FIXED POINT THEOREMS FOR GENERALIZED LIPSCHITZIAN SEMIGROUPS IN BANACH SPACES

BALWANT SINGH THAKUR and JONG SOO JUNG

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ABSTRACT. Fixed point theorems for generalized Lipschitzian semigroups are proved in \( p \)-uniformly convex Banach spaces and in uniformly convex Banach spaces. As applications, its corollaries are given in a Hilbert space, in \( L^p \) spaces, in Hardy space \( H^p \), and in Sobolev spaces \( H^{k,p} \), for \( 1 < p < \infty \) and \( k \geq 0 \).

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1. Introduction. Let \( K \) be a nonempty closed convex subset of a Banach space \( E \). A mapping \( T : K \rightarrow K \) is said to be Lipschitzian mapping if for each \( n \geq 1 \), there exists a positive real number \( k_n \) such that

\[
\|T^n x - T^n y\| \leq k_n \|x - y\| \quad (1)
\]

for all \( x, y \in K \). A Lipschitzian mapping is said to be nonexpansive if \( k_n = 1 \) for all \( n \geq 1 \), uniformly \( k \)-Lipschitzian if \( k_n = k \) for all \( n \geq 1 \), and asymptotically nonexpansive if \( \lim_n k_n = 1 \), respectively. These mappings were first studied by Geobel and Kirk [6] and Geobel, Kirk, and Thele [8]. Lifshitz [10] showed that in a Hilbert space \( H \), a uniformly \( k \)-Lipschitzian mapping \( T \) with \( k < \sqrt{2} \) has a fixed point. Downing and Ray [3] and Ishihara and Takahashi [9] verified that Lifshitz’s theorem is valid for uniformly Lipschitzian semigroup in Hilbert spaces.

Mizoguchi and Takahashi [14] introduced the notion of a submean on an appropriate space and, using a submean, they proved a fixed point theorem for uniformly Lipschitzian semigroup in a Hilbert space. Recently, Tan and Xu [21] generalized the result of Mizoguchi and Takahashi [14] to a Banach space setting and, also, proved a new fixed point theorem for uniformly \( k \)-Lipschitzian semigroup in a uniformly convex Banach space.

Now, we consider the following class of mappings, which we call generalized Lipschitzian mapping whose \( n \)th iterate \( T^n \) satisfies the following condition:

\[
\|T^n x - T^n y\| \leq a_n \|x - y\| + b_n (\|x - T^n x\| + \|y - T^n y\|) + c_n (\|x - T^n x\| + \|y - T^n y\|) \quad (2)
\]

for each \( x, y \in K \) and \( n \geq 1 \), where \( a_n, b_n, c_n \) are the nonnegative constants such that there exists an integer \( n_0 \) such that \( b_n + c_n < 1 \) for all \( n \geq n_0 \).
This class of generalized Lipschitzian mappings are more general than nonexpansive, asymptotically nonexpansive, Lipschitzian, and uniformly k-Lipschitzian mappings and it can be seen by taking \( b_n = c_n = 0 \).

In this paper, we prove some fixed point theorems for generalized Lipschitzian semigroups in \( p \)-uniformly convex Banach spaces and in uniformly convex Banach spaces. Next, we give its corollaries in a Hilbert space, in \( L^p \) spaces, in Hardy space \( H^p \), and in Sobolev spaces \( H^{k,p} \), for \( 1 < p < \infty \) and \( k \geq 0 \). Our results improve and extend results from \([9, 14, 21, 22]\).

