NONLINEAR VARIATIONAL EVOLUTION INEQUALITIES IN HILBERT SPACES

JIN-MUN JEONG, DOO-HOAN JEONG, and JONG-YEOUL PARK

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ABSTRACT. The regular problem for solutions of the nonlinear functional differential equations with a nonlinear hemicontinuous and coercive operator \( A \) and a nonlinear term \( f(\cdot,\cdot): x'(t) + Ax(t) + \partial \phi(x(t)) \ni f(t,x(t)) + h(t) \) is studied. The existence, uniqueness, and a variation of solutions of the equation are given.

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1. Introduction. Let \( H \) and \( V \) be two real separable Hilbert spaces such that \( V \) is a dense subspace of \( H \). Let the operator \( A \) be given a single-valued operator, which is hemicontinuous and coercive from \( V \) to \( V^* \). Here \( V^* \) stands for the dual space of \( V \). Let \( \phi: V \to (-\infty, +\infty] \) be a lower semicontinuous, proper convex function. Then the subdifferential operator \( \partial \phi: V \to V^* \) of \( \phi \) is defined by

\[
\partial \phi(x) = \{ x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), y \in V \},
\]

where \((\cdot, \cdot)\) denotes the duality pairing between \( V^* \) and \( V \). We are interested in the following nonlinear functional differential equation on \( H \):

\[
\frac{dx(t)}{dt} + Ax(t) + \partial \phi(x(t)) \ni f(t,x(t)) + h(t), \quad 0 < t \leq T, \\
x(0) = x_0,
\]

where the nonlinear mapping \( f \) is a Lipschitz continuous from \( \mathbb{R} \times V \) into \( H \). Equation (1.2) is caused by the following nonlinear variational inequality problem:

\[
\left( \frac{dx(t)}{dt} + Ax(t), x(t) - z \right) + \phi(x(t)) - \phi(z) \\
\leq (f(t,x(t)) + h(t), x(t) - z), \quad \text{a.e., } 0 < t \leq T, \quad z \in V,
\]

\[
x(0) = x_0.
\]

If \( A \) is a linear continuous symmetric operator from \( V \) into \( V^* \) and satisfies the coercive condition, then equation (1.2), which is called the linear parabolic variational inequality, is extensively studied in Barbu [5, Sec. 4.3.2] (also see [4, Sec. 4.3.1]). The existence of solutions for the semilinear equation with similar conditions for nonlinear term \( f \) have been dealt with in [1, 2, 6]. Using more general hypotheses for
nonlinear term \( f(\cdot, x) \), we intend to investigate the existence and the norm estimate of a solution of the above nonlinear equation on \( L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \), which is also applicable to optimal control problem. A typical example was given in the last section.

2. Perturbation of subdifferential operator. Let \( H \) and \( V \) be two real Hilbert spaces. Assume that \( V \) is a dense subspace in \( H \) and the injection of \( V \) into \( H \) is continuous. If \( H \) is identified with its dual space we may write \( V \subset H \subset V^* \) densely and the corresponding injections are continuous. The norm on \( V \) (respectively \( H \)) will be denoted by \( \| \cdot \| \) (respectively \( | \cdot | \)). The duality pairing between the element \( v_1 \) of \( V^* \) and the element \( v_2 \) of \( V \) is denoted by \( (v_1, v_2) \), which is the ordinary inner product in \( H \) if \( v_1, v_2 \in H \). For the sake of simplicity, we may consider

\[
\| u \| \leq | u | \leq \| u \|_*, \quad u \in V,
\]

where \( \| \cdot \|_* \) is the norm of the element of \( V^* \).

