

WEIGHTED NORM INEQUALITIES FOR THE \mathcal{H} -TRANSFORMATION

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ABSTRACT. In this paper we establish weighted norm inequalities for an integral transform whose kernel is a Fox function.

KEY WORDS AND PHRASES: Fox function, integral transformation, weighted norm inequalities.

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1. INTRODUCTION

The transformations we will investigate in this paper are the ones called \mathcal{H} -transformations. These transformations are defined by

$$\mathcal{H}(f)(x) = \int_0^\infty \mathfrak{H}_{p,q}^{m,n} \left(xt \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right) f(t) dt, \quad f \in C_0, \quad (1)$$

where $\mathfrak{H}_{p,q}^{m,n}$ denotes the Fox function ([9]) and as usual C_0 represents the class of complex valued functions on $(0, \infty)$ which are continuous and compactly supported. In the last years, the \mathcal{H} -transformation has been studied by several authors (see [6], [7], [14] and [18]) and it reduces to important integral transforms (Laplace, Hankel, Meijer, Hardy, ...) by specifying the involved parameters. In a previous paper [5] the authors (simultaneously to A. A. Kilbas, M. Saigo and S. A. Shlapakov [15], [16] and [17]), investigated the behavior of transformation (1) in certain weighted L_p spaces introduced by P. G. Rooney [21].

Weighted Fourier transform norm inequalities have been exhaustively studied (see [2], [3], [4], [10], [13], [20], amongst others). Inspired by the above works our aim in this paper is to give conditions on a positive Borel measure Ω on $(0, \infty)$, and on a measurable nonnegative function v on $(0, \infty)$ which are sufficient in order that the inequality

$$\left\{ \int_0^\infty |\mathcal{H}(f)(x)|^s d\Omega(x) \right\}^{\frac{1}{s}} \leq C \left\{ \int_0^\infty v(x) |f(x)|^r dx \right\}^{\frac{1}{r}}, \quad f \in C_0, \quad (2)$$

holds where $1 \leq r, s \leq \infty$ and C is a suitable positive constant. Also we analyze some special cases of (2). Moreover we establish some properties on Ω and v that are implied by (2).

We now introduce some notations that will be used throughout this paper. We need consider some parameters related to the \mathfrak{H} -function. Let $m, n, p, q \in \mathbb{N}$ being $0 \leq m \leq s, 0 \leq n \leq r$ and $r + s \geq 1$. Assume that $a_j, j = 1, \dots, r$ and $b_j, j = 1, \dots, s$, are real numbers and $\alpha_j, j = 1, \dots, r$, and $\beta_j, j = 1, \dots, s$, are positive real numbers. We define

$$\alpha = \begin{cases} \max \left\{ -\frac{b_j}{\beta_j}, j = 1, \dots, m \right\} & , \text{ for } m > 0 \\ -\infty & , \text{ for } m = 0 \end{cases}$$

$$\beta = \begin{cases} \min\left\{\frac{1-\alpha_j}{\alpha_j}, j = 1, \dots, n\right\} & , \text{ for } n > 0 \\ +\infty & , \text{ for } n = 0 \end{cases}$$

$$\mu = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j$$

$$\nu = \sum_{j=1}^q b_j - \sum_{j=1}^p a_j$$

$$\xi = \sum_{j=1}^n \alpha_j - \sum_{j=n+1}^p \alpha_j + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j$$

$$\eta = \prod_{j=1}^p \alpha_j^{-\alpha_j} \prod_{j=1}^q \beta_j^{\beta_j}.$$

Also we remember a result that was established in [5] and that will be very useful in the sequel

THEOREM A (Corollary 1 of [5]). If $\alpha < \gamma < \beta$ and if either

- (a) $\xi > 0$ or
- (b) $\xi = 0, \mu \neq 0$ and $\nu + \mu\gamma - \frac{1}{2}(q - p) < -1$

holds, then the function \mathfrak{H} is defined by

$$\mathfrak{H}(x) = \mathfrak{H}_{p,q}^{m,n} \left(x \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} x^{-s} h(s) ds \tag{3}$$

for every $x > 0$, where

$$h(s) = h_{p,q}^{m,n} \left(\begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| s \right) = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s) \prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)}.$$

Here the empty products as usual are understood as 1. Moreover

$$|\mathfrak{H}(x)| \leq C_\gamma x^{-\gamma} \tag{4}$$

for every $x > 0$, C_γ being a positive constant. Furthermore if $\alpha < \gamma < \beta, \xi = \mu = 0$ and $\nu - \frac{1}{2}(q - p) < -1$ then (3) and (4) hold for every $x > 0$ except for $x = \eta$. ■

In view of the above considerations we will assume in the sequel that our parameters satisfy one of the following four conditions, namely

- (i) $\xi > 0$
- (ii) $\xi = 0, \mu > 0$ and $\beta < -\frac{1}{\mu}[\nu + 1 + \frac{1}{2}(p - q)]$
- (iii) $\xi = 0, \mu < 0$ and $\alpha > -\frac{1}{\mu}[\nu + 1 + \frac{1}{2}(p - q)]$
- (iv) $\xi = 0, \mu = 0$ and $\nu + \frac{1}{2}(p - q) < -1$

Throughout this paper for every $1 \leq r \leq \infty$ we denote by r' the conjugate of r (that is, $r' = \frac{r}{r-1}$). Also when some of the exponents in our weighted inequality are infinite said inequality must be understood in the obvious form.

