FIRST EIGENVALUE OF SUBMANIFOLDS IN EUCLIDEAN SPACE

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(Received 23 April 1999)

ABSTRACT. We give some estimates of the first eigenvalue of the Laplacian for compact and non-compact submanifold immersed in the Euclidean space by using the square length of the second fundamental form of the submanifold merely. Then some spherical theorems and a nonimmersionsibility theorem of Chern and Kuiper type can be obtained.

Keywords and phrases. Laplacian, eigenvalue, submanifold.

2000 Mathematics Subject Classification. Primary 53C42.

1. Introduction. Let $M$ be an $n$-dimensional compact connected submanifold immersed in the Euclidean space $\mathbb{R}^{n+p}$. Denote by $\|\sigma\|^2$ and $\lambda_1$ the square length of the second fundamental form and the first eigenvalue of the Laplacian of $M$. It is well known that if $M$ is a standard hypersphere in the Euclidean space $\mathbb{R}^{n+1}$, then $\lambda_1 = n$. We find that $\|\sigma\|^2$ is equal to $n$ at the same time, i.e., $\lambda_1 = \|\sigma\|^2$. Inspiring the exterior rigidity of sphere, a natural problem appears: can you characterize those submanifolds immersed in $\mathbb{R}^{n+p}$ as $n$-sphere by $\lambda_1$ and $\|\sigma\|^2$?

The main goal of this paper is to give an affirmative answer for this question. In fact we can prove the following further result.

**Theorem 1.1.** Let $M$ be a compact submanifold immersed in the Euclidean space $\mathbb{R}^{n+p}$. Denote by $\|\sigma\|^2$ the square length of the second fundamental form and $\lambda_1$ the first eigenvalue of the Laplacian of $M$. Then $\lambda_1 \leq \max_M \|\sigma\|^2$. Furthermore, if $\lambda_1 \geq \|\sigma\|^2$ holds at any point of $M$, then $M$ is isometric to a sphere $S^n$.

According to Nash’s imbedded theorem, every Riemannian manifold can be isometrically imbedded in a Euclidean space of sufficiently large dimension. It is very significant to investigate the geometry of submanifold of the Euclidean space. For example, in the case of an $n$-dimensional compact hypersurface immersed in the sphere $S^{n+1}(c)$ with constant curvature $c$ in the Euclidean space $\mathbb{R}^{n+2}$, similar conclusion can be obtained immediately as follows.

**Theorem 1.2.** Let $M$ be a compact hypersurface immersed in the sphere $S^{n+1}(c)$. Denote by $\|\sigma\|^2$ the square length of the second fundamental form and $\lambda_1$ the first eigenvalue of the Laplacian of $M$. Then $\lambda_1 \leq nc + \max_M \|\sigma\|^2$. Furthermore, if $\lambda_1 \geq nc + \|\sigma\|^2$ holds at any point of $M$, then $\|\sigma\|^2 = 0$ and $M$ is isometric to a totally geodesic sphere $S^n(c)$.

In fact we set up a sharp estimate of the upper bound for the first eigenvalue of $M$ in $\mathbb{R}^{n+p}$ by using merely $\|\sigma\|^2$. A useful version of the lower bound for the Ricci
curvature of submanifold stated as a lemma will be given. The lemma can be applied
not only to the estimate of the first eigenvalue for both compact and non-compact
submanifolds in the Euclidean space $\mathbb{R}^{n+p}$, but in some propositions of the geometry
of submanifolds (see [2, 7]). As is well known, this type of theorems of compact
hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ was also proven by some authors such as
Reilly, Ros, and Deshmukh (see [4, 8, 9]). Deshmukh obtained similar results under
the condition that $M$ is a strictly convex hypersurface immersed in $\mathbb{R}^{n+1}$. We shall deal
with the more general case without the assumption of convexity of hypersurfaces.