**Remark 2.1.** If an operator \( A_0 \) is bounded linear from \( V \) to \( V^* \) and generates an analytic semigroup, then it is easily seen that

\[
H = \left\{ x \in V^*: \int_0^T \| A_0 e^{tA_0} x \|_*^2 \, dt < \infty \right\} \quad \text{for the time } T > 0.
\]

Therefore, in terms of the intermediate theory we can see that

\[
(V, V^*)_{1/2,2} = H,
\]

where \( (V, V^*)_{1/2,2} \) denotes the real interpolation space between \( V \) and \( V^* \).

We note that nonlinear operator \( A \) is said to be hemicontinuous on \( V \) if

\[
\text{w-lim}_{t \to 0} A(x + ty) = Ax \quad \text{for every } x, y \in V,
\]

where “w-lim” indicates the weak convergence on \( V \). Let \( A: V \to V^* \) be given a single valued and hemicontinuous from \( V \) to \( V^* \) such that

\[
A(0) = 0, \quad (Au - Av, u - v) \geq \omega_1 \| u - v \|^2 - \omega_2 | u - v |^2,
\]

\[
\| Au \|_* \leq \omega_3 (\| u \| + 1)
\]

for every \( u, v \in V \), where \( \omega_2 \in \mathbb{R} \) and \( \omega_1, \omega_3 \) are some positive constants. Here, we note that if \( A(0) \neq 0 \) we need the following assumption:

\[
(Au, u) \geq \omega_1 \| u \|^2 - \omega_2 | u |^2 \quad \text{for every } u \in V.
\]

It is also known that \( A + \omega_2 I \) is maximal monotone and \( R(A + \omega_2 I) = V^* \) where \( R(A + \omega_2 I) \) is the range of \( A + \omega_2 I \) and \( I \) is the identity operator.

First, let us be concerned with the following perturbation of subdifferential operator:

\[
\frac{dx(t)}{dt} + Ax(t) + \partial \phi(x(t)) \ni h(t), \quad 0 < t \leq T, \quad x(0) = x_0.
\]

To prove the regularity for the nonlinear equation \((1.2)\) without the nonlinear term \( f(\cdot, x) \) we apply the method in [5, Sec. 4.3.2].
PROPOSITION 2.1. Let \( h \in L^2(0, T; V^*) \) and \( x_0 \in V \) satisfying that \( \phi(x_0) < \infty \). Then (2.7) has a unique solution
\[
x \in L^2(0, T; V) \cap C([0, T]; H)
\] (2.8)
which satisfies
\[
\|x\|_{L^2 \cap C} \leq C_1 \left( 1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)} \right),
\] (2.9)
where \( C_1 \) is a constant and \( L^2 \cap C = L^2(0, T; V) \cap C([0, T]; H) \).

