Internat. J. Math. & Math. Sci.
VOL. 20 NO. 4 (1997) 681-688

LOCAL CONNECTIVITY AND MAPS ONTO NON-METRIZABLE ARCS

J. NIKIEL

American University of Beirut
Beirut, Lebanon
nikiel@layla.aub.ac.lb

and

L.B. TREYBIG

Texas A & M University
College Station, Texas 77843-3368, USA
treybig@math.tamu.edu

and

H.M. TUNCALI

Nipissing University
North Bay, Ontario, Canada P1B 8L7
muratt@einstein.unipissing.ca

(Received December 15, 1995 and in revised form July 6, 1996)

ABSTRACT. Three classes of locally connected continua which admit sufficiently many maps onto non-metric arcs are investigated. It is proved that all continua in those classes are continuous images of arcs and, therefore, have other quite nice properties.

KEY WORDS AND PHRASES: arc, locally connected continuum, monotonically normal, rim-countable, rim-finite, rim-metrizable, rim-scattered

1991 AMS SUBJECT CLASSIFICATION CODES: Primary 54F15, Secondary 54C05 54F05

INTRODUCTION

Let $\mathcal{C}$ denote the class of all Hausdorff continuous images of ordered continua. In the last three decades the class $\mathcal{C}$ has been studied extensively by a number of authors (see e.g. [2], [4], [6–8], [11–13], [16–22], [26] and [27]). Two results from this study have suggested that the investigation could naturally be extended to the larger class $\mathcal{R}_M$ of all rim-metrizable, locally connected continua. Namely, (1) in [8] in 1967 Mardešić proved that each element of $\mathcal{C}$ has a basis of open F_σ-sets with metrizable boundaries, and (2) in [4] in 1991 Grispolakis, Nikiel, Simone and Tymchatyn showed that if a set P is irreducible with respect to the property of being a compact set which separates the element X of $\mathcal{C}$, then P is metrizable.

In his 1989 thesis [23] and two subsequent papers [24] and [25] Tuncali began an investigation of the class $\mathcal{R}_M$ and continuous images of elements of that class. He showed that Treybig's product theorem of [18] which holds in $\mathcal{C}$ is no longer valid in $\mathcal{R}_M$. However, he proved that Mardešić's theorem for $\mathcal{C}$ on preservation of weight by light mappings is true in $\mathcal{R}_M$, [25]. He also considered the class $\mathcal{R}_S$ of all rim-scattered, locally connected continua, and the class $\mathcal{R}_C$ of all rim-countable, locally connected continua. Later, Nikiel, Tuncali and Tymchatyn gave an example to show that $\mathcal{R}_C$ is not a subclass of $\mathcal{C}$, [15]. Then, recently the authors of this paper showed the the continuous image of an element of $\mathcal{R}_M$ need not be in $\mathcal{R}_M$, [14]. Furthermore, Drozdovsky and Filippov proved in [3] that $\mathcal{R}_S$ is a larger class of spaces than $\mathcal{R}_C$.

Also, in 1973 Heath, Lutzer and Zenor, [5], showed that every linearly ordered ordered topological space and each of its Hausdorff continuous and closed images are monotonically normal. In [10] in 1986 Nikiel asked if every monotonically normal compactum is the continuous image of a compact ordered space. That problem still remains open. In what follows we let $\mathcal{R}_{MN}$ denote the class of monotonically normal, locally connected continua. Our first result is the following:

THEOREM 1. If $X \in \mathcal{R}_M \cup \mathcal{R}_S \cup \mathcal{R}_{MN}$ and for each pair of points $a, b \in X$ there exists a continuous onto map $f : X \to [c,d]$ such that $f(a) = c$, $f(b) = d$ and $[c,d]$ is a non-metrizable arc, then $X \in \mathcal{C}$.