As an application in the proof of Theorem 1.1, a new nonimmersibility theorem of
Chern and Kuiper type [3] can be obtained as follows.

**Theorem 1.3.** Let $M$ be an $n$-dimensional compact Riemannian manifold whose
Ricci curvature $\text{Ric}$ and scalar curvature $\text{R}$ satisfy $\text{Ric}(v,v) + R \geq 0$ and $R < n(n-1)\lambda^{-2}$
for each unit vector field $v$ and some constant $\lambda > 0$. Then no isometric immersion of
$M$ into the Euclidean space $\mathbb{R}^{n+1}$ is contained in a ball $B^{n+1}$ of radius $\lambda$.

Deshmukh and Al-Gwaiz [5] proved a similar result under the assumption that the
dimension of manifolds should be odd. Furthermore, when the dimension of $M$ is
odd say $2m-1$, the condition $R < 2(2m-1)(m-1)\lambda^{-2}$ in Theorem 1.3 is better than
$\text{Ric} < 2(m-1)\lambda^{-2}$ stated in [5].

2. Preliminaries. Let $M$ be a compact submanifold immersed in $\mathbb{R}^{n+p}$. Take a local
orthonormal frame field $\{e_1, \ldots, e_{n+p}\}$ in $\mathbb{R}^{n+p}$ around a point $p \in M$ such that when
restricting on $M$, $\{e_1, \ldots, e_n\}$ are tangent to $M$ and $\{e_{n+1}, \ldots, e_{n+p}\}$ are normal to $M$. Let
$\nabla, \bar{\nabla}$, and $\bar{\nabla}^{\perp}$ be the Riemannian connections on $\mathbb{R}^{n+p}$, $TM$, and $(TM)^{\perp}$, respectively.
The Gauss and Weingarten formulas are

$$
\bar{\nabla}_X Y = \nabla_X Y + B(X,Y), \quad \bar{\nabla}_X e_\alpha = -A_\alpha X + \bar{\nabla}_X e_\alpha,
$$

(2.1)

where $n + 1 \leq \alpha \leq n + p$, $X, Y$ are vector fields on $M$. Denote $H_\alpha$ the trace of the
Weingarten transformation $A_\alpha$, then the mean curvature of the immersion can be
written as

$$
H = \frac{1}{n} \sqrt{\sum_\alpha H_\alpha^2}.
$$

(2.2)

From the Gauss equation we have

$$
\text{Ric}(X,Y) = H_\alpha \langle A_\alpha(X), Y \rangle - \langle A_\alpha(X), A_\alpha(Y) \rangle,
$$

(2.3)

$$
\text{R} = n^2 H^2 - \|\sigma\|^2,
$$

(2.4)

where $\text{Ric}$ and $R$ are the Ricci curvature and the scalar curvature of $M$. We accept the
convention that the double indexes mean the summation.

Let $x: M \to \mathbb{R}^{n+p}$ be an isometric immersion. For $q \in M$, $x(q)$ also means the position
vector of $q$ with origin zero. The support function $\rho_\alpha: M \to R$ of the immersion
$x$ is given by

$$
\rho_\alpha(q) = \langle x, e_\alpha \rangle_q.
$$

(2.5)
We define $M \rightarrow R$ by $f = (1/2)\|x\|^2$ as Reilly did in [8]. Let us denote by $\nabla f$ the gradient of the function $f$. Then

\[ x = \nabla f + \rho_{\alpha}e_{\alpha}. \]  

**Proof of Theorem 1.1.** From the definition of the Riemannian curvature operator

\[ R(X, Y)Z = (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z), \]

we get

\[ R(e_i, e_j) \nabla f = (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}) \nabla f. \]  

Without loss of generality we suppose $\nabla_{e_i} e_j|_q = 0$ for $q \in M$. Hence

\[ \text{Ric}(\nabla f, \nabla f) = \langle (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}) \nabla f, e_i \rangle \langle \nabla f, e_j \rangle. \]  