2. WEIGHTED NORM INEQUALITIES FOR THE \mathcal{H} -TRANSFORM

We shall firstly give sufficient conditions on a positive function v on $(0, \infty)$ and on a positive Borel measure Ω on $(0, \infty)$ in order that the inequality

$$\left\{ \int_0^\infty |\mathcal{H}(f)(x)|^s d\Omega \right\}^{\frac{1}{s}} \leq C \left\{ \int_0^\infty v(x) |f(x)|^r dx \right\}^{\frac{1}{r}}, \quad f \in C_0,$$

holds, where $1 \leq r, s < \infty$ and C denotes a certain positive constant. When either $r = \infty$ or $s = \infty$ inequality (2) takes the obvious form. The employed procedure here is inspired by the one used by J. J. Benedetto and H. P. Heinig ([2] and [3]) in their studies about Fourier transforms

PROPOSITION 1. Assume that Ω is a positive Borel measure on $(0, \infty)$ and that v is a nonnegative measurable function on $(0, \infty)$ belonging to $L^1_{loc}(0, \infty)$.

If $1 \leq r \leq s \leq \infty$ and there exist $\alpha < a$, $b < \beta$ such that

$$B_1 = \sup_{x>0} \left\{ \int_0^x t^{-as} d\Omega(t) \right\}^{\frac{1}{s}} \left\{ \int_0^{\frac{1}{x}} t^{-ar'} v(t)^{1-r'} dt \right\}^{\frac{1}{r'}} < \infty$$

and

$$B_2 = \sup_{x>0} \left\{ \int_x^\infty t^{-bs} d\Omega(t) \right\}^{\frac{1}{s}} \left\{ \int_{\frac{1}{x}}^\infty t^{-br'} v(t)^{1-r'} dt \right\}^{\frac{1}{r'}} < \infty,$$

then (2) holds for every $f \in C_0$.

Also if $1 \leq s < r < \infty$ and there exist $\alpha < a$, $b < \beta$ such that

$$B'_1 = \int_0^\infty \left\{ \int_0^{\frac{1}{x}} z^{-as} d\Omega(z) \right\}^{\frac{1}{s}} \left\{ \int_0^x z^{-ar} v(z)^{1-r'} dz \right\}^{\frac{1}{r'}} x^{-ar'} v(x)^{1-r'} dx < \infty$$

and

$$B'_2 = \int_0^\infty \left\{ \int_{\frac{1}{x}}^\infty z^{-bs} d\Omega(z) \right\}^{\frac{1}{s}} \left\{ \int_x^\infty z^{-br} v(z)^{1-r'} dz \right\}^{\frac{1}{r'}} x^{-br'} v(x)^{1-r'} dx < \infty$$

where $\frac{1}{h} = \frac{1}{s} - \frac{1}{r}$, then (2) holds for every $f \in C_0$

PROOF. First we consider the case $1 < r \leq s < \infty$

Let $f \in C_0$. By virtue of (4) for every $\alpha < a$, $b < \beta$ there exists $C_{a,b} > 0$ such that

$$|\mathcal{H}(f)(x)| \leq C_{a,b} \left\{ \int_0^{\frac{1}{x}} (xt)^{-a} |f(t)| dt + \int_{\frac{1}{x}}^\infty (xt)^{-b} |f(t)| dt \right\}, \quad x > 0.$$

By using the Minkowski inequality we obtain

$$\begin{aligned} \left\{ \int_0^\infty |\mathcal{H}(f)(x)|^s d\Omega(x) \right\}^{\frac{1}{s}} &\leq C_{a,b} \left[\left\{ \int_0^\infty \left\{ \int_0^{\frac{1}{x}} t^{-a} |f(t)| dt \right\}^s x^{-as} d\Omega(x) \right\}^{\frac{1}{s}} \right. \\ &\quad \left. + \left\{ \int_0^\infty \left\{ \int_{\frac{1}{x}}^\infty t^{-b} |f(t)| dt \right\}^s x^{-bs} d\Omega(x) \right\}^{\frac{1}{s}} \right] = C_{a,b} (J_1 + J_2). \end{aligned} \quad (5)$$