We note that a large class of examples satisfying the properties of X above can be constructed as follows: In [1] in 1945 Arens studied the class $\mathcal{L}$ of linear homogeneous continua, that is the class of arcs which are order isomorphic to each of their subarcs. Arens showed, that up to a homeomorphism, there exist at least $\aleph_1$ members of $\mathcal{L}$, including the real numbers interval $[0,1]$. Thus, some spaces X as in Theorem 1 could be obtained by pasting together copies of any $Z \in \mathcal{L}$.

If a subset B of a space P contains no dense–in–itself, non-empty subset, we say that B is *scattered.*

In this paper the definition of monotone normality we use is an equivalent one given in Lemma 2.2 (a) of [5]. It says that a space P is *monotonically normal* provided there is an operator G which assigns to each ordered pair (S,T) of mutually separated subsets of P an open set $G(S,T)$ such that

(i) $S \subset G(S,T) \subset \mathrm{cl}(G(S,T)) \subset P - T$, and

(ii) if (S',T') is also a pair of mutually separated sets such that $S \subset S'$ and $T' \subset T$, then $G(S,T) \subset G(S',T')$.

PROOF OF THEOREM 1. Suppose that X is not hereditarily locally connected. Then, there exists a subcontinuum C of X such that C fails to be connected im kleinen at the point p. Utilizing the ideas in Theorem 11, p. 90, of Moore [9], there exists a connected open set U containing p, a sequence $R_1, R_2, R_3, \ldots$ of connected open in X sets containing p, and a sequence

$G_1, G_2, G_3, \ldots$ of continua such that

(1) $U \supset \overline{R_1} \supset R_1 \supset \overline{R_2} \supset R_2 \supset \ldots$;

(2) $G_n \cap R_n \neq \emptyset$ and $G_n \cap R_{n+1} = \emptyset$ for $n = 1, 2, 3, \ldots$;

(3) each G_n is a component of $\overline{U} \cap C$ and $G_n \cap \mathrm{bd}(U) \neq \emptyset$ for $n = 1, 2, 3, \ldots$; and

(4) $G_n \cap G_m = \emptyset$ if $n \neq m$, and there exist mutually exclusive open sets $V_1, V_2, V_3, \ldots$ such that $G_n \subset V_n$ for $n = 1, 2, 3, \ldots$

For each positive integer n let H_n be a component of $G_n - R_1$ which intersects $\mathrm{bd}(R_1)$ and $\mathrm{bd}(U)$, and let $s_n \in H_n \cap \mathrm{bd}(R_1)$ and $t_n \in H_n \cap \mathrm{bd}(U)$. Let H_0 denote the limiting set of the sequence $H_1, H_2, H_3, \ldots$; which by definition is the set of all x such that every open set containing x intersects infinitely many sets H_n.

Let L_1 (resp. L_2) denote the limiting set of $\{s_1\}, \{s_2\}, \{s_3\}, \ldots$ (resp. $\{t_1\}, \{t_2\}, \{t_3\}, \ldots$). There exists $(s, t) \in L_1 \times L_2$ so that if V is a neighborhood of s and W is a neighborhood of t, then (s_n, t_n) belongs to $V \times W$ for infinitely many n.

We shall show that some component of H_0 contains $\{s, t\}$. If not, then H_0 is the union of two mutually separated sets S and T such that $s \in S$ and $t \in T$. There exist disjoint open sets V and W so that $S \subset V$ and $T \subset W$. Then (s_n, t_n) belongs to $V \times W$ for infinitely many n. Since each H_n is a continuum, $H_n \cap (X - (V \cup W)) \neq \emptyset$ for infinitely many n. It follows that some point of H_0 lies in $X - (V \cup W)$, a contradiction.