Integrating both sides of (2.9) and using the divergence theorem, it follows that

\[ \int_M \langle \nabla_{e_i} \nabla f, e_i \rangle^2 - \|\nabla \nabla f\|^2 - \text{Ric}(\nabla f, \nabla f) = 0. \]  

We have at $q$,

\[ \nabla_X \nabla f = \langle \nabla_X x, e_j \rangle e_j + \langle x, \nabla_X e_j \rangle e_j = X + \langle x, B(X, e_j) \rangle e_j = X + \rho_{\alpha}A_{\alpha}(X). \]

Hence

\[ \Delta f = n + \rho_{\alpha}H_{\alpha}. \]  

Integrating both sides of (2.12) and using Stokes theorem, we get

\[ \int_M n + \rho_{\alpha}H_{\alpha} = 0. \]  

When $p = 1$ the expression becomes the classical Minkowski formula. It follows from (2.11) that

\[ \langle \nabla_{e_i} \nabla f, e_i \rangle^2 = n^2 + 2n\rho_{\alpha}H_{\alpha} + (\rho_{\alpha}H_{\alpha})^2, \quad \|\nabla \nabla f\|^2 = n + 2\rho_{\alpha}H_{\alpha} + |\rho_{\alpha}A_{\alpha}|^2. \]  

Substituting (2.14) in (2.10), we reach

\[ \int_M \rho_{\alpha}\rho_{\beta}(H_{\alpha}H_{\beta} - \langle A_{\alpha}A_{\beta} \rangle) - \text{Ric}(\nabla f, \nabla f) = n(n - 1) \text{Vol} M, \]

where Vol$M$ expresses the volume of $M$. We take the center of mass of $M$ as the origin zero of $\mathbb{R}^{n+p}$. Then $\int_M x = 0$. According to the max-minimum principle we get

\[ n \text{Vol} M = -\int_M \langle \Delta x, x \rangle \geq \lambda_1 \int_M \|x\|^2, \]  

where $\lambda_1$ is the first eigenvalue of the Laplace-Beltrami operator on $M$. This completes the proof of Theorem 1.1.
in which $\lambda_1$ is the first eigenvalue of $M$. Using an orthogonal transformation to \{e_{n+1}, \ldots, e_{n+p}\}, we can make the symmetric matrix $(H_\alpha H_\beta - \langle A_\alpha, A_\beta \rangle)$ to be diagonal at $q \in M$. Without loss of generality, we may assume that $(H_\alpha H_\beta - \langle A_\alpha, A_\beta \rangle)$ is diagonal at $q$. By using the Schwartz inequality it follows that

$$\sum_\alpha \rho_\alpha^2 (H_\alpha - A_\alpha)^2 \leq (n-1) \sum_\alpha \rho_\alpha^2 \| A_\alpha \|^2 \leq (n-1) \| \sigma \|^2 \sum_\alpha \rho_\alpha^2. \quad (2.17)$$

We get from (2.15), (2.16), and (2.17)

$$\lambda_1 \| x \|^2 \leq \int_M \| \sigma \|^2 \sum_\alpha \rho_\alpha^2 - \frac{1}{n-1} \text{Ric}(\nabla f, \nabla f). \quad (2.18)$$

It follows from the following lemma that

$$\text{Ric}(\nabla f, \nabla f) \geq -\frac{\sqrt{n-1}}{2} \| \sigma \|^2 \| \nabla f \|^2. \quad (2.19)$$

Therefore,

$$\lambda_1 \int_M \| x \|^2 \leq \int_M \| \sigma \|^2 \left( \sum_\alpha \rho_\alpha^2 + \frac{1}{2\sqrt{n-1}} \| \nabla f \|^2 \right). \quad (2.20)$$