A straightforward change of variable leads to

$$J_1 = \left\{ \int_0^\infty \left\{ \int_0^{\frac{1}{x}} t^{-a} |f(t)| dt \right\}^s x^{-as} d\Omega(x) \right\}^{\frac{1}{s}} = \left\{ \int_0^\infty \left\{ \int_x^\infty h(u) du \right\}^s x^{-as} d\Omega(x) \right\}^{\frac{1}{s}}$$

where $h(u) = u^{a-2} |f(\frac{1}{u})|$, $u > 0$

Therefore from Theorem 4 (1.3.1) [19] one infers

$$J_1 \leq C_1 \left\{ \int_0^\infty h(t)^r v_1(t) dt \right\}^{\frac{1}{r}} = C_1 \left\{ \int_0^\infty |f(t)|^r v(t) dt \right\}^{\frac{1}{r}} \quad (6)$$

with $C_1 > 0$ and $v_1(t) = v(\frac{1}{t}) t^{2r-2-ar}$, $t > 0$, provided that $B_1 < \infty$.

On the other hand, we have

$$J_2 = \left\{ \int_0^\infty \left\{ \int_{\frac{1}{x}}^\infty t^{-b} |f(t)| dt \right\}^s x^{-bs} d\Omega(x) \right\}^{\frac{1}{s}} = \left\{ \int_0^\infty \left\{ \int_0^x g(t) dt \right\}^s x^{-bs} d\Omega(x) \right\}^{\frac{1}{s}}$$

where $g(t) = t^{b-2} |f(\frac{1}{t})|$, $t > 0$. Then by invoking Theorem 1 (1.3.1) [19] it follows

$$J_2 \leq C_2 \left\{ \int_0^\infty g(t)^r v_2(t) dt \right\}^{\frac{1}{r}} = C_2 \left\{ \int_0^\infty |f(t)|^r v(t) dt \right\}^{\frac{1}{r}} \tag{7}$$

where $v_2(t) = v(\frac{1}{t})t^{2r-2-br}$, $t > 0$, when $B_2 < \infty$

By combining (5), (6) and (7) we can immediately deduce (2)

When either $r = \infty$ or $s = \infty$ the proof can be made in a similar way

In the case $1 \leq s < r \leq \infty$ (2) can be established as the above case by invoking the Theorem 2 (1 3.2) [19]. ■

In the sequel we present some special cases of inequality (2). The following results are related to known weighted norm inequalities for other integral transforms due to P Heywood and P G Rooney ([11], [12]), N E. Aguilera and E O Harboure [1], B Muckenhoupt [20] and S A Emará and H P Heinig [8]

A generalization of Theorem 2.1 of [12] is the following

PROPOSITION 2. Let $\alpha < 1 - \eta < \beta$ and $1 \leq s \leq \infty$. Then

$$\left\{ \int_0^\infty |x^{1-\eta} \mathcal{H}(f)(x)|^s \frac{dx}{x} \right\}^{\frac{1}{s}} \leq C \int_0^\infty x^{\eta-1} |f(x)| dx, \quad f \in C_0, \tag{8}$$

for certain $C > 0$.

PROOF. This result, that also can be proved in a similar way to Theorem 2.1 of [12], is a consequence of Proposition 1. In effect if $1 - \eta < a < \beta$ we have

$$\left\{ \int_x^\infty t^{s(1-\eta-a)-1} dt \right\}^{\frac{1}{s}} \left\| t^{-a-\eta+1} \chi_{[\frac{1}{2}, \infty)}(t) \right\|_{\infty, t^{\eta-1} dt} = (s(1-\eta-a))^{\frac{1}{s}}, \quad x > 0 \text{ and } 1 \leq s < \infty$$

where $\| \cdot \|_{\infty, t^{\eta-1} dt}$ denotes the essential supremum respect to the measure $t^{\eta-1} dt$ and χ_E represents as usual the characteristic function associated to the measure set E .

In a similar way we can see that if $\alpha < b < 1 - \eta$ and $1 \leq s < \infty$. Then

$$\sup_{x>0} \left\{ \int_0^x t^{s(1-\eta-b)-1} dt \right\}^{\frac{1}{s}} \left\| t^{-b-\eta+1} \chi_{(0, \frac{1}{2}]}(t) \right\|_{\infty, t^{\eta-1} dt} < \infty.$$

Hence according to Proposition 1 (8) holds for every $1 \leq s < \infty$

When $s = \infty$ the result can be proved analogously ■

We now investigate the inequality (2) when $d\Omega = u(x)dx$ being u is a measurable nonnegative function on $(0, \infty)$, $v = 1$ and $r = s$.

PROPOSITION 3. Let $1 \leq r \leq 2$, $\alpha < 0$ and $\frac{1}{2} < \beta$. If u is a locally integrable nonnegative function on $(0, \infty)$ for which there exists a constant $M > 0$ such that for every measurable set $E \int_E u(x) dx \leq M|E|^{r-1}$ is satisfied, then

$$\int_0^\infty u(x) |\mathcal{H}(f)(x)|^r dx \leq C \int_0^\infty |f(x)|^r dx, \quad f \in C_0, \tag{9}$$

for a certain $C > 0$.