2. Preliminaries. Let \( p > 1 \) and denote by \( \lambda \) the number in \([0, 1]\) and by \( w_p(\lambda) \) the function \( \lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda) \). The functional \( \| \cdot \|^p \) is said to be uniformly convex (cf. Zalinescu [24]) on the Banach space \( E \) if there exists a positive constant \( c_p \) such that, for all \( \lambda \in [0, 1] \) and \( x, y \in E \), the following inequality holds:

\[
\| \lambda x + (1 - \lambda) y \|^p \leq \lambda \| x \|^p + (1 - \lambda) \| y \|^p - w_p(\lambda) \cdot c_p \cdot \| x - y \|^p. \tag{3}
\]

Xu [23] proved that the functional \( \| \cdot \|^p \) is uniformly convex on the whole Banach space \( E \) if and only if \( E \) is \( p \)-uniformly convex, i.e., there exists a constant \( c_p > 0 \) such that the modulus of convexity (see [7]) \( \delta_E(\epsilon) \geq c_p \cdot \epsilon^p \) all \( 0 \leq \epsilon \leq 2 \).

Let \( G \) be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that, for each \( a \in G \), the mapping \( t \rightarrow a \cdot t \) and \( t \rightarrow t \cdot a \) from \( G \) onto itself are continuous. A semitopological semigroup \( G \) is left reversible if any two closed right ideals of \( G \) have nonempty intersection. In this case, \((G, \leq)\) is a directed system when the binary relation "\( \leq \)" on \( G \) is defined by \( a \leq b \) if and only if \( \{ a \} \cup \overline{aG} \supseteq \{ b \} \cup \overline{bG} \), where \( \overline{D} \) is the closure of set \( D \). Examples of left reversible semigroups include commutative and all left amenable semigroups.

Let \( m(G) \) be the Banach space of bounded real valued functions on \( G \) with the supremum norm. Suppose \( X \) is a subspace of \( m(G) \) containing constants. Following Mizoguchi and Takahashi [14], we say that a real valued function \( \mu \) on \( X \) is a submean on \( X \) if the following conditions are satisfied:

- (i) \( \mu(f + g) \leq \mu(f) + \mu(g) \) for all \( f, g \in X \);
- (ii) \( \mu(\alpha f) = \alpha \mu(f) \) for all \( f \in X \) and \( \alpha \geq 0 \);
- (iii) if \( f, g \in X \) with \( f \leq g \), then \( \mu(f) \leq \mu(g) \); and
- (iv) \( \mu(c) = c \) for every constant \( c \).

If \( \mu \) is a submean on \( X \) and \( f \in X \), then we denote by either \( \mu(f) \) or \( \mu_t(f(t)) \), according to time and circumstances, the value of \( \mu \) at \( f \). For \( a \in G \) and \( f \in m(G) \), we define \( (l_a f)(t) = f(at) \) and \( (r_a f)(t) = f(ta) \) for all \( t \in G \). Let \( X \) be a subspace of \( m(G) \) containing constants which is \( l_G \)-invariant, i.e., \( l_a(X) \subseteq X \) for all \( a \in G \). Then a submean \( \mu \) on \( X \) is said to be left invariant if \( \mu(f) = \mu(l_a f) \) for every \( a \in G \) and \( f \in X \). A right invariant submean is defined similarly. A submean is called invariant if it is left and right invariant. Let \( K \) be a closed convex subset of a Banach space \( E \). Then a collection \( \mathcal{F} = \{ T_s : s \in G \} \) of mappings of \( K \) into itself is said to be a generalized Lipschitzian semigroup on \( K \) if the following conditions are satisfied:

- (i) \( T_s t x = T_s T_t x \) for all \( s, t \in G \) and \( x \in K \);
- (ii) for each \( x \in K \), the mapping \( t \rightarrow T_t x \) from \( G \) into \( K \) is continuous; and
(iii) for each \( s \in G \)

\[
\|T_s x - T_s y\| \leq a_s \|x - y\| + b_s (\|x - T_s x\| + \|y - T_s y\|) + c_s (\|x - T_s y\| + \|y - T_s x\|),
\]

(4)

for \( x, y \in K \), where \( a_s, b_s, c_s > 0 \) such that there exists a \( t_1 \in G \) such that \( b_s + c_s < 1 \) for all \( s \geq t_1 \).