Then we reach

$$\lambda_1 \leq \max_M \| \sigma \|^2. \quad (2.21)$$

If the equality in (2.21) holds, then the equalities in (2.17), (2.19), and (2.20) also appear. Hence $\nabla f = 0$, $\| \sigma \|^2 = \text{constant}$ and $M$ lies in a sphere $S^{n+p-1}$. From (2.17) we get $\sum_\alpha \rho_\alpha^2 \| A_\alpha \|^2 = \| \sigma \|^2 \sum_\alpha \rho_\alpha^2$, so it concludes that for some $\alpha$, say $\alpha = n+1$, $\| A_{n+1} \|^2 = \| \sigma \|^2$ and $\| A_{n+2} \|^2 = \cdots = \| A_{n+p} \|^2 = 0$. Then $M$ lies in a totally geodesic $S^{n+1}$. As $M$ is isometrically a closed submanifold in the Euclidean sphere, $M$ should be isometric to a sphere in $\mathbb{R}^{n+1}$ with radius $r = n/\| \sigma \|^2$. This ends the proof of Theorem 1.1. \[ \square \]

**Remark 2.1.** It is an interesting fact that one can find the upper bounds of the first eigenvalue for some kind of hypersurfaces by using Theorem 1.1. For example, as well known, the Clifford hypersurfaces $M_p \times M_q = S^p(1/(\sqrt{1+\lambda^2})) \times S^q(\lambda/(\sqrt{1+\lambda^2}))$, where integers $p+q = n$, are compact hypersurfaces in $S^{n+1}$ with constant $\| \sigma \|^2 = n + p\lambda^2 + q/\lambda^2$ (see [3]), then we have $\lambda_1(M_p \times M_q) \leq n + p\lambda^2 + q/\lambda^2$.

3. **Lemma and Corollaries Results.** We need the following lemma.

**Lemma 3.1.** Let $M$ be an $n$-dimensional submanifold immersed in a Riemannian manifold $N^{n+p}$. Denote by $\text{Ric}$ and $\| \sigma_N \|^2$ the functions on $M$ that assign to each point of $M$ the minimum Ricci curvature and the square length of the second fundamental form at the point, respectively. If all the sectional curvatures of $N^{n+p}$ are bounded below by $k$, then

$$\text{Ric} \geq (n-1)k - \frac{\sqrt{n-1}}{2} \| \sigma_N \|^2. \quad (3.1)$$
Proof. It is known from Cai and Leung (see [2, 7]) that
\[
\text{Ric} \geq \frac{n-1}{n} \left\{ nk + nH^2 - \| \varphi \|^2 - \frac{n-2}{\sqrt{n-1}} \sqrt{nH^2} \| \varphi \| \right\},
\]
where $H$ is the mean curvature of the immersion and $\| \varphi \|^2 = \| \sigma_N \|^2 - nH^2$ (see [1]). Let us consider the quadratic form with eigenvalues $\pm n/2 \sqrt{n-1}$:
\[
F(x, y) = x^2 - \frac{n}{2\sqrt{n-1}} xy - y^2.
\]
By using an orthogonal transformation, $F(x, y)$ can be written as
\[
F(x, y) = \frac{n}{2\sqrt{n-1}} (u^2 - v^2).
\]
Let $x = \sqrt{nH^2}$, $y = \| \varphi \|$ then $x^2 + y^2 = \| \sigma_N \|^2$. It follows from $x^2 + y^2 = u^2 + v^2$ that
\[
\text{Ric} \geq (n-1)k + \frac{\sqrt{n-1}}{2} (u^2 - V^2) \geq (n-1)k - \frac{\sqrt{n-1}}{2} \| \sigma_N \|^2.
\]
Thus we derived the conclusion.

In the case of complete non-compact submanifolds in $\mathbb{R}^{n+p}$, Gage (see [6]) proved that $\lambda_1 \leq -(n-1)/4 \text{Ric}$. Together with Lemma 3.1, we obtain the following corollary.