PROOF. Our proof is essentially the same one given in Theorem 1 of [1]. Let $1 < r < 2$ we define the operator

$$(Tf)(x) = \begin{cases} u^{-\frac{1}{2}}(x) \mathcal{H}(f)(x) & , \text{ if } u(x) \neq 0 \\ 0 & , \text{ if } u(x) = 0 \end{cases}, \quad f \in C_0$$

where $b = \frac{2}{2-r}$

Since $\alpha < 0 < \beta$, then by (4) \mathfrak{H} is a bounded function on $(0, \infty)$. Hence, according to Theorem 2 of [1] we obtain

$$\int_{\{x \mid |Tf(x)| > \lambda\}} u^b(x) dx \leq \int_{\{x \mid u^{\frac{1}{2}}(x) \leq \frac{C_1}{\lambda} \int_0^\infty |f(x)| dx\}} u^b(x) dx \leq \frac{C_2}{\lambda} \int_0^\infty |f(x)| dx$$

where $C_i, i = 1, 2$, are positive constants. Thus T is a weak type (1,1) operator, on measure spaces $((0, \infty), dx)$ and $((0, \infty), u^b(x) dx)$

Moreover by virtue of Proposition 3 of [5] \mathcal{H} is a bounded operator from $L_2(0, \infty)$ into itself because $\alpha < \frac{1}{2} < \beta$. Therefore

$$\int_0^\infty |Tf(x)|^2 u^b(x) dx \leq C \int_0^\infty |f(x)|^2 dx$$

with $C > 0$, and T is a strong type (2,2) operator between the spaces under consideration

Now by the Marcinkiewicz interpolation theorem we obtain the desired result for $1 < r < 2$

Finally, note that if $r = 1$ then $\int_0^\infty u(x) dx < \infty$ and (9) holds trivially because $\alpha < 0 < \beta$ and by (4). Moreover if $r = 2$ then u is bounded function on $(0, \infty)$ and since $\alpha < \frac{1}{2} < \beta$ (4) leads to (9). ■

By proceeding as in §7 of [1] we can deduce from Proposition 3 conditions for a function v that imply inequality (2) holds when Ω is the Lebesgue measure on $(0, \infty)$ and $r = s$

We now give conditions for u that are deduced from (9)

PROPOSITION 4. Let $1 < r < \infty$. Assume that one of the following two conditions is satisfied:

(i) There exists $j_0 \in \mathbb{N}, 1 \leq j_0 \leq p$, such that $-\frac{\alpha_{j_0}}{\alpha_{j_0}} > \max\{\alpha, 1 - \frac{1}{r}\}$ and

$$\inf_{0 < x < 1} \left| \mathfrak{H}_{p,q}^{m,n} \left(\begin{matrix} (a'_1, \alpha_1), \dots, (a'_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| x \right) \right| = K_1 > 0 \tag{10}$$

where $a'_{j_0} = a_{j_0} + 1$ and $a'_j = a_j, 1 \leq j \leq p, j \neq j_0$.

(ii) There exists $j_0 \in \mathbb{N}, 1 \leq j_0 \leq q$, such that $\frac{1-b_{j_0}}{\beta_{j_0}} > \max\{\beta, 1 - \frac{1}{r}\}$ and

$$\inf_{0 < x < 1} \left| \mathfrak{H}_{p,q}^{m,n} \left(\begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b'_1, \beta_1), \dots, (b'_q, \beta_q) \end{matrix} \middle| x \right) \right| = K_2 > 0$$

where $b'_{j_0} = b_{j_0} - 1$ and $b'_j = b_j, 1 \leq j \leq q, j \neq j_0$

Then there exists a positive constant L such that

$$\int_0^a u(x) dx \leq C a^{1-r}, \quad \text{holds for every } a > 0, \tag{11}$$

provided that (9) holds.

PROOF. We will establish the result when (i) is satisfied with $n + 1 \leq j_0 \leq p$. The proof in the other cases can be made in a similar way

It is easy to see that

$$\begin{aligned} & \frac{d}{dx} \left[x^{-a_{j_0}} \mathfrak{H}_{p,q}^{m,n} \left(\begin{matrix} (a'_1, \alpha_1), \dots, (a'_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| x^{\alpha_{j_0}} \right) \right] \\ &= -x^{-(a_{j_0}+1)} \mathfrak{H}_{p,q}^{m,n} \left(\begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| x^{\alpha_{j_0}} \right), \quad x > 0 \end{aligned} \tag{12}$$

being $a'_{j_0} = a_{j_0} + 1$ and $a'_j = a_j, j = 1, \dots, p, j \neq j_0$

For $a > 0$ fixed, define

$$f_a(x) = \begin{cases} x^{-\frac{\alpha_{j_0} + \alpha_{j_0}}{\alpha_{j_0}}}, & 0 < x \leq \frac{1}{a} \\ 0 & , x > \frac{1}{a} . \end{cases}$$