Let $f : X \to [c, d]$ be a continuous map onto a non-metrizable arc $[c, d]$, where $f(s) = c$ and $f(t) = d$. There is an increasing sequence $n_1, n_2, n_3, \ldots$ of positive integers such that

(1) $f(s_{n_i}) \geq f(s_{n_{i+1}})$ and $f(t_{n_i}) \leq f(t_{n_{i+1}})$ for $i = 1, 2, \ldots$;

(2) $f(s_{n_i}) \to c$ and $f(t_{n_i}) \to d$; and

(3) $[f(s_{n_i}), f(t_{n_i})]$ is not metrizable for $i = 1, 2, \ldots$

Let $c' = f(s_{n_1})$ and $d' = f(t_{n_1})$.

Our proof now divides into three cases.

CASE 1. $X \in \mathcal{R}_M$. For each $n \geq 2$ let M_n denote a metrizable closed set lying in $X - \bigcup_{k=1}^{n} H_k$ such that if $1 \leq i < j \leq n$, then H_i and H_j are separated in X by M_n. Let D_n denote a countable set dense in M_n for $n = 2, 3, \ldots$ We intend to show that $f\left(\bigcup_{k=2}^{\infty} D_k\right)$ is dense in $[c, d]$, which would mean that $[c, d]$ is separable, and therefore metric, a contradiction.

Let $x \in]c, d[$ and let $c < u < x < v < d$ in the natural ordering of $[c, d]$. The components of $f^{-1}(]u, v[)$ which have limit points in both $f^{-1}(u)$ and $f^{-1}(v)$ can be labeled $P_1, P_2, \ldots, P_{n_0}$. Let N_0 be an integer such that if $i \geq N_0$ then $s_{n_i} \in f^{-1}([c, u[)$ and $t_{n_i} \in f^{-1}(]v, d])$. There exist two of $N_0, N_0 + 1, \ldots, N_0 + n_0$, say i and j, such that H_{n_i} and H_{n_j} both intersect the same P_ℓ, which must then intersect some D_m. Therefore, $\bigcup_{k=2}^{\infty} f(D_k)$ intersects $]u, v[$.

CASE 2. $X \in \mathcal{R}_{MN}$. For each $i = 1, 2, \ldots$ let Q_i denote a component of $H_{n_i} \cap f^{-1}([c', d'])$ which intersects $f^{-1}(c')$ and $f^{-1}(d')$, and let Q_0 denote the limiting set of $Q_1, Q_2, Q_3, \ldots$ We note that some component of Q_0 intersects both $f^{-1}(c')$ and $f^{-1}(d')$ since every map onto an arc is weakly confluent.

By Remark 2.3 (c) of [5], $Z = \bigcup_{n=0}^{\infty} Q_n$ is monotonically normal; so let $\mathcal{G}$ be a monotone normality operator on Z as in the earlier definition. For each closed set F in $[c', d']$ let $Q_F = \{x : f(x) \in F \text{ and } x \in Z - Q_0\}$, and let $R_F = \{x : f(x) \in [c', d'] - F \text{ and } x \in Q_0\}$. Now, Q_F and R_F are mutually separated subsets of Z; so for each positive integer n, let $T(F, n) = \{y \in [c', d'] : y = f(x) \text{ for some } x \in Q_n \cap \mathcal{G}(Q_F, R_F)\}$. It can be shown that T is a stratification for $[c', d']$. Since each stratifiable compact space is metrizable, $[c', d']$ is metrizable, a contradiction.

CASE 3. $X \in \mathcal{R}_S$. For each $i = 1, 2, 3, \ldots$ let K_i denote a component of $H_{n_i} \cap f^{-1}([c', d'])$ which intersects $f^{-1}(c')$ and $f^{-1}(d')$.

We have to consider some subcases.

CASE 3A. $[c', d']$ contains uncountably many mutually exclusive open sets.

CASE 3A$_1$. $[c', d']$ does not satisfy the first axiom of countability. Thus, without loss of generality, assume that there is a subset $\{d_\alpha : \alpha < \omega_1\}$ of $[c', d']$ such that $\alpha_1 < \alpha_2$ implies that $d_{\alpha_1} < d_{\alpha_2}$ in $[c', d']$, and $d_\alpha \to d'$.