The following lemma is needed to prove the main result:

**Lemma 1** [22, Lem. 2.1]. Let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty closed convex subset of \( E \), and \( \{x_t : t \in G\} \) a bounded family of elements of \( E \). Also, suppose that for every \( x \) in \( K \), the function \( f \) on \( G \), defined by

\[
f(t) = \|x_t - x\|^p, \quad t \in G,
\]

(5)

belongs to \( X \). Set

\[
r(x) = \mu_t \|x_t - x\|^p, \quad x \in K
\]

(6)

and

\[
r = \inf \{r(x) : x \in K\}.
\]

(7)

Then there exists a unique point \( z \) in \( K \) such that

\[
r + c_p \|z - x\|^p \leq r(x)
\]

(8)

for all \( x \) in \( K \), where \( c_p \) is the constant appearing in (3).

**3. Main results.** Now, we prove the first result of this paper.

**Theorem 1.** Let \( K \) be a nonempty closed convex subset of a \( p \)-uniformly convex Banach space \( E \), \( X \) an \( l_G \)-invariant subspace of \( m(G) \) containing constants which has left invariant submean \( \mu \), and \( \mathcal{S} = \{T_s : s \in G\} \) a generalized Lipschitzian semigroup on \( K \). Suppose that there exists an \( x_0 \) in \( K \) such that \( \{T_s x_0 : x \in G\} \) is bounded and that, for every \( u, v \in K \), the function \( f \) on \( G \) defined by

\[
f(t) = \|T_t u - v\|^p, \quad t \in G,
\]

(9)

and the function \( g \) on \( G \) defined by

\[
g(t) = 2^{p-1}(\alpha_t^p + \beta_t^p), \quad t \in G
\]

(10)

belong to \( X \). Then, if \( 2^{p-1}\{\mu_t(\alpha_t^p + \beta_t^p)\} < 1 + c_p \), where \( \alpha_t = (a_t + b_t + c_t)/(1 - b_t - c_t) \), \( \beta_t = (2b_t + 2c_t)/(1 - b_t - c_t) \), and \( c_p \) is the constant appearing in (3), there exists a \( z \in K \) such that \( T_s z = z \) for all \( s \in G \).
PROOF. Since \(\{T_s x_0 : s \in G\}\) is bounded, it follows that \(\{T_s x : s \in G\}\) is bounded for every \(x \in K\). By Lemma 1, we inductively construct a sequence \(\{x_n\}_{n=1}^{\infty}\) in \(K\) in the following manner:

\[
\mu_t||T_t x_{n-1} - x_n||^p = \min_{y \in K} \mu_t||T_t x_{n-1} - y||^p
\]

(11) for \(n = 1, 2, \ldots\). It follows from Lemma 1 that

\[
c_p||x_n - y||^p \leq \mu_t||T_t x_{n-1} - y||^p - \mu_t||T_t x_{n-1} - x_n||^p
\]

(12) for all \(y \in K\) and \(n \geq 1\). Since \(T\) is generalized Lipschitzian, we get, after a simple calculation,

\[
||T_s x - T_s y|| \leq \alpha_s||x - y|| + \beta_s||y - T_s y||
\]

(13) for each \(x, y \in K\) and \(s \in G\), where \(\alpha_s = (a_s + b_s + c_s)/(1 - b_s - c_s)\) and \(\beta_s = (2b_s + 2c_s)/(1 - b_s - c_s)\). By putting \(y = T_s x_n\) into (12), we have

\[
c_p||x_n - T_s x_n||^p \leq \mu_t||T_t x_{n-1} - T_s x_n||^p - \mu_t||T_t x_{n-1} - x_n||^p
\]

\[
= \mu_t||T_t x_{n-1} - T_s x_n||^p - \mu_t||T_t x_{n-1} - x_n||^p
\]

\[
= \mu_t||T_t x_{n-1} - T_s x_n||^p - \mu_t||T_t x_{n-1} - x_n||^p
\]

\[
\leq \mu_t [\alpha_s||T_s x_{n-1} - x_n|| + \beta_s||x_n - T_s x_n||]^p - \mu_t||T_t x_{n-1} - x_n||^p
\]