By using (12) we can write

$$\begin{aligned} (\mathcal{H}f_a)(x) &= \int_0^{\frac{1}{a}} t^{-\frac{\alpha_{j_0} + \alpha_{j_0}}{\alpha_{j_0}}} \mathfrak{F}_{p,q}^{m,n} \left(\begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| tx \right) dt \\ &= \alpha_{j_0} x^{\frac{\alpha_{j_0}}{\alpha_{j_0}}} \int_0^{\left(\frac{1}{a}\right)^{\frac{1}{\alpha_{j_0}}}} \frac{d}{dv} \left[-v^{-\alpha_{j_0}} \mathfrak{F}_{p,q}^{m,n} \left(\begin{matrix} (a'_1, \alpha_1), \dots, (a'_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| v^{\alpha_{j_0}} \right) \right] dv \\ &= -\alpha_{j_0} a^{\frac{\alpha_{j_0}}{\alpha_{j_0}}} \mathfrak{F}_{p,q}^{m,n} \left(\begin{matrix} (a'_1, \alpha_1), \dots, (a'_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| \frac{x}{a} \right) \end{aligned}$$

because

$$\lim_{v \rightarrow 0^+} v^{-\alpha_{j_0}} \mathfrak{F}_{p,q}^{m,n} \left(\begin{matrix} (a'_1, \alpha_1), \dots, (a'_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| v^{\alpha_{j_0}} \right) = 0. \tag{13}$$

Since $\alpha < -\frac{\alpha_{j_0}}{\alpha_{j_0}}$ to see (13) it is sufficient to take into account (4) Hence, by virtue of (10)

$$\begin{aligned} \int_0^a u(x) dx &\leq K_1^{-r} \int_0^a u(x) \left| \mathfrak{F}_{p,q}^{m,n} \left(\begin{matrix} (a'_1, \alpha_1), \dots, (a'_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| \frac{x}{a} \right) \right|^r dx \\ &= \left(K_1 \alpha_{j_0} a^{\frac{\alpha_{j_0}}{\alpha_{j_0}}} \right)^{-r} \int_0^a u(x) |\mathcal{H}(f_a)(x)|^r dx. \end{aligned}$$

Similarly from (9) one deduces

$$\int_0^a u(x) dx \leq C \left(K_2 \alpha_{j_0} a^{\frac{\alpha_{j_0}}{\alpha_{j_0}}} \right)^{-r} \int_0^{\frac{1}{a}} x^{-\frac{(\alpha_{j_0} + \alpha_{j_0})r}{\alpha_{j_0}}} dx = C (K_2 \alpha_{j_0})^{-r} a^{r-1}.$$

Thus the proof is finished ■

Note that if $r = 1$ (11) implies that u is integrable over $(0, \infty)$ When $r = 2$, u is bounded on $(0, \infty)$ provided that (11) holds Also if $r > 2$ and (11) is satisfied then $u = 0$, a.e $(0, \infty)$

B Muckenhoupt [20] investigated sufficient conditions for the measurable functions u and v that guarantee that the inequality (2), with $d\Omega(x) = u(x)dx$, holds when the \mathcal{H} -transformation is replaced by the Fourier transform Also he studied the converse problem proving that, in some cases, the above cited conditions are necessary Later P Heywood and P.G Rooney [11] analyzed weighted norm inequalities for the Hankel transformation in a similar way We now use an analogous procedure to extend the results in [11] to the \mathcal{H} -transformation (note that this transform reduces to the Hankel transformation when the parameters take on suitable values)

It will be used to recall some definitions of [11]. For every $\eta \in \mathbb{R}$, $1 \leq r < \infty$ and for every v nonnegative measurable function on $(0, \infty)$, the space $\mathcal{L}_{\eta,v,r}$ is constituted by all those measurable functions f on $(0, \infty)$ such that

$$\|f\|_{\eta,v,r} = \left\{ \int_0^\infty |x^\eta v(x) f(x)|^r \frac{dx}{x} \right\}^{\frac{1}{r}} < \infty.$$

The space $\mathcal{L}_{\eta,v,r}$ is a Banach space when it is endowed with the topology associated to the norm $\| \cdot \|_{\eta,v,r}$ Also, if u and v are nonnegative measurable functions on $(0, \infty)$ we say that $(u, v) \in A(r, s, \delta)$ with $\delta \in \mathbb{R}$ and $1 < r, s < \infty$ when there exist positive constants B and C for which

$$\left[\int_{u(x) > B\omega} \{x^\delta u(x)\}^s \frac{dx}{x} \right]^{\frac{1}{s}} \left[\int_{v(x) < \omega} \left\{ \frac{x^\delta}{v(x)} \right\}^r \frac{dx}{x} \right]^{\frac{1}{r}} \leq C$$

for every $\omega > 0$

In Propositions 4-8 [5] we established some conditions on the parameters involved in the \mathfrak{H} -function in order that the \mathcal{H} -transformation can be extended to the space $\mathcal{L}_{\eta,r}$ as a bounded operator from $\mathcal{L}_{\eta,r}$ into $\mathcal{L}_{1-\eta,s}$. In the following Proposition the above results are improved. We prove that under suitable conditions the \mathcal{H} -transformation can be extended to $\mathcal{L}_{\eta,v,r}$ as a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$. We only stated the result corresponding Proposition 8 of [5] although similar results corresponding to Propositions 4-7 of [5] can be established.