Let K_0 denote the limiting set of $K_1, K_2, K_3, \ldots$ Let Q denote a component of K_0 which intersects both $f^{-1}(c')$ and $f^{-1}(d')$. For each $\alpha < \omega_1$ let W_α denote a connected open set such that W_α contains a point x_α of $Q \cap f^{-1}(]d_\alpha, d_{\alpha+1}[)$, and $\overline{W_\alpha} \subset f^{-1}(]d_\alpha, d_{\alpha+1}[)$.

There exists a positive integer n_0 and a cofinal subsequence $\{d_{\alpha_\beta}\}$ of d_α such that $W_{\alpha_\beta} \cap K_{n_0} \neq \emptyset$ for all α_β. For each $\gamma < \omega_1$ let L_γ denote the closure of the set $\bigcup_{\beta \geq \gamma} W_{\alpha_\beta}$. Let $L = \bigcap_{\gamma < \omega_1} L_\gamma$. Observe that if $y \in L$, then each open neighborhood of y intersects uncountably many sets W_{α_β}. Let W be a component of L. Note that $W \cap K_{n_0} \neq \emptyset \neq Q \cap W$ and $W \subset f^{-1}(d')$. Thus, W is a non-degenerate continuum.

Let M_0 and M_1 be connected open sets such that $\overline{M_0} \cap \overline{M_1} = \emptyset$ and $M_i \cap W \neq \emptyset$ for $i = 0, 1$. Let $\mathcal{G}_1 = \{M_0, M_1\}$.

Now suppose that $\mathcal{G}_n$ has been chosen and consists of 2^n mutually exclusive connected open sets such that if $G, G' \in \mathcal{G}_n$ and $G \neq G'$, then $\overline{G} \cap \overline{G'} = \emptyset$ and $G \cap W \neq \emptyset \neq G' \cap W$. For each $G' \in \mathcal{G}_n$ let G'_0 and G'_1 be mutually exclusive connected open sets such that $\overline{G'_0} \cap \overline{G'_1} = \emptyset$, $\overline{G'_0} \cup \overline{G'_1} \subset G'$ and $G'_0 \cap W \neq \emptyset \neq G'_1 \cap W$. Let $\mathcal{G}_{n+1} = \{F : F = G'_0 \text{ or } F = G'_1 \text{ for some } G' \in \mathcal{G}_n\}$. For each n let $H'_n = \bigcup \mathcal{G}_n$ and let $H = \bigcap_{n=1}^{\infty} H'_n$.

There exists $\delta_0 < \omega_1$ such that $G' \cap f^{-1}(d_{\delta_0}) \neq \emptyset$ for each $G' \in \bigcup_{j=1}^{\infty} \mathcal{G}_j$. There exists a closed scattered set S in X which separates $f^{-1}([c, d_{\delta_0}])$ from $f^{-1}(d')$. However, $S \cap H$ contains a perfect set because $S \cap H$ can be mapped onto a Cantor set, and it is well known that a scattered set cannot be mapped continuously onto a perfect set. This is a contradiction.

CASE 3A$_2$. $[c', d']$ satisfies the first axiom of countability at each point. Let $\{]c_\alpha, d_\alpha[: \alpha < \omega_1\}$ denote an uncountable collection of mutually exclusive open intervals in $]c', d'[$. Using the local connectivity of X we find that for each α there exists only a finite number, say n_α, of components of $f^{-1}(]c_\alpha, d_\alpha[)$ which have limit points in both $f^{-1}(c_\alpha)$ and $f^{-1}(d_\alpha)$. Some integer $N_0 = n_\alpha$ repeats for uncountably many α's; so we may suppose without loss of generality that

$n_\alpha = N_0$ for each $\alpha < \omega_1$.