PROOF. Substituting \( v(t) = e^{\omega_2 t} x(t) \) we can rewrite (2.7) as follows:
\[
\frac{d v(t)}{dt} + (A + \omega_2 I) v(t) + e^{-\omega_2 t} \partial \phi(v(t)) \ni e^{-\omega_2 t} h(t), \quad 0 < t \leq T,
\]
\[
v(0) = e^{\omega_2 t} x_0.
\] (2.10)
Then the regular problem for (2.7) is equivalent to that for (2.10). Consider the operator \( L : D(L) \subset H \rightarrow H \)
\[
L v = \{Av + e^{-\omega_2 t} \partial \phi(v) + \omega_2 v \} \cap H \quad \forall v \in D(L),
\]
\[
D(L) = \{v \in V; [Av + e^{-\omega_2 t} \partial \phi(v) + \omega_2 v] \cap H \neq \emptyset \}. \tag{2.11}
\]
Since \( A + \omega_2 I \) is a monotone, biconvex and bounded operator from \( V \) into \( V^* \) and \( e^{-\omega_2 t} \partial \phi \) is maximal monotone, we infer in [4, Cor. 1.1 of Ch. 2] that \( L \) is maximal monotone. Then in [5, Thm. 1.4] (also see [4, Thm. 2.3, Cor. 2.1]), for every \( x_0 \in D(L) \) and \( h \in W^{1,1}([0, T]; H) \), the Cauchy problem (2.10) has a unique solution \( v \in W^{1,\infty}([0, T]; H) \). Let us assume that \( x_0 \in D(L) \) and \( h \in W^{1,2}(0, T; H) \). Multiplying (2.7) by \( x - x_0 \) and using (2.5) and the maximal monotonicity of \( \partial \phi \) it holds
\[
\frac{1}{2} \frac{d}{dt} \|x(t) - x_0\|^2 + \omega_1 \|x(t) - x_0\|^2 \\
\leq \omega_2 \|x(t) - x_0\| + (h(t) - Ax_0 - \partial \phi(x_0), x(t) - x_0).
\] (2.12)
Since
\[
(h(t) - Ax_0 - \partial \phi(x_0), x(t) - x_0) \leq \|h(t) - Ax_0 - \partial \phi(x_0)\|_{e} \|x(t) - x_0\| \\
\leq \frac{1}{2c} \|h(t) - Ax_0 - \partial \phi(x_0)\|^2 + \frac{c}{2} \|x(t) - x_0\|^2
\] (2.13)
for every real number \( c \), so using Gronwall’s inequality, the inequality (2.12) implies that
\[
\|x\|_{L^2(0, T; V) \cap C([0, T]; H)} \leq C_1 \left( 1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)} \right).
\] (2.14)
for some positive constant \( C_1 \), that is,
\[
\|x\|_{L^2(0, T; V) \cap C([0, T]; H)} \leq C_1 \left( 1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)} \right).
\] (2.15)
Hence we have proved (2.9). Let \( x_0 \in V \) such that \( \partial \phi(x_0) < \infty \) and \( h \in L^2(0, T; V^*) \). Then there exist sequences \( \{x_{0n}\} \subset D(L) \) and \( \{h_n\} \subset W^{1,2}(0, T; H) \) such that \( x_{0n} \rightarrow x_0 \)
in $V$ and $h_n \to h$ in $L^2(0,T;V^*)$ as $n \to \infty$. Let $x_n \in W^{1,\infty}(0,T;H)$ be the solution of (2.7) with initial value $x_{0n}$ and with $h_n$ instead of $h$. Since $\partial \phi$ is monotone, we have

$$\frac{1}{2} \frac{d}{dt} \left| x_n(t) - x_m(t) \right|^2 + \omega_1 \left| x_n(t) - x_m(t) \right|^2$$

$$< (h_n(t) - h_m(t), x_n(t) - x_m(t)) + \omega_2 \left| x_n(t) - x_m(t) \right|^2$$

$$\leq \frac{1}{2c} \left| h_n(t) - h_m(t) \right|^2 + c \left| x_n(t) - x_m(t) \right|^2$$

$$+ \omega_2 \left| x_n(t) - x_m(t) \right|^2,$$

a.e., $t \in (0,T)$

for every real number $c$. Therefore, if we choose $\omega_1 - (c/2)$ then by integrating over $[0,T]$ and using Gronwall’s inequality it follows that

$$\left| x_n(t) - x_m(t) \right| + 2 \left( \omega_1 - \frac{c}{2} \right) \left| x_n(t) - x_m(t) \right|_{L^2(0,T;V)}$$

$$\leq e^{2\omega_2 T} \left( |x_{0n} - x_{0m}| + c^{-1} \left| h_n - h_m \right|_{L^2(0,T;V^*)} \right),$$

and hence, we have that $\lim_{n \to \infty} x_n(t) = x(t)$ exists in $H$. Furthermore, $x$ satisfies (2.7). Indeed, for all $0 \leq s < t \leq T$ and $y \in \partial \phi(x)$, multiplying (2.7) by $x(t) - x$ and integrating over $[s,t]$ we have

$$\frac{1}{2} \left( |x(t) - x|^2 - |x(s) - x|^2 \right) \leq \int_s^t (h(\tau) - Ax - y, x(\tau) - x) d\tau$$