(14) or

\[
(c_p - 2p^{-1} \beta_s^p)||x_n - T_s x_n||^p \leq (2p^{-1} \alpha_s^p - 1) \cdot \mu_t||T_t x_{n-1} - x_n||^p.
\]

(15) Therefore, we have

\[
\mu_t||x_n - T_s x_n||^p \leq A \cdot \mu_t||T_t x_{n-1} - x_n||^p
\]

(16) where \(A = (2p^{-1} \alpha_s^p - 1)/(c_p - 2p^{-1} \beta_s^p) < 1\) by the assumption of the theorem. Since

\[
\mu_t||T_t x_{n-1} - x_n||^p \leq \mu_t||T_t x_{n-1} - x_{n-1}||^p
\]

(17) by (11), it follows from (13) that

\[
\mu_t||T_t x_{n-1} - x_n||^p \leq A \cdot \mu_t||T_t x_{n-1} - x_{n-1}||^p
\]

\[
\leq A^n \mu_t||T_t x_0 - x_0||^p.
\]

(18) Noticing that

\[
||x_n - x_{n-1}||^p \leq 2p^{-1} \left(||x_n - T_t x_{n-1}||^p + ||T_t x_{n-1} - x_{n-1}||^p\right),
\]

(19) we get

\[
||x_n - x_{n-1}||^p \leq 2p^{-1} \left(\mu_t||x_n - T_t x_{n-1}||^p + \mu_t||T_t x_{n-1} - x_{n-1}||^p\right)
\]

\[
\leq 2p \mu_t||T_t x_{n-1} - x_{n-1}||^p
\]

\[
\leq 2p A^{n-1} \mu_t||T_t x_0 - x_0||^p,
\]

(20)
which shows that \( \{x_n\} \) is a Cauchy sequence and, hence, convergent. Let \( z = \lim_{n \to \infty} x_n \).

Then, for each \( s \in G \), we have
\[
\|z - T_s z\|^p \leq \left( \|z - x_n\| + \|x_n - T_s x_n\| + \|T_s x_n - T_s z\| \right)^p
\]
\[
\leq \left( (1 + \alpha_s) \|z - x_n\| + (1 + \beta_s) \|x_n - T_s x_n\| \right)^p
\]
\[
\leq 2^{p-1} \left[ (1 + \alpha_s)^p \|z - x_n\| + (1 + \beta_s)^p \cdot A \cdot \mu_t \|x_n - T_s x_n\|^p \right]
\]
\[
\to 0 \quad \text{as} \quad n \to \infty.
\] (21)

Therefore, \( T_s z = z \) for all \( s \in G \) and the proof is complete.

Let \( E \) be a Banach space, \( K \) a nonempty closed convex subset of \( E \), and \( G \) an unbounded subset of \( [0, \infty) \) such that
\[
t + h \in G \quad \text{for all} \quad t, h \in G
\] (22)

and
\[
t - h \in G \quad \text{for all} \quad t, h \in G \quad \text{with} \quad t > h
\] (23)

(e.g., \( G = [0, \infty) \) or \( G = N \), the set of nonnegative integers). Suppose \( \mathcal{G} = \{T_s : s \in G\} \) is a generalized uniformly Lipschitzian semigroup on \( K \), i.e., a family of self-mappings of \( K \) satisfying the conditions:

(i) \( T_{s+h} x = T_s T_h x \) for all \( s, h \in G \) and \( x \in K \);

(ii) for each \( x \in K \), the mappings \( s \to T_s x \) from \( G \) onto \( K \) is continuous when \( G \) has the relative topology of \( [0, \infty) \); and

(iii)
\[
\|T_s x - T_s y\| \leq a \|x - y\| + b \left( \|x - T_s x\| + \|y - T_s y\| \right) + c \left( \|x - T_s y\| + \|y - T_s x\| \right)
\] (24)

for all \( x, y \in K \) and \( s \in G \), where \( a, b, c \) are nonnegative constants such that \( b + c < 1 \).