There exists a closed scattered set S such that S separates K_i from K_j for each pair i, j such that $1 \leq i < j \leq N_0 + 1$. Thus, since for each α, each set K_i where $1 \leq i \leq N_0 + 1$ has the property that some component of $K_i \cap f^{-1}(]c_\alpha, d_\alpha[)$ has limit points in both $f^{-1}(c_\alpha)$ and $f^{-1}(d_\alpha)$, it follows that S must intersect each $f^{-1}(]c_\alpha, d_\alpha[)$.

Since $[c', d']$ is first countable, there exist collections $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots$ such that (1) each $\mathcal{G}_n$ consists of 2^n mutually exclusive closed intervals in $[c', d']$, and (2) each element of each $\mathcal{G}_n$ contains exactly two elements of $\mathcal{G}_{n+1}$ and contains uncountably many elements of $\{]c_\alpha, d_\alpha[: \alpha < \omega_1\}$.

For each positive integer n let $L'_n = \bigcup \mathcal{G}_n$, and let $L' = \bigcap_{n=1}^\infty L'_n$. We find that $S \cap f^{-1}(L')$ contains a perfect set, a contradiction.

CASE 3B. $[c', d']$ is not metrizable and does not contain uncountably many mutually exclusive open sets (i.e., it is a Souslin line). Thus, $[c', d']$ satisfies the first axiom of countability. If there exists a collection of metrizable open intervals whose union is dense in $[c', d']$, we find that $[c', d']$ is metrizable since it is separable. Hence, without loss of generality we may assume that $[c', d']$ contains no metrizable subinterval.

Similarly as above, for each $]x, y[\subset [c', d']$ we let n_{xy} denote the number of components of $f^{-1}(]x, y[)$ with limit points in both $f^{-1}(x)$ and $f^{-1}(y)$.

CASE 3B$_1$. Suppose there exists a positive integer N_0 and a subinterval $]x, y[$ of $[c', d']$ such that if $x \leq z < w \leq y$, then $n_{zw} \leq N_0$. Let S be a closed scattered set such that if $1 \leq i < j \leq N_0 + 1$, then S separates K_i from K_j. Using the ideas from Case 3A$_2$ we find that if $x \leq z < w \leq y$, then $S \cap f^{-1}(]z, w[) \neq \emptyset$. Therefore, $f(S) \supset [x, y]$, which contradicts the well-known fact that a scattered compactum can not be mapped onto a perfect set.

CASE 3B$_2$. Assume that for every $]x, y[\subset [c', d']$ there exists an interval $]z, w[\subset]x, y[$ such that $n_{zw} > n_{xy}$.

For each positive integer n let $\mathcal{G}_n$ be maximal relative to the property of being a collection of mutually exclusive open intervals lying in $[c', d']$ such that if $]x, y[\in \mathcal{G}_n$ then $n_{xy} = n$. Note that each $\mathcal{G}_n$ is at most countable. Let S_n denote the set of all end-points of intervals which belong to $\mathcal{G}_n$. We are going to show that $\bigcup_{n=1}^\infty S_n$ is dense in $[c', d']$, and thus obtain a contradiction.

Let $]x, y[\subset [c', d']$. There exists $]z, w[\subset]x, y[$ such that $n_{zw} > n_{xy}$. Thus, $x \neq z$ or $y \neq w$. By maximality of $\mathcal{G}_{n_{zw}}$, there exists $]s, t[\in \mathcal{G}_{n_{zw}}$ such that $]s, t[\cap]z, w[\neq \emptyset$. But $]s, t[\not\supset]x, y[$, and so $s \in]x, y[$ or $t \in]x, y[$. Therefore, the set $\bigcup_{n=1}^\infty S_n$ is dense in $[c', d']$, a contradiction.

The consideration of subcases 1, 2 and 3 is concluded and we return now to the main proof. Since X is hereditarily locally connected, it is the continuous image of an arc by [12].