$$+ \omega_2 \int_s^t |x(\tau) - x|^2 d\tau,$$

and, therefore,

$$\frac{1}{t-s} \left( x(t) - x(s), x(s) - x \right) \leq \frac{1}{t-s} \int_s^t (h(\tau) - Ax - y, x(\tau) - x) d\tau$$

$$+ \frac{\omega_2}{t-s} \int_s^t |x(\tau) - x|^2 d\tau.$$

This implies

$$\left( \frac{d}{dt} x(t), x(t) - x \right) \leq (h(t) - Ax - y + \omega_2 (x(t) - x), x(t) - x),$$

a.e., $t \in (0,T)$, that is,

$$\left( \frac{d}{dt} x(t) - h(t) - \omega_2 x(t) + (Ax + y + \omega_2 x), x(t) - x \right) \leq 0.$$
Then equation (2.7) has a unique solution
\[ x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \cap C([0, T]; H) \] (2.24)
which satisfies
\[ \|x\|_{L^2 \cap W^{1,2} \cap C} \leq C \left( 1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)} \right). \] (2.25)

**Proof.** From (2.7) and (2.23) it follows that
\[ \left\| \frac{d}{dt} x(t) \right\|_* + \omega_1 \|x(t)\| \leq \omega_2 \left| x(t) \right| + M \left| x(t) \right| + h(t) \|_{\ast}. \] (2.26)
Hence, by virtue of (2.15) we have that
\[ \|x\|_{W^{1,2}(0, T; H)} \leq C_2 \left( 1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)} \right). \] (2.27)

**Remark 2.2.** If \( V \) is compactly imbedded in \( H \), the imbedding \( L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset L^2(0, T; H) \) is compact in Aubin [3, Rem. 1, Thm. 2]. Hence, the mapping \( h \to x \) is compact from \( L^2(0, T; H) \) to \( L^2(0, T; H) \).

### 3. Nonlinear integrodifferential equation.
Let \( f : [0, T] \times V \to H \) be a nonlinear mapping satisfying the following variational evolution inequality:
\[ |f(t, x) - f(t, y)| \leq L \|x - y\|, \quad f(t, 0) = 0 \] (3.1)
for a positive constant \( L \).

**Theorem 3.1.** Let (2.5) and (3.1) be satisfied. Then (1.2) has a unique solution
\[ x \in L^2(0, T; V) \cap C([0, T]; H). \] (3.2)
Furthermore, there exists a constant \( C_2 \) such that
\[ \|x\|_{L^2 \cap C} \leq C_2 \left( 1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)} \right). \] (3.3)
If \( (x_0, h) \in V \times L^2(0, T; V^*) \), then \( x \in L^2(0, T; V) \cap C([0, T]; H) \) and the mapping
\[ V \times L^2(0, T; V^*) \ni (x_0, h) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H) \] (3.4)
is continuous.

**Proof.** Let \( y \in L^2(0, T; V) \). Then from (3.1), \( f(\cdot, y(\cdot)) \in L^2(0, T; H) \). Thus, by virtue of Proposition 2.1 we know that the problem
\[ \frac{dx(t)}{dt} + Ax(t) + \partial \phi(x(t)) \ni f(t, y(t)) + h(t), \quad 0 < t \leq T, \]
\[ x(0) = x_0 \] (3.5)
has a unique solution \( x_y \in L^2(0, T; V) \cap C([0, T]; H) \), where \( x_y \) is the solution of (3.5).
Let us choose a constant \( c > 0 \) such that
\[ \omega_1 - \frac{c}{2} > 0, \] (3.6)
and let us fix $T_0 > 0$ so that

$$(2c\omega_1 - c^2)^{-1}e^{2\omega_2 T_0}L < 1.$$  \hspace{1cm} (3.7)