For the rest of this paper, \( \lim_t \) and \( \overline{\lim}_t \) always stand for \( \lim_{t \to \infty, t \in G} \) and \( \overline{\lim}_{t \to \infty, t \in G} \) respectively.

The normal structure coefficient \( N(E) \) of \( E \) is defined (cf. [2]) by
\[
N(E) = \inf \left\{ \frac{\text{diam} K}{r_K(K)} : K \text{ is a bounded convex subset of } E \text{ consisting of more than one point} \right\},
\] (25)

where \( \text{diam} K = \sup \{\|x - y\| : x, y \in K\} \) is the diameter of \( K \) and \( r_K(K) = \inf_{x \in K} \{\sup_{y \in K} \|x - y\|\} \) is the Chebyshev radius of \( K \) relative to itself. \( E \) is said to have uniformly normal structure if \( N(E) > 1 \). It is known that a uniformly convex Banach space has the uniformly normal structure and for a Hilbert space \( H \), \( N(H) = \sqrt{2} \).

Recently, Pichugov [15] (cf. Prus [17]) showed that
\[
N(L^p) = \min \{2^{1/p}, 2^{(p-1)/p}\}, \quad 1 < p < \infty.
\] (26)

Some estimate for normal structure coefficient in other Banach spaces may be found in [18].
Suppose $E$ is a uniformly convex Banach space. Then it is easily seen that the equation
\[ \xi^2 \delta_E^{-1} \left( 1 - \frac{1}{\xi} \right) \tilde{N}(E) = 1 \] (27)
has a unique solution $\xi > 1$, where $\tilde{N}(E) = N(E)^{-1}$.

Now, recall the definition of an asymptotic center. Let $K$ be a nonempty closed convex subset of a Banach space $E$ and $\{x_t : t \in G\}$ be a bounded family of elements of $E$. Then the asymptotic radius and asymptotic center of $\{x_t : t \in G\}$ with respect to $K$ are the number
\[ r_K(\{x_t\}) = \inf_{y \in K} \lim_{t} \|x_t - y\| \] (28)
and the (possibly empty) set
\[ A_K(\{x_t\}) = \{ y \in K : \lim_{t} \|x_t - y\| = r_K(\{x_t\}) \}, \] (29)
respectively. It is easy to see that if $E$ is reflexive, then $A_K(\{x_t\})$ is nonempty bounded closed and convex and if $E$ is uniformly convex, then $A_K(\{x_t\})$ consists of a single point.

We need the following lemma to prove our next theorem.

**Lemma 2** [22, Lem. 3.4]. Let $E$ be a Banach space with uniformly normal structure. Then for every bounded family $\{x_t : t \in G\}$ of elements of $E$, there exists $y$ in $\overline{co}(\{x_t : t \in G\})$ such that
\[ \lim_{t} \|x_t - y\| \leq \tilde{N}(E) A(\{x_t\}), \] (30)
where $\overline{co}(D)$ is the closure of the convex hull of $D \subseteq E$ and
\[ A(\{x_t\}) = \lim_{t} \left( \sup \{ \|x_i - x_j\| : t \leq i, j \in G\} \right) \] (31)
is the asymptotic diameter of $\{x_t\}$.

Now, we are in position to prove our next theorem.

**Theorem 2.** Let $E$ be a uniformly convex Banach space, $K$ a nonempty closed convex subset of $E$, and $\mathcal{F} = \{T_s : s \in G\}$ a generalized uniformly Lipschitzian semigroup on $K$ with $(\alpha + \beta) < \xi$, where $\xi > 1$ is the unique solution of (27), $\alpha = (a + b + c)/(1 - b - c)$ and $\beta = (2b + 2c)/(1 - b - c)$. Suppose there is an $x_0$ in $K$ such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists $z$ in $K$ such that $T_s z = z$ for all $s$ in $G$.