PROPOSITION 5. Let $1 < r \leq s < \infty$, $\xi > 0$ and $\alpha < 1 - \eta < \beta$. Suppose that $(u, v) \in A(\tau, s, 1 - \eta - \sigma)$, with $\alpha < \sigma < \beta$. Then the \mathcal{H} -transformation can be extended to $\mathcal{L}_{\eta,v,r}$ as a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$.

PROOF. This result can be proved as Theorem 1 of [11]. It is sufficient to take into account that $|\mathfrak{H}(x)| \leq C_\sigma x^{-\sigma}$, $x > 0$, with $\alpha < \sigma < \beta$ and for certain $C_\sigma > 0$. By using this inequality instead of (2.5) of [11] and Proposition 8 of [5] instead of Lemma 1 of [11] the proof of our result follows as the one of Theorem 1 of [11].

On the other hand this result can be proved also by invoking Proposition 1 because if $(u, v) \in A(\tau, s, 1 - \eta - \sigma)$ being $\alpha < 1 - \eta$, $\sigma < \beta$ then the conditions $B_i < \infty$, $i = 1, 2$, in Proposition 1 are satisfied when $d\Omega$ and v are replaced by $x^{(1-\eta-\sigma)s-1}u(x)^s dx$ and $x^{(1-\eta-\sigma)r-1}v(x)^r$, respectively. ■

Our next objective is to establish a partial converse to Proposition 5

LEMMA 1. Let $1 < r \leq s < \infty$ and $0 < \eta < 1$. Assume that u and v are nonnegative measurable functions on $(0, \infty)$ such that u is decreasing, $\lim_{x \rightarrow \infty} u(x) = 0$ and v is increasing. Also suppose that

$$\inf_{0 < x < 1} \mathfrak{H}_{p,q}^{m,n} \left(\begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \middle| x \right) = C_1 > 0. \tag{14}$$

Then there exists a positive constant $B > 0$ for which

$$\sup\{x : u(x) > B\omega\} \cdot \sup\{x : v(x) < \omega\} \leq 1,$$

for every $\omega > 0$, provided that \mathcal{H} is a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$.

PROOF. This result will be proved when we see that if

$$\sup\{x : u(x) > B\omega\} \cdot \sup\{x : v(x) < \omega\} > 1,$$

for some $\omega > 0$, then B is less than a positive constant only depending on τ, s and η , the lemma then holds with any larger value of B .

Let $B, \omega > 0$. For simplicity denote

$$M = M(B, \omega) = \sup\{x : u(x) > B\omega\}.$$

Since $\lim_{x \rightarrow \infty} u(x) = 0$, $M(B, \omega) < \infty$. Assume now $M(B, \omega) \cdot \sup\{x : v(x) < \omega\} > 1$ and define the function

$$f(x) = \begin{cases} 1 & , \text{ if } 0 < x < \frac{1}{M} \\ 0 & , \text{ if } x > \frac{1}{M} \end{cases}$$

It is clear that $f \in \mathcal{L}_{\eta,v,r}$ and one has

$$\|f\|_{\eta,v,r} = \left\{ \int_0^{\frac{1}{M}} |x^\eta v(x)|^r \frac{dx}{x} \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{\frac{1}{M}} \omega^r x^{\eta r - 1} dx \right\}^{\frac{1}{r}} = \frac{\omega}{M^\eta (\eta r)^{\frac{1}{r}}} \tag{15}$$

because $v(x) \leq \omega$, for every $x \in (0, \frac{1}{M})$. Since $\mathcal{H}f \in \mathcal{L}_{1-\eta,u,s}$ then by virtue of (14) and since

$u(x) \geq \omega B$, for every $x \in (0, M)$ we can write

$$\begin{aligned} \|\mathcal{H}f\|_{1-\eta,u,s} &\geq \left\{ \int_0^M \left| x^{1-\eta} u(x) \int_0^{\frac{1}{M}} \mathfrak{H}(xt) dt \right|^s \frac{dx}{x} \right\}^{\frac{1}{s}} \\ &\geq \frac{C_1}{M} \left\{ \int_0^M |x^{1-\eta} u(x)|^s \frac{dx}{x} \right\}^{\frac{1}{s}} > \frac{C_1 B \omega}{M} \left\{ \int_0^M x^{(1-\eta)s-1} dx \right\}^{\frac{1}{s}} = \frac{C_1 B \omega}{M^n (s(1-\eta))^{\frac{1}{s}}} \end{aligned} \tag{16}$$

for a suitable $K > 0$.