Let $x_i, i = 1, 2,$ be the solution of (3.5) corresponding to $y_i.$ Then, by the monotonicity of $\partial \phi,$ it follows that

$$(\dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)) + (Ax_1(t) - Ax_2(t), x_1(t) - x_2(t))$$

$$\leq (f(t, y_1(t)) - f(t, y_2(t)), x_1(t) - x_2(t)), \hspace{1cm} (3.8)$$

and hence, using the assumption (2.5), we have that

$$\frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 + \omega_1 |x_1(t) - x_2(t)|^2$$

$$\leq \omega_2 |x_1(t) - x_2(t)|^2 + \|f(t, y_1(t)) - f(t, y_2(t))\|_s |x_1(t) - x_2(t)|.$$  \hspace{1cm} (3.9)

Since

$$\|f(t, y_1(t)) - f(t, y_2(t))\|_s |x_1(t) - x_2(t)|$$

$$\leq \frac{1}{2c} \|f(t, y_1(t)) - f(t, y_2(t))\|_s^2 + \frac{c}{2} |x_1(t) - x_2(t)|^2$$ \hspace{1cm} (3.10)

for every $c > 0$ and by integrating (3.9) over $(0, T_0)$ we have

$$|x_1(T_0) - x_2(T_0)|^2 + (2\omega_1 - c) \int_0^{T_0} \|x_1(t) - x_2(t)|^2 dt$$

$$\leq \frac{1}{c} \|f(t, y_1(t)) - f(t, y_2(t))\|_{L^2(0, T_0; V^*)}^2 + 2\omega_2 \int_0^{T_0} |x_1(t) - x_2(t)|^2 dt,$$ \hspace{1cm} (3.11)

and by Gronwall’s inequality,

$$\|x_1 - x_2\|_{L^2(0, T_0; V)} \leq (2c\omega_1 - c^2)^{-1}e^{2\omega_2 T_0}\|f(t, y_1) - f(t, y_2)\|_{L^2(0, T_0; V^*)}^2.$$ \hspace{1cm} (3.12)

Thus, from (3.1) it follows that

$$\|x_1 - x_2\|_{L^2} \leq (2c\omega_1 - c^2)^{-1}e^{2\omega_2 T_0}L\|y_1 - y_2\|_{L^2(0, T_0; V)}.$$ \hspace{1cm} (3.13)

Hence we have proved that $y - x$ is a strictly contraction from $L^2(0, T_0; V)$ to itself if condition (3.7) is satisfied. It shows that (1.2) has a unique solution in $[0, T_0].$

Let $y$ be the solution of

$$\frac{dy(t)}{dt} + Ay(t) + \partial \phi(y(t)) \equiv 0, \quad 0 < t \leq T_0, \quad y(0) = x_0.$$  \hspace{1cm} (3.14)

Then, since

$$\frac{d}{dt} (x(t) - y(t)) + (Ax(t) - Ay(t)) + (\partial \phi(x(t)) - \partial \phi(y(t))) \equiv f(t, x(t)) + h(t),$$ \hspace{1cm} (3.15)

multiplying by $x(t) - y(t)$ and using the monotonicity of $\partial \phi,$ we obtain