**Proof.** By induction, we define a sequence $\{x_n\}_0^\infty$ in $K$ in the following manner:
\[ x_{n+1} = A_K(\{T_s x_n : t \in G\}) \] (32)
for $n = 0, 1, \ldots$, i.e., $x_{n+1}$ is the unique point in $K$ such that
\[ \lim_{t} \|T_t x_n - x_{n+1}\| = \inf_{y \in K} \lim_{t} \|T_t x_n - y\|. \] (33)
Write \( r_n = r_K(\{T_tx_n\}) \). Then by Lemma 2, we have
\[
\begin{align*}
  r_n &= \lim_t \|T_t x_n - x_{n-1}\| \\
  &\leq \tilde{N}(E) \cdot A\left(\big\{T_t x_n\big\}_{t \in G}\right) \\
  &= \tilde{N}(E) \lim_t \left\{ \sup \{\|T_i x_n - T_j x_n\| : t \leq i, j \in G\} \right\} \\
  &\leq \tilde{N}(E)(\alpha + \beta) \cdot d(x_n),
\end{align*}
\]
that is,
\[
  r_n \leq (\alpha + \beta) \cdot \tilde{N}(E) d(x_n),
\]
where \( d(x_n) = \sup \{\|T_t x_n - x_n\| : t \in G\} \). We may assume that \( d(x_n) > 0 \) for all \( n \geq 0 \). Let \( n \geq 0 \) be fixed and let \( \epsilon > 0 \) be small enough. First, choose \( j \in G \) such that
\[
\|T_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \epsilon
\]
and then choose \( s_0 \) in \( G \) so large that
\[
\|T_s x_n - x_{n+1}\| < r_n + \epsilon
\]
and
\[
\|T_s x_n - T_j x_{n+1}\| \leq \alpha \|T_{s-j} x_n - x_{n+1}\| + \beta \|T_j x_n - x_{n}\| \leq (\alpha + \beta)(r_n + \epsilon)
\]
for all \( s \geq s_0 \). It, then, follows that
\[
\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \leq (\alpha + \beta)(r_n + \epsilon) \left(1 - \delta_E \left(\frac{d(x_{n+1}) - \epsilon}{(\alpha + \beta)(r_n + \epsilon)}\right)\right)
\]
for \( s \geq s_0 \) and, hence,
\[
\begin{align*}
  r_n &\leq \lim_s \|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \\
  &\leq (\alpha + \beta)(r_n + \epsilon) \left(1 - \delta_E \left(\frac{d(x_{n+1}) - \epsilon}{(\alpha + \beta)(r_n + \epsilon)}\right)\right).
\end{align*}
\]
Taking the limit as \( \epsilon \to 0 \), we get
\[
  r_n \leq (\alpha + \beta) \cdot r_n \left(1 - \delta_E \left(\frac{d(x_{n+1})}{(\alpha + \beta)r_n}\right)\right)
\]
which together with (35) leads to the conclusion
\[
  d(x_{n+1}) \leq (\alpha + \beta)^2 \tilde{N}(E) \delta_E^{-1} \left(1 - \frac{1}{(\alpha + \beta)}\right) d(x_n).
\]
Hence,
\[
  d(x_n) \leq Ad(x_{n-1}) \leq A^n d(x_0),
\]
where \( A = (\alpha + \beta)^2 \tilde{N}(E) \delta_F^{-1} (1 - (1/(\alpha + \beta))) < 1 \) by assumption. Noticing that

\[
\|x_{n+1} - x_n\| \leq \lim_t \|T_t x_n - x_{n+1}\| + \lim_t \|T_t x_n - x_n\| \leq r_n + d(x_n) \leq 2d(x_n),
\]

we see from (43) that \( \{x_n\} \) is a Cauchy sequence and, hence, strongly convergent. Let \( z = \lim_{n \to \infty} x_n \). Then we have, for each \( s \in G \),

\[
\|z - T_sz\| \leq \|z - x_n\| + \|T_s x_n - x_n\| + \|T_s x_n - T_sz\| \leq (1 + \alpha)\|z - x_n\| + (1 + \beta)d(x_n) \to 0 \quad \text{as} \quad n \to \infty.
\]

This completes the proof.