Moreover for a certain $C > 0$

$$\|\mathcal{H}f\|_{1-\eta,u,s} \leq C \|f\|_{\eta,v,r} \tag{17}$$

By combining (15), (16) and (17) one concludes that

$$B \leq \frac{C[s(1-\eta)]^{\frac{1}{s}}}{C_1(\eta r)^{\frac{1}{s}}}$$

Note that the constant in the right hand side of the last inequality is positive since $0 < \eta < 1$. Thus the proof is complete. ■

PROPOSITION 6. Let $1 < r \leq s < \infty$ and $0 < \eta < 1$. Assume that u and v are measurable nonnegative functions on $(0, \infty)$ such that u is decreasing, $\lim_{x \rightarrow \infty} u(x) = 0$, v is increasing and $\int_{v(x) < \omega} \left\{ \frac{x^{1-\eta}}{v(x)} \right\}^r \frac{dx}{x} < \infty$, for every $\omega > 0$. Then $(u, v) \in A(r, s, 1 - \eta)$ provided that \mathcal{H} is a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$ and (14) holds

PROOF. We define for every $\omega > 0$ the function

$$f_\omega(x) = \begin{cases} x^{\frac{\eta-1}{1-r}} v(x)^{-r} & \text{if } 0 < v(x) < \omega \\ 0 & \text{otherwise} \end{cases}$$

It is not hard to show that

$$\|f_\omega\|_{\eta,v,r} = \left\{ \int_{v(x) < \omega} \left\{ \frac{x^{1-\eta}}{v(x)} \right\}^r \frac{dx}{x} \right\}^{\frac{1}{r}}$$

and $f_\omega \in \mathcal{L}_{\eta,v,r}$, for every $\omega > 0$

But since \mathcal{H} is a bounded operator from $\mathcal{L}_{\eta,v,r}$ into $\mathcal{L}_{1-\eta,u,s}$, there exists a positive constant $C > 0$ such that

$$\|\mathcal{H}f_\omega\|_{1-\eta,u,s} \leq C \|f_\omega\|_{\eta,v,r}, \quad \omega > 0.$$

Hence

$$\left\{ \int_{u(x) > B\omega} |x^{1-\eta} u(x) \mathcal{H}(f_\omega)(x)|^s \frac{dx}{x} \right\}^{\frac{1}{s}} \leq \|\mathcal{H}f_\omega\|_{1-\eta,u,s} \leq C \left\{ \int_{v(x) < \omega} \left\{ \frac{x^{1-\eta}}{v(x)} \right\}^r \frac{dx}{x} \right\}^{\frac{1}{r}}, \quad \omega > 0 \tag{18}$$

where B denotes the constant given in Lemma 1.

Moreover, according to Lemma 1, if $\omega, x, t > 0$, $u(x) > B\omega$ and $v(t) < \omega$, then

$$xt \leq \sup\{x : u(x) > B\omega\} \sup\{t : v(t) < \omega\} \leq 1.$$

Hence (14) leads to

$$\begin{aligned} &\left\{ \int_{u(x) > B\omega} |x^{1-\eta} u(x) \mathcal{H}(f_\omega)(x)|^s \frac{dx}{x} \right\}^{\frac{1}{s}} \\ &= \left\{ \int_{u(x) > B\omega} \left| x^{1-\eta} u(x) \int_{v(t) < \omega} t^{\frac{\eta-1}{1-r}} \mathfrak{H}(xt) v(t)^{-r} dt \right|^s \frac{dx}{x} \right\}^{\frac{1}{s}} \\ &\geq C_1 \left\{ \int_{u(x) > B\omega} |x^{1-\eta} u(x)|^s \frac{dx}{x} \right\}^{\frac{1}{s}} \int_{v(t) < \omega} \left\{ \frac{t^{1-\eta}}{v(t)} \right\}^r \frac{dt}{t}, \quad \omega > 0. \end{aligned} \tag{19}$$

By combining (18) and (19) we conclude that $(u, v) \in A(r, s, 1 - \eta)$. ■

S A Emara and H P Heinig [8] established interpolation theorems (Theorems 1 and 2 of [8]) that they employed to study the behavior of the Hankel and K -transformations on weighted L_p -spaces. We can use such interpolation theorems to obtain new weighted norm inequalities for the \mathcal{H} -transform. The weight functions that appear in this inequality are in the class $F_{r,s}^*$ that we are going to define. Let u and v be nonnegative measurable functions defined on $(0, \infty)$ and let u^* and $(\frac{1}{v})^*$ be the equimeasurable decreasing rearrangements of u and $\frac{1}{v}$, respectively. We say that $(u, v) \in F_{r,s}^*$ if