$$\frac{1}{2} \frac{d}{dt} |x(t) - y(t)|^2 + \omega_1 |x(t) - y(t)|^2$$

$$\leq \omega_2 |x(t) - y(t)|^2 + \|f(t, x(t)) + h(t)\|_s |x(t) - y(t)|.$$ \hspace{1cm} (3.16)
Therefore, putting
\[ N = (2c \omega_1 - c^2)^{-1} e^{2\omega_2 T_0}, \] (3.17)
from (3.1), it follows that
\[ \| x - y \|_{L^2(0,T_0;V)} \leq N \| f(\cdot,x) + h \|_{L^2(0,T_0;V^*)} \leq NL \| x \|_{L^2(0,T_0;V)} + N \| h \|_{L^2(0,T_0;V^*)}, \] (3.18)
and hence
\[ \| x \|_{L^2(0,T_0;V)} \leq \frac{1}{1 - NL} \| y \|_{L^2(0,T_0;V)} + N \| h \|_{L^2(0,T_0;V^*)} \leq C_1 (1 + \| x_0 \| + N \| h \|_{L^2(0,T_0;V^*)}) \] (3.19)
for some positive constant $C_2$. Since condition (3.7) is independent of the initial values, the solution of (1.2) can be extended to the interval $[0,nT_0]$ for natural number $n$, i.e., for the initial value $x(nT_0)$ in the interval $[nT_0,(n+1)T_0]$, as analogous estimate (3.19) holds for the solution in $[0,(n+1)T_0]$. Furthermore, similar to (2.12) and (2.15) in Section 2, the estimate (3.3) is easily obtained.

Now we prove the last result. If $(x_0,h) \in V \times L^2(0,T;V^*)$ then $x$ belongs to $L^2(0,T;V)$. Let $(x_0,i,h_i) \in V \times L^2(0,T;V^*)$ and $x_1$ be the solution of (1.2) with $(x_0,i,h_i)$ in place of $(x_0,u)$ for $i = 1,2$. Multiplying (1.2) by $x_1(t) - x_2(t)$, we have
\[ \frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 + \omega_1 \| x_1(t) - x_2(t) \|^2 \leq \omega_2 |x_1(t) - x_2(t)|^2 + \| f(t,x_1(t)) - f(t,x_2(t)) \|_V \| x_1(t) - x_2(t) \| \] (3.20)
\[ + \| h_1(t) - h_2(t) \|_V \| x_1(t) - x_2(t) \|. \]
If $\omega_1 - c/2 > 0$, we can choose a constant $c_1 > 0$ so that
\[ \omega_1 - \frac{c}{2} - \frac{c_1}{2} > 0, \] (3.21)
\[ \| h_1(t) - h_2(t) \|_V \| x_1(t) - x_2(t) \| \leq \frac{1}{2c_1} \| h_1(t) - h_2(t) \|_V^2 + \frac{c_1}{2} \| x_1(t) - x_2(t) \|^2. \]
Let $T_1 < T$ be such that
\[ 2\omega_1 - c - c_1 = c^{-1} e^{2\omega_2 T_1} L > 0. \] (3.22)
Integrating (3.20) over $[0,T_1]$, where $T_1 < T$ and as seen in the first part of the proof, it follows that
(2\omega_1 - c - c_1)||x_1 - x_2||^2_{L^2(0, T_0; V)} \\
\leq e^{2\omega_2 T_1} \left\{ ||x_0 - x_0|| + \frac{1}{c} ||f(t, t) - f(t, t)||^2_{L^2(0, T_0; V^*)} + \frac{1}{c_1} ||h_1 - h_2||_{L^2(0, T_0; V^*)} \right\} \\
\leq e^{2\omega_2 T_1} \left\{ ||x_0 - x_0|| + \frac{1}{c} L ||x_1 - x_2||^2_{L^2(0, T_0; V^*)} + \frac{1}{c_1} ||h_1 - h_2||_{L^2(0, T_0; V^*)} \right\}.  \\
(3.23)

Putting 
N_1 = 2\omega_1 - c - c_1 - c^{-1} e^{2\omega_2 T_1} L,  \\
(3.24)

we have 
||x_1 - x_2||_{L^2} \leq \frac{e^{2\omega_2 T_1}}{N_1} \left( ||x_0 - x_0|| + \frac{1}{c_1} ||h_1 - h_2|| \right).  \\
(3.25)