THEOREM 2. If X is as in Theorem 1, then

(a) X is rim-finite,

(b) every subcontinuum G of X has the property that some point or a pair of points separates G, and

(c) each closed set irreducible with respect to the property of being a compact set which separates X is metrizable.

PROOF. The claims (a), (b) and (c) follow from [19], [18] and [4], respectively, because X contains no non-degenerate metric continuum.

Given a locally connected continuum X, for each pair of distinct points a, b of X let $[X, a, b]$ denote the class of all continuous maps $f : X \to P$ such that $P = f(X)$ is a non-metric arc with end-points c and d and $f(a) = c$ and $f(b) = d$. Also, introduce a relation $\sim$ on X in the following way: $a \sim b$ if and only if $a = b$ or $[X, a, b] = \emptyset$.

THEOREM 3. Suppose that X is a locally connected continuum. Then $\sim$ is an equivalence relation on X, and if X also satisfies the first axiom of countability, then equivalence classes of $\sim$ are closed and the set $\mathcal{E}$ of equivalence classes of $\sim$ is upper semi-continuous.

PROOF. $\sim$ is easily seen to be reflexive and symmetric, so suppose that $a \sim b$ and $b \sim c$ hold, but that there exists $f \in [X, a, c]$ such that $f(X)$ is a non-metric arc $[d, e]$ with $f(a) = d$ and $f(c) = e$.

CASE 1. $f(b) = d$. Then $f \in [X, b, c]$, a contradiction.

CASE 2. $f(b) = e$ - analogous to Case 1.

CASE 3. $d < f(b) < e$. Then one of the arcs $[d, f(b)]$ and $[f(b), e]$ is non-metric, so suppose $[d, f(b)]$ is non-metric. Define $r : [d, e] \to [d, f(b)]$ so that $r(x) = x$ if $x \in [d, f(b)]$ and $r(x) = f(b)$ if $x \in [f(b), e]$. Clearly, $r \circ f \in [X, a, b]$, a contradiction.

Let us now show that each equivalence class $G \in \mathcal{E}$ is closed if X is first countable. Let $G \in \mathcal{E}$ and suppose that $x \in \overline{G} - G$. There exists a countable basis $U_1, U_2, \ldots$ of open neighborhoods of x in X and a sequence $x_1, x_2, \ldots$ of points of G such that $x_i \in U_i$ for $i = 1, 2, \ldots$ Let $f : X \to [c, d]$ be a continuous map onto a non-metric arc $[c, d]$, where $f(x_1) = c$ and $f(x) = d$. Since each $[f(x_1), f(x_i)]$ is a metric subarc of $[c, d]$, it follows that $[c, d]$ is the closure of a countable union of metric arcs. Consequently, $[c, d]$ is separable, and therefore metrizable, a contradiction. Thus G is closed in X.

It remains to show that $\mathcal{E}$ is upper semi-continuous if X is first countable. Let the element G of $\mathcal{E}$ be a subset of an open set U. Suppose that for each open set V such that $G \subset V \subset U$, there is an element G_V of $\mathcal{E}$ so that $V \cap G_V \neq \emptyset$ and $G_V \not\subset U$. Thus, for some point x of G there is a countable basis $U_1, U_2, \ldots$ of open neighborhoods of x such that for each U_i there is an element G_i of $\mathcal{E}$ with the property that $G_i \cap U_i \neq \emptyset \neq G_i \cap (X - U)$.

There is a point y of $X - U$ so that every neighborhood of y intersects G_i for infinitely many i. We may assume without loss of generality that there exists $y_i \in G_i \cap (X - U)$ for each i, and that the points y_i converge to y. Let $z_i \in U_i \cap G_i$ for $i = 1, 2, \ldots$

There exists $f \in [X, x, y]$ such that $f : X \to [c, d]$, where $[c, d]$ is a non-metric arc, $f(x) = c$ and $f(y) = d$. Since the points $f(z_i)$ converge to c, and the points $f(y_i)$ converge to d, and each arc $[f(z_i), f(y_i)]$ is metric, we find that $[c, d]$ is metric - a contradiction.

