) - F(t), v - u(t) \right\rangle \geq 0, \tag{2.6}$$

where A is a coercive continuous linear operator induced by $a(\cdot, \cdot)$. By the well-known existence theorem of the evolution variational inequality (cf. [2, 3, 4, 8, 9]) for each fixed $z \in C$, (2.6) has a unique solution w , which we denote by $w = S(z)$. This defines the map

$$S : C \rightarrow H, \quad w = S(z). \tag{2.7}$$

It is clear that (2.7) is the selection map associated with (2.3) and the existence of its solution is equivalent to the fixed point problem

$$u \in C : S(u) = u. \tag{2.8}$$

The existence of a fixed point is guaranteed by the Schauder-Tychonov fixed point theorem if we show that $S(C)$ is bounded and S is a weakly continuous map with respect to the weak topology of H . By the hypothesis of the theorem, we find that every $w = S(z)$, $w \in C$, satisfies

$$\left\| \frac{\partial w}{\partial t} \right\| \geq \mu \|w^* - w\|^2, \tag{2.9}$$

where v in (2.6) is replaced by w^* . This shows that $S(C)$ is bounded.

We now prove that $S : C \rightarrow C$ is continuous for the weak topology of H . Let z_h be a sequence in C , converging weakly to z and let $w_h \in K(z_h)$. Since $\|w_h\|$ is bounded, there is a subsequence, say w_h , which converges weakly to some w . By (i), $w \in K(z)$. Furthermore, by (ii), for every $v \in K(z)$, there exists $v_h \in K(z_h)$ such that v_h converges to v strongly.

Replacing u by v_h in (2.6) and taking the limit as $h \rightarrow \infty$, we get

$$\left\langle \frac{\partial w}{\partial t} + Aw(t) - F(t), v - w(t) \right\rangle \geq 0. \tag{2.10}$$

Therefore $w = S(z)$. By the uniqueness of solution of (2.6), w_h converges weakly to w . By restricting S to the set $C^* = C \cap \{z : \|z\| \leq r\}$, $r \gg 0$, which is a weak compact subset of H , we see that S maps C^* into itself provided r is sufficiently large. This proves the existence of a fixed point of S and consequently a solution of (2.3). \square

REMARK 2.2. Very recently our attention has been drawn to a preprint of December 1997 by Kunze (University of Köln) and Monteiro Marques (University of Lisbon) entitled "On parabolic quasi-variational inequalities and state-dependent sweeping process" where a result similar to Theorem 2.1 has been proved by a different method. This paper has now appeared [7]. In a recent paper of Lions [8], it is observed in Remark 6.3 that one can extend the methods of the present paper to some quasi-variational inequalities.

It may be observed that the stationary case models the implicit Signorini problem

$$\begin{aligned} -\Delta u + u &= \varphi \quad \text{in } \Omega, \\ u - \left(h - \int_{\Gamma} \varphi \frac{\partial u}{\partial n} \right) &\geq 0, \quad \frac{\partial u}{\partial n} \geq 0, \\ \left[u - \left(h - \int_{\Gamma} \varphi \frac{\partial u}{\partial n} \right) \right] \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.11}$$

Many important physical phenomena can be interpreted in this form.

The number of solutions of (2.3) and its numerical analysis like error estimation, stability and convergence are open questions even in the stationary case. Mosco [11, page 29] has made certain suggestions in this connection. Kočvara and Outrata [6] and Outrata and Zowe [12] have studied the existence and numerical solutions of certain elliptic quasi-variational inequalities. Numerical solutions of (2.3), in some special cases, have been considered by Prigozhin [13].

3. Evolution quasi-variational inequality of Han-Reddy-Schroeder type. Han, Reddy, and Schroeder [5] have studied the following evolution variational inequality problem and its application to elastoplasticity:

Find $u : [0, T] \rightarrow H$, $u(0) = 0$ such that for almost all $t \in [0, T]$, $\dot{u}(t) \in K$ and

$$a(u(t), v - \dot{u}(t)) + j(\dot{u}(t)) \geq \langle F(t), v - \dot{u}(t) \rangle, \quad \forall v \in K. \quad (3.1)$$

We prove here an existence theorem for the following evolution quasi-variational inequality problem:

Find $u \in K(u) \cap C$, $u(0) = 0$ such that for almost all $t \in [0, T]$,

$$a(u(t), v - \dot{u}(t)) \geq \langle F(t), v - \dot{u}(t) \rangle, \quad \forall v \in K(u). \quad (3.2)$$

THEOREM 3.1. *Under the hypothesis of Theorem 2.1, inequality (3.2) has at least one solution.*

PROOF OF THEOREM 3.1. The problem of evolution quasi-variational inequality can be expressed as the following family of variational inequalities indexed by the vector $z \in C$:

$$w \in K(z), \quad a(w, v - \dot{w}(t)) \geq \langle F(t), v - \dot{w}(t) \rangle, \quad \text{for every } v \in K(z). \quad (3.3)$$

By [5, Theorem 4.3], (3.2) has a unique solution w , denoted by $w = S(z)$ which defines the map

$$S : C \rightarrow C, \quad w = S(z), \quad (3.4)$$

that is, $\dot{z}(t)$ is associated with every z and is itself mapped to $P_K \dot{z}(t) - \rho(Az - F(t))$, where A is the bounded linear operator induced by $a(\cdot, \cdot)$, namely

$$\langle Az, v \rangle = a(z, v). \quad (3.5)$$

By the arguments used in the proof of Theorem 2.1, it can be proved that S has a fixed point which proves the existence of a solution of (3.2). \square

It may be observed that the uniqueness, stability and finite element analysis of this class of evolution variational equalities are open problems.

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