Suppose \((x_{0n}, h_n) \rightarrow (x_0, h)\) in \(V \times L^2(0, T; V^*)\), and let \(x_n\) and \(x\) be the solutions of (1.2) with \((x_{0n}, h_n)\) and \((x_0, h)\), respectively. Then, by virtue of (3.25) and (3.20), we see that \(x_n \rightarrow x\) in \(L^2(0, T_1, V) \cap C([0, T_1]; H)\). This implies that \(x_n(T_1) \rightarrow x(T_1)\) in \(V\). Therefore the same argument shows that \(x_n \rightarrow x\) in 

\[ L^2(T_1, \min\{2T_1, T\}; V) \cap C([T_1, \min\{2T_1, T\}]; H). \]  \\
(3.26)

Repeating this process, we conclude that \(x_n \rightarrow x\) in \(L^2(0, T; V) \cap C([0, T]; H)\).

If \(\partial \phi\) satisfies the growth condition (2.23) as is seen in Corollary 2.1, we can obtain the following result.

**Corollary 3.1.** Let (2.5), (3.1), and the growth condition (2.23) be satisfied. Then (1.2) has a unique solution 

\[ x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H). \]  \\
(3.27)

Furthermore, there exists a constant \(C_2\) such that 

\[ ||x||_{L^2(0, T; V) \cap W^{1,2}(0, T; V^*)} \leq C_2 \left( 1 + ||x_0|| + ||h||_{L^2(0, T; V^*)} \right). \]  \\
(3.28)

If \((x_0, h) \in V \times L^2(0, T; V^*)\), then \(x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*)\) and the mapping 

\[ V \times L^2(0, T; V^*) \ni (x_0, h) \mapsto x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \]  \\
(3.29)

is continuous.

**Example.** Let \(\Omega\) be a region in an \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) with boundary \(\partial \Omega\) and closure \(\overline{\Omega}\). For an integer \(m \geq 0\), \(C^m(\Omega)\) is the set of all \(m\)-times continuously differentiable functions in \(\Omega\), and \(C^m_0(\Omega)\) is its subspace consisting of functions with compact supports in \(\Omega\). If \(m \geq 0\) is an integer and \(1 \leq p \leq \infty\), \(W^{m, p}(\Omega)\) is the set of all functions \(f\) whose \(m\)-th derivative \(D^m f\) up to degree \(m\) in the distribution sense belong to \(L^p(\Omega)\). As usual, the norm of \(W^{m, p}(\Omega)\) is given by 

\[ ||f||_{m, p} = \left( \sum_{|\alpha| \leq m} ||D^\alpha f||_p \right)^{1/p} = \left\{ \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha f(x)|^p \, dx \right\}^{1/p}, \]  \\
(3.30)

where \(1 \leq p < \infty\) and \(D^0 f = f\). In particular, \(W^{0, p}(\Omega) = L^p(\Omega)\) with the norm \(|| \cdot \||_{0, p}^\circ\).
$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p^\prime = p/(p-1)$, $1 < p < \infty$, $W^{-m,p}(\Omega)$ stands the dual space $W_0^{m,p}(\Omega)$ of $W_0^{m,p}(\Omega)$ whose norm is denoted by $\| \cdot \|_{-m,p}$.

We take $V = W_0^{m,2}(\Omega)$, $H = L^2(\Omega)$ and $V^* = W^{-m,2}(\Omega)$ and consider a nonlinear differential operator of the form

$$Ax = \sum_{|\alpha| \leq m} (-D)^\alpha A_\alpha(u,x,...,D^m x),$$

(3.31)

where $A_\alpha(u,\xi)$ are real functions defined on $\Omega \times \mathbb{R}^N$ and satisfy the following conditions:

1. $A_\alpha$ are measurable in $u$ and continuous in $\xi$. There exists $k \in L^2(\Omega)$ and a positive constant $C$ such that

$$A_\alpha(u,0) = 0, \quad |A_\alpha(u,\xi)| \leq C(|\xi| + k(u)), \quad \text{a.e., } u \in \Omega,$$

(3.32)

where $\xi = (\xi_\alpha; |\alpha| \leq m)$.