THEOREM 4. Suppose that $X \in \mathcal{R}_M \cup \mathcal{R}_S \cup \mathcal{R}_{MN}$ and X is first countable. Let $\mathcal{H}$ be the family of all components of sets in $\mathcal{E}$. Then $X/\mathcal{H}$ is the continuous image of an arc.

PROOF. Since $\mathcal{E}$ is upper semi-continuous, $\mathcal{H}$ is upper semi-continuous as well (see e.g. [28]). Thus, $\mathcal{H}$ is an upper semi-continuous decomposition of X into closed sets and the quotient space $X/\mathcal{H}$ is a locally connected continuum.

If $X/\mathcal{H}$ is hereditarily locally connected, we apply the main result of [12] to obtain the desired conclusion.

Otherwise, in $X/\mathcal{H}$ there is a subcontinuum C such that C fails to be connected im kleinen at a point P. There is thus an open set W in $X/\mathcal{H}$ such that $P \in W$ but the component of $W \cap C$ containing P contains no relatively open subset of C containing P. Let Q denote the element of $\mathcal{E}$ containing P. There is a closed subset S of X such that $S \subset \bigcup W - Q$ and S separates P from $\mathrm{bd}(\bigcup W)$ in X. Let $\phi : X \to X/\mathcal{H}$ denote the natural map and let $B = \phi(S)$. Let U denote the component of $X/\mathcal{H} - B$ which contains P. Using the facts that $\sim$ is upper semi-continuous and that $Q \cap S = \emptyset$, we let $R_1, R_2, \ldots; G_1, G_2, \ldots; V_1, V_2 \ldots$ be subsets of $X/\mathcal{H}$ similarly as in the proof of Theorem 1, except for the additional condition that no element of $\mathcal{E}$ intersects $\mathrm{cl}(\bigcup R_1)$ and $\mathrm{bd}(\bigcup U)$.

Now, let $s_1, s_2, \ldots$ and $t_1, t_2, \ldots$ be such that $s_i \in (\bigcup G_i) \cap (\bigcup \mathrm{bd}(R_1))$ and $t_i \in (\bigcup G_i) \cap (\bigcup \mathrm{bd}(U))$ for $i = 1, 2, \ldots$ Since X is first countable, we may assume without loss of generality that the points s_i converge to some point s, and the points t_i converge to some point t, and the limiting set L of $\bigcup G_1,\ \bigcup G_2,\ \bigcup G_3,\ \ldots$ is a continuum containing s and t.

There is an $f \in [X, s, t]$ such that $f(X)$ is a non-metric arc $[c, d]$ with $f(s) = c$ and $f(t) = d$. We may now obtain a contradiction as in the proof of Theorem 1.

ACKNOWLEDGMENT. H.M. Tuncali was partially supported by an NSREC grant.

REFERENCES

[1] Arens, R., "On the construction of linear homogeneous continua," *Bol. Soc. Mat. Mexicana* **2** (1945), 33-36.

[2] Cornette, J.L., "Image of a Hausdorff arc is cyclically extensible and reducible," *Trans. Amer. Math. Soc.* **199** (1974), 255-267.

[3] Drozdovsky S.A. and Filippov, V.V., "An example of a rim-scattered locally connected continuum which is not rim-countable (in Russian)," *Mat. Sbornik* **185**, No. 10 (1994), 27-38.

[4] Grispolakis, J., Nikiel, J., Simone J.N. and Tymchatyn, E.D., "Separators in continuous images of ordered continua and hereditarily locally connected continua," *Canadian Math. Bull.* **36** (1993), 154-163.