As a consequence of Theorem 2, we have the following result.

**Corollary 1.** Let \( K \) be a nonempty bounded closed convex subset of a uniformly convex Banach space \( E \) and let \( T : K \to K \) be a generalized uniformly Lipschitzian mapping with \((\alpha + \beta) < \xi \) (\( \xi \) is as in Theorem 2). Then \( T \) has a fixed point.

If we take \( b = c = 0 \) in Theorem 2, then we have the following result from Theorem 2:

**Corollary 2** [22, Thm. 3.5]. Let \( E \) be a uniformly convex Banach space, \( K \) a non-empty closed convex subset of \( E \), and \( \mathcal{F} = \{T_s : s \in G\} \) a uniformly \( k \)-Lipschitzian semigroup on \( K \) with \( k < \xi \), where \( \xi > 1 \) is the unique solution of (27). Suppose there is an \( x_0 \) in \( K \) such that \( \{T_s x_0 : s \in G\} \) is bounded. Then there exists \( z \) in \( K \) such that \( T_sz = z \) for all \( s \) in \( G \).

4. Some applications. Since a Hilbert space \( H \) is 2-uniformly convex and the following equality holds:

\[
\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2
\]

for all \( x, y \) in \( H \) and \( \lambda \in [0, 1] \).

By Theorem 1 and (46), we immediately obtain the following:

**Corollary 3.** Let \( E \) be a nonempty closed convex subset of a Hilbert space \( H \), \( X \) be an \( l_G \)-invariant subspace of \( m(G) \) containing constants which has left invariant submean \( \mu \), and \( \mathcal{F} = \{T_s : s \in G\} \) be a generalized Lipschitzian semigroup on \( K \). Suppose that there exists an \( x_0 \) in \( K \) such that \( \{T_s x_0 : s \in G\} \) is a generalized Lipschitzian semigroup on \( K \). Suppose that there exists an \( x_0 \) in \( K \) such that \( \{T_s x_0 : s \in G\} \) is bounded and that for every \( u, v \) in \( K \), then the function \( f \) on \( G \) defined by

\[
f(t) = \|T_t u - v\|^2, \quad t \in G
\]

and the function \( g \) on \( G \) defined by

\[
g(t) = 2(\alpha_t^2 + \beta_t^2), \quad t \in G
\]

belong to \( X \). Then, if \( \{\mu_t(\alpha_t^2 + \beta_t^2)\} < 1 \), where \( \alpha_t = (a_t + b_t + c_t)/(1 - b_t - c_t) \) and \( \beta_t = (2b_t + 2c_t)/(1 - b_t - c_t) \), there exists \( z \) in \( K \) such that \( T_sz = z \) for all \( s \) in \( G \).
If $1 < p \leq 2$, then we have for all $x, y$ in $L^p$ and $\lambda \in [0, 1]$
\[\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)(p-1)\|x-y\|^2 \quad (49)\]
(the inequality (49) is contained in [12, 20]).

Assume that $2 < p < \infty$ and $t_p$ is the unique zero of the function $g(x) = -x^{p-1} + (p-1)x + p-2$ in the interval $(1, \infty)$. Let
\[c_p = (p-1)(1+t_p)^{2-p} = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}. \quad (50)\]
Then we have the following inequality
\[\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda) \cdot c_p \cdot \|x-y\|^p \quad (51)\]
for all $x, y$ in $L^p$ and $\lambda \in [0, 1]$. (The inequality (51) is essentially due to Lim [11].)

By Theorem 1 and inequality (49) and (51), we immediately obtain the following result.