2. For every $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N$ and for almost every $u \in \Omega$ the following condition holds:

$$\sum_{|\alpha| \leq m} (A_\alpha(u,\xi) - A_\alpha(u,\eta))(\xi_\alpha - \eta_\alpha) \geq \omega_1 ||\xi - \eta||_{m,2} - \omega_2 ||\xi - \eta||_{0,2},$$

(3.33)

where $\omega_2 \in \mathbb{R}$ and $\omega_1$ is a positive constant.

Let the sesquilinear form $a : V \times V \rightarrow \mathbb{R}$ be defined by

$$a(x,y) = \sum_{|\alpha| \leq m} \int_\Omega A_\alpha(u,x,...,D^m x)D^\alpha y \text{ } du,$$

(3.34)

Then by Lax-Milgram theorem we know that the associated operator $A : V \rightarrow V^*$, defined by

$$(Ax,y) = a(x,y), \quad x,y \in V,$$

(3.35)

is monotone and semicontinuous and satisfies conditions (2.5) in Section 2.

Let $g(t,u,x,p), p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t,r) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ and a real constant $1 \leq \gamma$ such that

$$g(t,u,0,0) = 0,$$

$$|g(t,u,x,p) - g(t,u,x,q)| \leq \rho(t,|x|)(1 + |p|^\gamma - |q|^\gamma)|p - q|,$$

$$|g(t,u,x,p) - g(t,u,y,p)| \leq \rho(t,|x| + |y|)(1 + |p|^\gamma)|x - y|.$$

(3.36)

Let

$$f(t,x)(u) = g(t,u,x,Dx,D^2 x,...,D^m x).$$

(3.37)

Then noting that

$$||f(t,x) - f(t,y)||_{0,2}^2 \leq 2 \int_\Omega |g(t,u,x,p) - g(t,u,x,q)|^2 \text{ } du$$

$$+ 2 \int_\Omega |g(t,u,x,q) - g(t,u,y,q)|^2 \text{ } du,$$

(3.38)
where \( p = (Dx, \ldots, D^m x) \) and \( q = (Dy, \ldots, D^m y) \), it follows from (3.36) that
\[
\left\| f(t, x) - f(t, y) \right\|^2_{0,2,2} \leq L \left( \|x\|_{m,2}, \|y\|_{m,2} \right) \|x - y\|_{m,2},
\]
(3.39)
where \( L(\|x\|_{m,2}, \|y\|_{m,2}) \) is a constant depending on \( \|x\|_{m,2} \) and \( \|y\|_{m,2} \).

Let \( \phi : V \to (-\infty, +\infty] \) be a lower semicontinuous, proper convex function. Then for \( x_0 \in W^m_0(\Omega) \) satisfying that \( \phi(x_0) < \infty \) and \( h \in L^2(0, T; W^{-m,2}(\Omega)) \), (1.2) is caused by the following nonlinear variational inequality problem:

\[
\left( \frac{dx(t)}{dt} + Ax(t), x(t) - z \right) + \phi(x(t)) - \phi(z) \\
\leq (f(t, x(t)) + h(t), x(t) - z), \quad a.e., 0 < t \leq T, \quad z \in W^m_0(\Omega),
\]
(3.40)
has a unique solution
\[
x \in L^2\left(0, T; W^m_0(\Omega) \right) \cap C\left([0, T]; L^2(\Omega) \right).
\]
(3.41)

References


Jeong: Division of Mathematical Sciences, Pukyong National University, Pusan 608-737, Korea
E-mail address: jmjeong@pknu.dolphin.ac.kr

Jeong: Dongeui Technical Junior College, Pusan 614-053, Korea

Park: Department of Mathematics, Pusan National University, Pusan 609-739, Korea
E-mail address: jyepark@hyowon.pusan.ac.kr
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Lead Guest Editor
Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor
Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie

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