**Corollary 3.2.** Let $M$ be an $n$-dimensional complete non-compact submanifold immersed in $\mathbb{R}^{n+p}$. Then
\[
\lambda_1(M) \leq \frac{n-1}{8} \sqrt{n-1} \sup_M \| \sigma \|^2.
\]

Now we consider the case of $p = 1$ in which $M$ is a closed hypersurface immersed in $\mathbb{R}^{n+1}$. By using Lemma 3.1 and (2.4) in (2.15) we get
\[
(n-1)\lambda_1 \leq \int_M \left( \frac{(R \sigma^2 + (\sqrt{n-1}/2) \| \sigma \|^2 \| \nabla f \|^2)}{\sigma^2 + \| \nabla f \|^2} \right) \leq \max_M \left( R, \frac{\sqrt{n-1}}{2} \| \sigma \|^2 \right).
\]
Hence we obtain the following corollary.

**Corollary 3.3.** Let $M$ be a closed hypersurface immersed in $\mathbb{R}^{n+1}$. Then
\[
\lambda_1 \leq \max_M \left\{ \frac{R}{n-1}, \frac{1}{2\sqrt{n-1}} \| \sigma \|^2 \right\}
\]
and the equality holds if and only if $M$ is isometric to a sphere $S^n(r)$ with radius $r$.

As is well known, a hypersurface in $\mathbb{R}^{n+1}$ possessing the non-negative Ricci curvature implies that it is a convex hypersurface of $\mathbb{R}^{n+1}$. Thus we can easily get from (2.15) and (2.16) the following.

**Corollary 3.4.** Let $M$ be a closed convex hypersurface immersed in $\mathbb{R}^{n+1}$. If $R \leq (n-1)\lambda_1$ holds for all points of $M$. Then $M$ is isometric to a sphere $S^n(r)$.

**Remark 3.5.** Here we only need the condition $\text{Ric} \geq 0$ rather than $\text{Ric} > 0$ which was assumed by Deshmukh in [4]. We should point out that [4, Theorem 2] is very obvious by means of the expression $\Delta f = n(1 + \rho H)$ and the property of harmonic functions on the compact Riemannian manifold.
**Proof of Theorem 1.3.** Suppose that there exists an isometric immersion $x : M \to \mathbb{R}^{n+1}$ such that $x(M)$ is contained in a ball $B^{n+1}$ of $\mathbb{R}^{n+1}$ with radius $\lambda$. For $p = 1$ from (2.4), (2.15) becomes

$$
\int_M \rho^2 R - \text{Ric}(\nabla f, \nabla f) - n(n - 1) = 0. \quad (3.9)
$$

Now, we observe that the vector field $\nabla f$ is not identically zero on $M$. For if $\nabla f \equiv 0$, then $f = (1/2)\rho^2$ on $M$. We conclude that $M$ is a sphere with radius $r$. So $R = n(n - 1)\|x\|^2$, it contradicts the hypothesis $R < n(n - 1)\lambda^{-2}$. Then we can let $v = \nabla f / \|\nabla f\|$ is the unit position vector field defined on the open subset of $M$ where $\nabla f$ is non-zero. Using $\|x\|^2 = \|\nabla f\|^2 + \rho^2$ in the integral formula (3.9), we obtain

$$
\int_M \|\nabla f\|^2 (\text{Ric}(v, v) + R) + n(n - 1) - \|x\|^2 R = 0. \quad (3.10)
$$

From this hypothesis of the theorem it follows that $\text{Ric}(v, v) + S \geq 0$ and $\|x\|^2 R \leq \lambda^2 R < n(n - 1)$, we obtain a contradiction to (3.10). This ends the proof of Theorem 1.3.

**Acknowledgement.** The author is grateful to the School of Mathematics, University of Bristol for their hospitality during his visit in 1998. Projects were supported by the National Natural Science Foundation of China.

**References**


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