[5] Heath, R.W., Lutzer D.J. and Zenor, P.L., "Monotonically normal spaces," *Trans. Amer. Math. Soc.* **178** (1973), 481-493.

[6] Mardešić, S., "On the Hahn-Mazurkiewicz theorem in non-metric spaces," *Proc. Amer. Math. Soc.* **11** (1960), 929-937.

[7] Mardešić, S., "Mapping ordered continua onto product spaces," *Glasnik Mat.* **15** (1960), 85-89.

[8] Mardešić, S., "Images of ordered compacta are locally peripherally metric," *Pacific J. Math.* **23** (1967), 557-568.

[9] Moore, R.L., "Foundations of point set topology" (rev. ed.), *Amer. Math. Soc. Colloquium Publications*, **vol. 13**, Providence, RI, 1962.

[10] Nikiel, J., "Some problems on continuous images of compact ordered spaces," *Questions Answers Gen. Topology* **4** (1986/87), 117-128.

[11] Nikiel, J., "Images of arcs - a nonseparable version of the Hahn-Mazurkiewicz theorem," *Fund. Math.* **129** (1988), 91-120.

[12] Nikiel, J., "The Hahn-Mazurkiewicz theorem for hereditarily locally connected continua," *Topology Appl.* **32** (1989), 307-323.

[13] Nikiel, J., "On continuous images of arcs and compact orderable spaces," *Topology Proc.* **14** (1989), 163-193 and 279-280.

[14] Nikiel, J., Treybig, L.B. and Tuncali, H.M., "A rim-metrizable continuum," *Proc. Amer. Math. Soc.* **123** (1995), 281-286.

[15] Nikiel, J., Tuncali, H.M. and Tymchatyn, E.D., "A locally connected rim-countable continuum which is the continuous image of no arc," *Topology Appl.* **42** (1991), 83-93.

[16] Pearson, B.J., "Mapping an arc onto a dendritic continuum," *Colloq. Math.* **30** (1974), 237-243.

[17] Simone, J.N., "Metric components of continuous images of ordered compacta," *Pacific J. Math.* **69** (1977), 269-274.

[18] Treybig, L.B., "Concerning continuous images of compact ordered spaces," *Proc. Amer. Math. Soc.* **15** (1964), 866-871.

[19] Treybig, L.B., "Concerning continua which are continuous images of compact ordered spaces," *Duke Math. J.* **32** (1965), 417-422.

[20] Treybig, L.B., "Separation by finite sets in connected, continuous images of ordered compacta," *Proc. Amer. Math. Soc.* **74** (1979), 326-328.

[21] Treybig, L.B., "A characterization of spaces that are the continuous image of an arc," *Topology Appl.* **24** (1986), 229-239.

[22] Treybig, L.B. and Ward, Jr., L.E., "The Hahn-Mazurkiewicz problem," in: Topology and order structures I, *Math. Centre Tracts*, **vol. 142**, Amsterdam, 1981, p. 95-105.

[23] Tuncali, H.M., "Some generalizations of the Hahn-Mazurkiewicz theorem," Ph.D. Thesis, University of Saskatchewan, Saskatoon, 1989.

[24] Tuncali, H.M., "Analogues of Treybig's product theorem," *Proc. Amer. Math. Soc.* **108** (1990), 855-858.

[25] Tuncali, H.M., "Concerning continuous images of rim-metrizable continua," *Proc. Amer. Math. Soc.* **113** (1991), 461-470.

[26] Ward, Jr., L.E., "A generalization of the Hahn-Mazurkiewicz theorem," *Proc. Amer. Math. Soc.* **58** (1976), 369-374.

[27] Ward, A.J., "Some properties of images of ordered compacta with special reference to topological limits," unpublished.

[28] Whyburn, G.T., "Analytic topology," *Amer. Math. Soc. Colloquium Publications*, **vol. 28**, Providence, RI, 1942.