$$\sup_{\omega > 0} \left\{ \int_0^{\frac{1}{\omega}} u^*(t)^s dt \right\}^{\frac{1}{s}} \left\{ \int_0^{\omega} \left[\left(\frac{1}{v} \right)^*(t) \right]^r dt \right\}^{\frac{1}{r}} < \infty \quad (20)$$

holds for every $1 < r \leq s < \infty$, and when $1 < s < r < \infty$ the conditions

$$\int_0^{\infty} \left\{ \left\{ \int_0^{\frac{1}{x}} u^*(t)^s dt \right\}^{\frac{1}{s}} \left\{ \int_0^x \left[\left(\frac{1}{v} \right)^*(t) \right]^r dt \right\}^{\frac{1}{r}} \right\}^h \left(\frac{1}{v} \right)^*(x)^r dx < \infty \quad (21)$$

$$\int_0^{\infty} \left\{ \left\{ \int_{\frac{1}{x}}^{\infty} [t^{-\frac{1}{2}} u^*(t)]^s dt \right\}^{\frac{1}{s}} \left\{ \int_x^{\infty} \left[t^{-\frac{1}{2}} \left(\frac{1}{v} \right)^*(t) \right]^r dt \right\}^{\frac{1}{r}} \right\}^h \left\{ \left(\frac{1}{v} \right)^*(x) x^{-\frac{1}{2}} \right\}^r dx < \infty \quad (22)$$

hold, where $\frac{1}{h} = \frac{1}{s} - \frac{1}{r}$. Moreover if (20), (21) and (22) hold when u^* and $(\frac{1}{v})^*$ are replaced by u and $\frac{1}{v}$, respectively, then we write $(u, v) \in F_{r,s}$.

PROPOSITION 7. Assume that $1 < r, s < \infty$, $\alpha < 0$ and $\frac{1}{2} < \beta$. Then

$$\left\{ \int_0^{\infty} |u(x)\mathcal{H}(f)(x)|^s dx \right\}^{\frac{1}{s}} \leq C \left\{ \int_0^{\infty} |v(x)f(x)|^r dx \right\}^{\frac{1}{r}}, \quad f \in C_0, \quad (23)$$

holds for a certain $C > 0$, provided that $(u, v) \in F_{r,s}$.

PROOF. Since $\alpha < 0 < \beta$, according to (4) we can write

$$\sup_{x > 0} |\mathcal{H}f(x)| \leq C \int_0^{\infty} |f(x)| dx, \quad f \in L_1(0, \infty)$$

for a certain $C > 0$, and then \mathcal{H} is a bounded operator from $L_1(0, \infty)$ into $L_{\infty}(0, \infty)$.

Moreover, \mathcal{H} is a bounded operator from $L_2(0, \infty)$ into itself because $\alpha < \frac{1}{2} < \beta$ (Proposition 3 of [5]).

Hence from Theorems 1 and 2 of [8] we can infer that the inequality (23) is satisfied. ■

We now prove a result that is a (partial) converse to Proposition 7. Note that here no monotonicity assumptions on the weights need be made.

PROPOSITION 8. Let $1 < r \leq s < \infty$ and let u and v be nonnegative measurable functions on $(0, \infty)$. Assume that (14) holds and that $\int_0^{\omega} v(x)^{-r'} dx < \infty$, for every $\omega > 0$. Then $(u, v) \in F_{r,s}$ when (23) is satisfied.

PROOF. Firstly we define for every $\omega > 0$ the function

$$f_{\omega}(x) = \begin{cases} v(x)^{-r'} & , \text{ if } 0 < x < \omega \\ 0 & , \text{ if } x > \omega. \end{cases}$$

From (14) one deduces

$$\begin{aligned} \int_0^{\infty} u(x) |\mathcal{H}(f_{\omega})(x)|^s dx &= \int_0^{\infty} \left| u(x) \int_0^{\infty} \mathfrak{H}(xt) f_{\omega}(t) dt \right|^s dx \\ &\geq \int_0^{\frac{1}{\omega}} \left| u(x) \int_0^{\omega} \mathfrak{H}(xt) v(t)^{-r'} dt \right|^s dx \geq M \int_0^{\frac{1}{\omega}} u(x)^s dx \left\{ \int_0^{\omega} v(t)^{-r'} dt \right\}^s, \quad \omega > 0 \end{aligned}$$

for a certain $M > 0$. Moreover,

$$\int_0^{\infty} |f_{\omega}(x)v(x)|^r dx = \int_0^{\omega} v(x)^{-r} dx, \quad \omega > 0.$$

Since (23) holds we can write

$$\left\{ M \int_0^{\frac{1}{\omega}} u(x)^s dx \left\{ \int_0^{\omega} v(t)^{-r} dt \right\}^s \right\}^{\frac{1}{s}} \leq C \left\{ \int_0^{\omega} v(t)^{-r} dt \right\}^{\frac{1}{r}}, \quad \omega > 0.$$

Thus we conclude that $(u, v) \in F_{r,s}$. ■

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