A NOTE ON CENTRALIZERS

HOWARD E. BELL

(Received 3 January 2000)

ABSTRACT. For prime rings $R$, we characterize the set $U \cap C_R([U,U])$, where $U$ is a right ideal of $R$ and we apply our result to obtain a commutativity-or-finiteness theorem. We include extensions to semiprime rings.

Keywords and phrases. Prime rings, semiprime rings, centralizers.

2000 Mathematics Subject Classification. Primary 16N60, 16U80.

Let $R$ be an arbitrary ring with center $Z$. For $x, y \in R$, denote by $[x, y]$ the commutator $xy - yx$; and for an arbitrary nonempty subset $S$ of $R$, denote by $[S, S]$ the set $\{[x, y] \mid x, y \in S\}$. Denote by $C_R(S)$ the centralizer of $S$ in $R$—i.e., the set $\{x \in R \mid [x, s] = 0 \text{ for all } s \in S\}$.

It is proved in [2] that if $R$ is semiprime and $I$ is a nonzero ideal of $R$, then $C_R([I,I]) \subseteq C_R(I)$. It follows that $C([I,I]) \cap I \subseteq Z$, since in a semiprime ring $R$ the center of a nonzero right ideal is contained in the center of $R$. The first goal of this note is to study the subring $H = C_R([U,U]) \cap U$, where $R$ is prime or semiprime and $U$ is a nonzero right ideal. The information obtained is used to prove commutativity-or-finiteness results extending [1, Theorem 3].

1. Preliminaries. We shall use standard notation for annihilators—that is, for a nonempty subset $S$ of $R$, $A_I(S)$ and $A(S)$ will be the left and two-sided annihilators of $S$. A subring $S$ will be said to have finite index in $R$ if $(S, +)$ is of finite index in $(R, +)$. We shall use without explicit mention the commutator identities $[x y, z] = x[y, z] + [x, z] y$ and $[x, y z] = y[x, z] + [x, y] z$.

We begin with a revealing example.

**Example 1.1.** Let $F$ be an arbitrary field, let $R$ be the ring of $2 \times 2$ matrices over $F$, and let $U = e_{11} R$. Then $R$ is prime, $U$ is a right ideal, and $[U, U] = Fe_{12}$. Note that $C_R([U,U]) \cap U = Fe_{12} = A([U,U]) \cap U$, and note that this set does not centralize $U$. Thus, the result in [2] for two-sided ideals does not hold for one-sided ideals, even in the case of prime rings.

2. The case of $R$ prime

**Theorem 2.1.** Let $R$ be a prime ring, $U$ a right ideal of $R$, and $H = C_R([U,U]) \cap U$. Then either $H = U \cap Z$, or $H$ is a zero ring and $H = A([U,U]) \cap U$. In any case, $H$ is a commutative subring of $R$. 
Proof. We begin as in the proof of [2, Lemma 1]. Let \( z \in C_R([U, U]) \). Then for all \( x, y \in U \), \( z[x, y] = [x, y]z \); hence \( z[x, y] = x[z, y]z = xz[x, y] \) and therefore \( [z, x][x, y] = 0 \). Replacing \( y \) by \( yz \), we get \( [z, x]U[z, x] = \{0\} \) for all \( x \in U \); and since \( [z, x]U \) is a nilpotent right ideal, we have \( [z, x]U = \{0\} \) for all \( z \in C_R([U, U]) \) and \( x \in U \). Taking \( z \in H \), we obtain \( [z, x]z = 0 = z[z, x] \) for all \( z \in H \) and \( x \in U \); and replacing \( x \) by \( xy \) for arbitrary \( r \in R \) yields \( zU[z, r] = \{0\} \), hence

\[ zUR[z, r] = \{0\} \text{ for all } z \in H \text{ and } r \in R. \tag{2.1} \]

Since \( R \) is prime, (2.1) shows that either \( z \in Z \) or \( zU = \{0\} \); hence \( H = (H \cap Z) \cup (H \cap A_1(U)) \). Since the abelian group \( H \) cannot be the union of two proper subgroups, we have \( H = H \cap Z \) or \( H = H \cap A_1(U) \), so that \( H \subseteq Z \) or \( H \subseteq A_1(U) \). In the first case, \( H \) is clearly equal to \( U \cap Z \), so suppose \( H = A_1(U) \). Since \( H \subseteq U \), \( H^2 = \{0\} \); moreover, \( H \subseteq A_1((U, U)) \cap C_R([U, U]) \), so \( H = A((U, U)) \) and hence \( H = A((U, U)) \cap U \).

We now proceed to a commutativity-or-finiteness result.

**Theorem 2.2.** Let \( R \) be a prime ring and \( U \) a right ideal of finite index in \( R \). If \([U, U]\) is finite, then \( R \) is either finite or commutative.

**Proof.** Suppose that \([U, U] = \{x_1, x_2, \ldots, x_m\}\). For each \( i = 1, 2, \ldots, m \) define \( \Phi_i : U \to U \) by \( \Phi_i(x) = [x_1, x] \) for all \( x \in U \). Then \( \Phi_i(U) \) is finite, hence \( \ker \Phi_i \) is of finite index in \( U \). Letting \( H = \bigcap_{i=1}^m \ker \Phi_i \), we see that \( H = U \cap C_R([U, U]) \) and that \( H \) is of finite index in \( U \). Now \( U \) is of finite index in \( R \), so \( H \) is of finite index in \( R \). It follows by a theorem of Lewin [3] that \( H \) contains an ideal \( I \) of \( R \) which is also of finite index in \( R \). If \( I = \{0\} \), then \( R \) is finite; if \( I \neq \{0\} \), Theorem 2.1 implies that \( R \) has a nonzero commutative ideal and hence \( R \) is commutative.

**3. The case of \( R \) semiprime.** Let \( R \) be semiprime, \( U \) a right ideal, and \( H = U \cap C_R([U, U]) \). Let \( \{P_\alpha : \alpha \in \Lambda\} \) be a collection of prime ideals such that \( \cap P_\alpha = \{0\} \). Now (2.1) holds in \( R \), hence for each \( \alpha \in \Lambda \) and each \( z \in H \), either \( [z, R] \subseteq P_\alpha \) or \( zU \subseteq P_\alpha \). Since each of these conditions defines an additive subgroup of \( H \), we see that \( [H, R] \subseteq P_\alpha \) or \( HU \subseteq P_\alpha \); therefore \( [H, H] \subseteq P_\alpha \) for all \( \alpha \in \Lambda \). Thus \( [H, H] = \{0\} \)—that is, \( H \) is a commutative subring of \( R \).

Revisiting the proof of Theorem 2.2, we see that in the semiprime case, either \( R \) is finite or \( R \) contains a nonzero commutative ideal \( I \). But in a semiprime ring, a commutative ideal is central; hence we have the following extension of Theorem 2.2.

**Theorem 3.1.** Let \( R \) be a semiprime ring and \( U \) a right ideal of finite index in \( R \). If \([U, U]\) is finite, then either \( R \) is finite or \( R \) contains a nonzero central ideal.

**References**


HOWARD E. BELL: DEPARTMENT OF MATHEMATICS, BROCK UNIVERSITY, ST. CATHARINES, ONTARIO L2S 3A1, CANADA

_E-mail address: hbell@spartan.ac.brocku.ca_
Special Issue on
Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/jamds/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>June 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>September 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>December 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be

Hindawi Publishing Corporation
http://www.hindawi.com