**Corollary 4.** Let $K$ be a closed convex subset of an $L^p$ space, $1 < p < \infty$, $X$ be an $l_G$-invariant subspace of $m(G)$ containing constants which has a left invariant submean $\mu$, and $\mathcal{F} = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on $K$. Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in K$ and that for every $u, v$ in $K$, the functions $f$ and $g$ on $G$ defined as in Theorem 1 belong to $X$. If $2\mu_s(\alpha_s^2 + \beta_s^2) < p$ when $1 < p \leq 2$ and $2^{p-1}\mu_s(\alpha_s^{p-1} + \beta_s^{p-1}) < 1 + c_p$ when $p > 2$, where $\alpha_s = (a_s + b_s + c_s)/(1-b_s-c_s)$ and $\beta_s = (2b_s + 2c_s)/(1-b_s-c_s)$, then there exists $z \in K$ such that $T_sz = z$ for all $s \in G$.

Let $H^p, 1 < p < \infty$, denote the Hardy space [5] of all functions $x$ analytic in the unit disk $|z| < 1$ of the complex plane and such that
\[\|x\| = \lim_{r \to 1^-} \left(\frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta\right)^{1/p} < \infty. \quad (52)\]

Now, let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega), k \geq 0, 1 < p < \infty$, the Sobolev space [1, p. 149] of distribution $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = a_1 + \cdots + a_n \leq k$ equipped with the norm
\[\|x\| = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha x(\omega)|^p \, d\omega\right)^{1/p}. \quad (53)\]

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha), \alpha \in \Lambda$, be a sequence of positive measure spaces, where index set $\Lambda$ is finite or countable. Given a sequence of linear subspaces $X_\alpha$ in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}, 1 < p < \infty$ and $q = \max\{2, p\}$ [13], the linear space of all sequences $x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm
\[\|x\| = \left(\sum_{\alpha \in \Lambda} (\|x_\alpha\|_{L^{q,p}})^q\right)^{1/q}. \quad (54)\]
where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L_p = (S_1, \Sigma_1, \mu_1)$ and $L_q = (S_2, \Sigma_2, \mu_2)$, where $1 < p < \infty$, $q = \max\{2, p\}$ and $(S_i, \Sigma_i, \mu_i)$ are positive measure spaces. Denote by $L_p(L_p)$ the Banach spaces \[ ] of all measurable $L_p$-value functions $x$ on $S_2$ such that \[ \| x \| = \left( \int_{S_2} (\| x(s) \|_p)^q \mu_2(ds) \right)^{1/q}. \] (55)

These spaces are $q$-uniformly convex with $q = \max\{2, p\}$ [16, 19] and the norm in these spaces satisfies
\[ \| \lambda x + (1 - \lambda) y \| \leq \lambda \| x \| + (1 - \lambda) \| y \| - d \cdot w_q(\lambda) \cdot \| x - y \|_q \] (56)
with a constant \[ d = d_p = \begin{cases} \frac{p-1}{8} & \text{for } 1 < p \leq 2 \\ \frac{1}{p-2p} & \text{for } 2 < p < \infty. \end{cases} \] (57)

Now, from Theorem 1, we have the following result.

**Corollary 5.** Let $K$ be a closed convex subset of the space $E$, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_q(p)$, or $E = L_q(L_p)$, and $1 < p < \infty$, $q = \max\{2, p\}$, $k \geq 0$, $X$ be an $l_G$-invariant subspace of $m(G)$ containing constants which has a left invariant submean $\mu$, and $f = \{T_s : s \in G\}$ be a generalized Lipschitzian semigroup on $K$. Suppose that $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in K$ and that for every $u, v$ in $K$, the functions $f$ and $g$ on $G$ defined as in Theorem 1 belong to $X$. If $2^{q-1} \mu_1(\alpha_1^q + \beta_1^q) < 1 + d$, where $\alpha_1 = (a_1 + b_1 + c_1)/(1 - b_1 - c_1)$ and $\beta_1 = (2b_1 + 2c_1)/(1 - b_1 - c_1)$, then there exists $z \in K$ such that $T_s z = z$ for all $s \in G$.

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**References**


Thakur: Govt. B. H. S. Gariaband, Dist. Raipur, M. P. 493889, India

Jung: Department of Mathematics, Dong-A University, Pusan 604–714, Korea

E-mail address: jungjs@seunghak.donga.ac.kr
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Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie