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Research Article

Periodic Solutions of Evolution m-Laplacian Equations with a Nonlinear Convection Term

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We show the existence and gradient estimates of periodic solutions in the case of $0 \leq \alpha < m+1$ to the evolution m-Laplacian equations of form $u_t - \operatorname{div}\{|\nabla \mathrm{u}|^m \nabla u\} + \mathbf{b}(u)\cdot\nabla u = f(t)u^\alpha + h(x,t)$, in $\Omega \times \mathbb{R}^1$ with the Dirichlet boundary value condition.

1. Introduction and main results

In this paper, we are concerned with the existence and gradient estimates for periodic solutions of the evolution m-Laplacian equations with a nonlinear convection term and with the Dirichlet boundary value condition

$$\begin{gathered} u_t - \operatorname{div}\left\{|\nabla u|^m \nabla u\right\} + \mathbf{b}(u)\cdot\nabla u = f(t)u^\alpha + h(x,t), \quad \text{in } \Omega \times \mathbb{R}^1, \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^1, \\ u(x,t+\omega) = u(x,t), \quad \text{in } \Omega \times \mathbb{R}^1, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\omega > 0$, $m > 1$, and $\mathbf{b}(u)$ is a nonlinear vector field such that $|\mathbf{b}(u)| \leq k|u|^\beta$, with some $k > 0$, $0 \leq \beta \leq m-1$. $f(t)$ and $h(x,t)$ are ω-periodic (in t) functions.

Equation (1.1) is a class of degenerate parabolic equations and appears to be relevant in the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations, see [1, 2] for instance. The term $\mathbf{b}(u)\cdot\nabla u$ describes an effect of convection with a velocity field $\mathbf{b}(u)$.

In the last two decades, periodic parabolic equations have been the subject of extensive study (see [3–11]). In Particular, Nakao [7] considered the following equation:

$$u_t - \Delta\beta(u) + B(x,t,u) = f(x,t), \tag{1.2}$$

where B and f are periodic in t with common period $\omega > 0$, $\beta(u)$ satisfies $\beta'(u) > 0$ except for $u = 0$ and $B(x,t,u)u \geq -b_0|u|$ with some constant $b_0 \geq 0$. The existence and L^∞ estimates of periodic solutions were established.

When $\mathbf{b}(u) = \mathbf{0}$ and $f(t)u^\alpha$ replaced by $g(x,u)$ with $g(x,u)u \leq k_0|u|^{\beta+1} + k_1|u|, 0 \leq \beta \leq m+1$, Nakao and Ohara [8] obtained the existence and $\|\nabla u(t)\|_\infty$ estimate of periodic solutions of (1.1).

For $\mathbf{b}(u) = \mathbf{0}$ and $h(x,t) = 0$, applying the topological degree theory, Wang et al. [9] discussed the existence of periodic solutions of (1.1) in the case of strongly nonlinear sources $(m+1 < \alpha < m+1+(m+2)/N)$.

The object of this paper is to prove the existence of periodic solutions in the case of $0 \leq \alpha < m+1$ and to derive an estimates of $\|\nabla u(t)\|_\infty$ for the problem (1.1). For the proof of our result, we employ Moser's technique as in [12] and make some devices as in [8] to obtain the existence of periodic solutions. Leray-Shauder fixed point theorem instead of approximate method used in [8] is applied to prove the existence of periodic solutions. To derive estimates of $\nabla u(t)$, we must treat the terms $\mathbf{b}(u) \cdot \nabla u$ and $f(t)u^\alpha$ at the same time very carefully. To our best knowledge, this result is not found in others work.

Let $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ denote $L^p = L^p(\Omega)$ and $W^{m,p} = W^{m,p}(\Omega)$ norms, respectively, $1 \leq p \leq \infty$.

Due to the degeneracy of the equations considered, problem (1.1) has no classical solutions in general, and thus we consider its weak solutions in the following sense.

Definition 1.1. Assume that $h(x,t) \in E = C_\omega(\overline{Q})$, the set of all functions in $C(\overline{\Omega} \times \mathbb{R}^1)$ which are periodic in t with period ω, where $Q = \Omega \times (0,\omega)$. A function u is said to be a periodic solution of problem (1.1) if

$$u \in L^{m+2}(0,\omega; W_0^{1,m+2}(\Omega)) \cap C_\omega(\overline{Q}), \tag{1.3}$$

and u satisfies

$$\iint_Q \{-u\,\varphi_t + |\nabla u|^m \nabla u \cdot \nabla\varphi - \mathbf{B}(u) \cdot \nabla\varphi - f(t)u^\alpha\varphi - h(x,t)\varphi\}\,dx\,dt = 0 \tag{1.4}$$

for any $\varphi \in C_0^1(0,\omega; C_0^1(\Omega))$ with $\varphi(x,0) = \varphi(x,\omega)$, where $\mathbf{B}(u) = \int_0^u \mathbf{b}(s)ds$ is set.

We assume

(H1) $\mathbf{b}(u) = (b_1(u), b_2(u), \ldots, b_N(u))$ is an $\mathbb{R}^N$-valued function on $\mathbb{R}^1$, satisfying

$$|\mathbf{b}(u)| \leq k|u|^\beta \tag{1.5}$$

for some $0 \leq \beta < m-1$ and $k > 0$, or $\beta = m-1$, and $k > 0$ is sufficiently small.

(H2) $h(x,t) \in C_\omega(\overline{Q}) \cap L^\infty(0,\omega; W_0^{1,\infty}(\Omega))$, $h(x,t) > 0$ for $(x,t) \in \Omega \times \mathbb{R}^1$ and we set $M_0 = \sup_t \|h(t)\|_\infty$, $M_1 = \sup_t \|\nabla h(t)\|_\infty$.

(H3) $f(t) \in L^\infty(0,\omega)$ is periodic in t with period ω. We also assume that $0 \leq \alpha < m+1$.

(H4) $\partial\Omega$ is of C^2 class and the mean curvature $H(x)$ at $x \in \partial\Omega$ is nonpositive with respect to the outward normal.

Remark 1.2. (H4) is satisfied in particular if Ω is convex. Without (H4), we cannot control the boundary integral which appears in the estimation of $\|\nabla u(t)\|_\infty$.

Our main results of this paper read as follows.

Theorem 1.3. *Under the assumptions (H1)–(H3), $N > 1$, problem (1.1) admits at least one solution $u(t)$, which satisfies*

$$u(t) \in L^\infty(0,\omega; W_0^{1,m+2}(\Omega)) \cap C_\omega(\overline{Q}), \quad u_t \in L^2(Q). \tag{1.6}$$

Theorem 1.4. *Under the assumptions (H1)–(H4), the solution $u(t)$ of problem (1.1) further belongs to $L^\infty(0,\omega; W_0^{1,\infty}(\Omega))$, and satisfies*

$$\sup_t \|\nabla u(t)\|_\infty \leq C_1 < \infty, \tag{1.7}$$

where C_1 is a constant, depending on M_0, M_1, and α.

For the proof of theorems, we use the following lemmas.

Lemma 1.5 [12] (Gagliardo-Nirenberg). *Let $\beta \geq 0$, $N > p \geq 1$, $\beta + 1 \leq q$, and $1 \leq r \leq q \leq (\beta+1)Np/(N-p)$, then for u such that $|u|^\beta u \in W^{1,p}(\Omega)$,*

$$\|u\|_q \leq C^{1/(\beta+1)} \|u\|_r^{1-\theta} \| \mid u \mid^\beta u\|_{1,p}^{\theta/(\beta+1)} \tag{1.8}$$

with $\theta = (\beta+1)(r^{-1} - q^{-1})/\{N^{-1} - p^{-1} + (\beta+1)r^{-1}\}$, where C is a constant independent of q, r, β, and θ.

Lemma 1.6 [8]. *Let $y(t) \in C^1(\mathbb{R}^1)$ be a nonnegative ω periodic function satisfying the differential inequality*

$$y'(t) + Ay^{1+\alpha}(t) \leq By(t) + C, \quad t \in \mathbb{R}^1, \tag{1.9}$$

with some $\alpha > 0$, $A > 0$, $B \geq 0$, and $C \geq 0$. Then

$$y(t) \leq \max\{1, (A^{-1}(B+C))^{1/\alpha}\}. \tag{1.10}$$

The paper is organized as follows. Section 2 is devoted to the proof of the existence of periodic solutions for problem (1.1) by using the Leray-Shauder fixed point theorem, which is different from that adopted in [8, 9]. Subsequently, we present the proof of Theorem 1.4 in Section 3.

2. The proof of Theorem 1.3

Our result will be proved by means of parabolic regularization. Namely, we consider the regularized equations

$$u_t - \operatorname{div}\{(|\nabla u|^2 + \varepsilon)^{m/2} \nabla u\} + \mathbf{b}(u) \cdot \nabla u = f(t)u^\alpha + h(x,t), \quad (x,t) \in Q, \tag{2.1}$$

where $\varepsilon > 0$. The desired solution $u(t)$ of problem (1.1) will be obtained as a limit point of the approximate solutions $u_\varepsilon(t)$ of (2.1). To prove the existence of the approximate solutions $u_\varepsilon(t)$, we apply the Leray-Shauder fixed point theorem. For our purpose, we need the following a priori estimate.

Proposition 2.1. *Let u_0 be a periodic solution of the equation*

$$u_t - \operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\} + \mathbf{b}(u)\cdot\nabla u = \tau f(t)u^\alpha + \tau h(x,t), \quad (x,t)\in Q, \tag{2.2}$$

with $\tau\in[0,1]$, and u_0 satisfying the Dirichlet boundary value condition of (1.1). Then there exists a constant $C_0>0$ independent of τ and ε such that

$$||u_0(t)||_\infty \le C_0. \tag{2.3}$$

Proof. We only consider $N>m+2$, the other case can be treated similarly.

Multiplying (2.2) by $|u|^{p-2}u$ ($p>2$), integrating by parts, and noticing that

$$\begin{aligned}\int_\Omega \mathbf{b}(u)\cdot\nabla u|u|^{p-2}u\,dx &= \int_\Omega\sum_{i=1}^N b_i(u)|u|^{p-2}u\frac{\partial u}{\partial x_i}\,dx\\ &= \sum_{i=1}^N\int_\Omega\left(\int_0^u b_i(s)|s|^{p-2}s\,ds\right)_{x_i}dx\\ &= \sum_{i=1}^N\int_{\partial\Omega}\left(\int_0^u b_i(s)|s|^{p-2}s\,ds\right)\cos(\mathbf{n},x_i)\,ds\\ &= 0,\end{aligned} \tag{2.4}$$

we have

$$\frac{1}{p}\frac{d}{dt}||u(t)||_p^p+\varepsilon(p-1)\left(\frac{m+2}{p+m}\right)^{m+2}||\nabla u^{(p+m)/(m+2)}||_{m+2}^{m+2} \le C\big(f(t)\|u\|_{p+\alpha-1}^{p+\alpha-1}+\|h\|_p\|u\|_p^{p-1}\big). \tag{2.5}$$

If $1\le\alpha<m+1$, by Hölder's inequality and Lemma 1.5, we have

$$\begin{aligned}\|u\|_{p+\alpha-1}^{p+\alpha-1} &= \int_\Omega |u|^{\theta_1}|u|^{\theta_2}\,dx \le C\|u\|_p^{\theta_1}\|u\|_q^{\theta_2}\\ &\le C\|u\|_p^{\theta_1}||\nabla u^{(p+m)/(m+2)}||_{m+2}^{\theta_2(m+2)/(p+m)}\\ &\le \frac{\varepsilon}{2M_0}(p-1)\left(\frac{m+2}{p+m}\right)^{m+2}||\nabla u^{(p+m)/(m+2)}||_{m+2}^{m+2}+C\big(\|u\|_p^{\theta_1}\big)^{r/\theta_1}p^\sigma,\end{aligned} \tag{2.6}$$

where we set $q=(p+m)N/(N-m-2)$, $\theta_1=p[q-(p+\alpha-1)]/(q-p)$, $\theta_2=q(\alpha-1)/(q-p)$, $r<p$, which imply $\theta=1$ in Gagliardo-Nirenberg inequality, and $\sigma>0$ is a constant independent of p.

If $0<\alpha<1$, by Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\|u\|_{p+\alpha-1}^{p+\alpha-1} &= \int_\Omega |u|^{\alpha p}|u|^{(1-\alpha)(p-1)}\,dx \le \left(\int_\Omega |u|^p\,dx\right)^\alpha\left(\int_\Omega |u|^{p-1}\,dx\right)^{1-\alpha}\\ &\le \|u\|_p^p + C\|u\|_p^{p-1}.\end{aligned} \tag{2.7}$$

If $\alpha = 0$, then we use Hölder's inequality to obtain

$$\|u\|_{p+\alpha-1}^{p+\alpha-1} \le \left(\int_\Omega |u|^p dx\right)^{(p-1)/p} \left(\int_\Omega dx\right)^{1/p} \le \max\{1, |\Omega|^{1/2}\} \|u\|_p^{p-1}. \tag{2.8}$$

It follows from (2.5)–(2.8) that

$$\frac{d}{dt}||u(t)||_p^p + C_1 p^{-m} ||\nabla u^{(p+m)/(m+2)}||_{m+2}^{m+2} \le C(M_0)\left(p^{\sigma+1}||u(t)||_p^p + 1\right), \tag{2.9}$$

where we set

$$p_1 = m+2, \qquad p_n = (m+2)p_{n-1} - m, \qquad \alpha_n = (p_n + m)\theta_n^{-1} - p_n (> m),$$
$$\theta_n = \frac{1 - p_{n-1}p_n^{-1}}{1 + N^{-1} - (m+2)^{-1}} = \frac{N[(m+1)p_n - m]}{p_n[(m+1)N + m + 2]}. \tag{2.10}$$

By using Lemma 1.5, we have

$$\|u\|_{p_n} \le C\|u\|_{p_{n-1}}^{1-\theta_n} ||\nabla u^{(p_n+m)/(m+2)}||_{m+2}^{\theta_n(m+2)/(p_n+m)}. \tag{2.11}$$

Set $p = p_n$ in (2.9) and by (2.11), we have

$$\begin{aligned} &\frac{d}{dt}||u(t)||_{p_n}^{p_n} + C_1 C^{-(p_n+m)\theta_n^{-1}} p_n^{-m} ||u(t)||_{p_{n-1}}^{(p_n+m)(\theta_n-1)/\theta_n} ||u(t)||_{p_n}^{(p_n+m)/\theta_n} \\ &\quad \le C(M_0)\left(p_n^{\sigma+1}||u(t)||_{p_n}^{p_n} + 1\right). \end{aligned} \tag{2.12}$$

Therefore,

$$\frac{d}{dt}||u(t)||_{p_n} + C_1 C^{-(p_n+m)\theta_n^{-1}} p_n^{-m-1} ||u(t)||_{p_{n-1}}^{m-\alpha_n} ||u(t)||_{p_n}^{\alpha_n+1} \le C(M_0)\left(p_n^\sigma ||u(t)||_{p_n} + 1\right). \tag{2.13}$$

Let $\chi_n \equiv \sup_t \|u(t)\|_{p_n}$, by Lemma 1.6, we obtain

$$\chi_n \le \max\left\{1, \left(C(M_0) C^{(p_n+m)\theta_n^{-1}} p_n^{m+\sigma+1} \chi_{n-1}^{\alpha_n - m}\right)^{1/\alpha_n} \equiv B_n^{1/\alpha_n}\right\}. \tag{2.14}$$

We set without loss of generality that $B_n^{1/\alpha_n} > 1$, which implies $\chi_n \le B_n^{1/\alpha_n}$. It is easy to verify that $\{\chi_n\}$ is bounded (see [7]), and

$$\sup_t ||u(t)||_\infty \le \overline{\lim_{n\to\infty}} \chi_n \le C(M_0) < \infty. \tag{2.15}$$

□

To prove the convergence of $u_\varepsilon(t)$, we need the following proposition.

Proposition 2.2. *Under the assumptions (H1)–(H3), the solution $u_\varepsilon(t)$ of (2.1) satisfies*

$$\int_0^\omega \|\nabla u\|_{m+2}^{m+2} dt \le C(M_0), \tag{2.16}$$

$$\int_0^\omega ||u_t(t)||_2^2 dt \le C(M_0), \tag{2.17}$$

where $C(M_0)$ denotes a constant depending on M_0 and independent of ε.

Proof. Multiplying (2.1) by u and integrating, we obtain

$$\begin{aligned}\int_0^\omega\int_\Omega uu_t\,dx\,dt+\int_0^\omega\int_\Omega|\nabla u|^{m+2}dx\,dt+\int_0^\omega\int_\Omega \mathbf{b}(u)\cdot\nabla uu\,dx\,dt\\=\int_0^\omega\int_\Omega fu^{\alpha+1}dx\,dt+\int_0^\omega\int_\Omega hu\,dx\,dt.\end{aligned}\tag{2.18}$$

By the periodicity, Hölder's inequality, and Poincare's inequality, we have

$$\begin{aligned}\int_0^\omega\|\nabla u(t)\|_{m+2}^{m+2}dt&\le\int_0^\omega\|f(t)\|_{p^*}\|u(t)\|_{m+2}^{\alpha+1}dt+\int_0^\omega\|h(t)\|_{m+2}\|u(t)\|_{m+2}^{m+1}dt\\&\le\left(\int_0^\omega\|f(t)\|_{p^*}^{p^*}dt\right)^{1/p^*}\left(\int_0^\omega\|u(t)\|_{m+2}^{m+2}dt\right)^{(\alpha+1)/(m+2)}\\&\quad+\left(\int_0^\omega\|h(t)\|_{m+2}^{m+2}dt\right)^{1/(m+2)}\left(\int_0^\omega\|u(t)\|_{m+2}^{m+2}dt\right)^{(m+1)/(m+2)}\\&\le C(M_0)\left[\left(\int_0^\omega\|\nabla u(t)\|_{m+2}^{m+2}dt\right)^{(\alpha+1)/(m+2)}\right.\\&\qquad\left.+\left(\int_0^\omega\|\nabla u(t)\|_{m+2}^{m+2}dt\right)^{(m+1)/(m+2)}\right],\end{aligned}\tag{2.19}$$

in which $p^*=(m+2)(m+1-\alpha)^{-1}$. Thus, we have

$$\int_0^\omega\|\nabla u\|_{m+2}^{m+2}dt\le C(M_0)<\infty.\tag{2.20}$$

In order to derive (2.17), we multiply (2.1) by u_t and integrate over $[o,\omega]\times\Omega$,

$$\int_0^\omega\|u_t(t)\|_2^2\,dt+\int_0^\omega\int_\Omega\mathbf{b}(u)\cdot\nabla uu_t\,dx\,dt\le\int_0^\omega\int_\Omega|u_th|\,dx\,dt+\int_0^\omega f(t)dt\int_\Omega u^\alpha u_t\,dx.\tag{2.21}$$

Hence, by (H1), (2.16), and Young's inequality, we have

$$\int_0^\omega\|u_t(t)\|_2^2\,dt\le C(M_0)<\infty.\tag{2.22}$$

□

Completion of the proof of Theorem 1.3. Now we apply the Leray-Schauder fixed point theorem to show the existence of periodic solutions of problem (1.1). To do this, we investigate the following regularized equation:

$$u_t-\operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\}+\mathbf{b}(u)\cdot\nabla u=g(x,t),\quad x\in\Omega,\ t>0,\tag{2.23}$$

where $g\in E=C_\omega(\overline{Q})$. By using Faedo-Galerkin method and Browder fixed pointed theorem, Crema and Boldrini [13] have proved that for any $g\in E$, the regularized problem has a solution $u\in L^\infty(0,\omega;W_0^{1,m+2}(\Omega))$ and $u_t\in L^2(Q)$.

We defined $T : E \to E$ by $Tg = u$, then the map T is continuous and compact.

In fact, by [14, Theorem 1.2 in page 42] and noticing the periodicity of u, we arrive at

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \|u\|_\infty \left(|x_1 - x_2| + \|u\|_\infty^{(p-2)/p} |t_1 - t_2|^{1/p} \right)^\beta \tag{2.24}$$

for every pair of points $(x_1,t_1),(x_2,t_2) \in \overline{Q}$, where the positive constants γ, β depend only on N, ε, m, $\|g\|_\infty$. Ascoli-Arezela theorem implies that T maps any bounded set of E into a compact set of E.

Next, suppose that $g_k \to g$ as $k \to \infty$ and denote $u_k = Tg_k$, then there exists a function $u \in E$ such that

$$u_k(x,t) \longrightarrow u(x,t) \quad \text{uniformly in } Q, \tag{2.25}$$

by taking some subsequence if necessary.

Noticing the fact that

$$\int_\Omega \mathbf{b}(u) \cdot \nabla u u \, dx = 0, \tag{2.26}$$

we can prove that $u = Tg$ by using the argument similar to [9].

Let $\Phi(v) = f(t)v_+^\alpha + h(x,t)$, by the conditions (H2)-(H3) and the estimate above, we can see that $T(\tau,\Phi(v))$ is also the complete continuous map for $\tau \in [0,1]$. Proposition 2.1 shows that if u_0 is a fixed point of $T(\tau,\Phi(v))$, then

$$\|u_0(t)\|_\infty \le C_0 \tag{2.27}$$

with $C_0 > 0$ is a constant independent of τ, ε. Hence, applying the Leray-Schauder fixed point theorem, we conclude that (2.1) admits a periodic solution u_ε.

Therefore, we can obtain a periodic solution $\{u(t)\}$ of the problem (1.1) as a limit point of $\{u_\varepsilon(t)\}$ (see [8, 12]).

3. The proof of Theorem 1.4

In this section, we will derive the estimates of $\|\nabla u(t)\|_\infty$ for an assumed smooth solution of the problem and prove Theorem 1.4.

Proposition 3.1. *Under the assumptions (H1)–(H4), the (smooth) periodic solution $u(t)$ of problem (1.1) satisfies*

$$\sup_t \|\nabla u(t)\|_\infty \le C_1 < \infty, \tag{3.1}$$

where C_1 is a constant only dependent on M_0, M_1, and α.

Proof. Multiplying (2.1) by $-\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}$ $(p>m+2)$, and integrating over Ω, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p &+ \int_\Omega \operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\}\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\ &= \int_\Omega \mathbf{b}(u)\cdot\nabla u \operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx - \int_\Omega f(t)u^\alpha \operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\ &\quad - \int_\Omega h(x,t)\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx.\end{aligned}\tag{3.2}$$

Further, integrating by parts, we obtain (see [12])

$$\begin{aligned}&\int_\Omega \operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\}\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\ &\quad\geq \int_\Omega |\nabla u|^{p+m-2}|D^2u|^2dx + \frac{p-2}{4}\int_\Omega |\nabla u|^{p+m-4}|\nabla(|\nabla u|^2)|^2dx\\ &\qquad - C(N-1)\int_{\partial\Omega}|\nabla u|^{p+m}H(x)ds.\end{aligned}\tag{3.3}$$

It follows from (H1)–(H3) and Young's inequality that

$$\begin{aligned}&\int_\Omega \mathbf{b}(u)\cdot\nabla u \operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\ &\quad\leq \int_\Omega |\nabla u|^{p+m-2}|D^2u|^2dx + \int_\Omega p^2|\mathbf{b}(u)|^2|\nabla u|^{p-m}dx\\ &\quad\leq \int_\Omega |\nabla u|^{p+m-2}|D^2u|^2dx + C_0p^2(1+\|\nabla u(t)\|_p^p),\end{aligned}\tag{3.4}$$

$$-\int_\Omega f(t)u^\alpha \operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx \leq C_1(\|\nabla u\|_p^{p-1}+\|\nabla u(t)\|_p^p),\tag{3.5}$$

$$-\int_\Omega h(x,t)\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx = \int_\Omega \nabla h\cdot\nabla u|\nabla u|^{p-2}dx \leq CM_1\|\nabla u\|_p^{p-1}.\tag{3.6}$$

We have from (3.2)–(3.6) and (H4) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{C_1}{p}\||\nabla u|^{(p+m)/2}\|_{1,2}^2\\ &\quad\leq C_1p^2(1+\|\nabla u(t)\|_p^p) + C_1\|\nabla u\|_p^{p-1} + \frac{C_1}{p}\int_\Omega |\nabla u|^{p+m}dx.\end{aligned}\tag{3.7}$$

For the third term of the right-hand side of (3.7), by Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\|\nabla u\|_{p+m}^{p+m} \leq \frac{1}{2}\||\nabla u|^{(p+m)/2}\|_{1,2}^2 + C\|\nabla u\|_{m+2}^{m+1}\|\nabla u\|_p^{p-1}.\tag{3.8}$$

Therefore, (3.7) can be rewritten as

$$\frac{d}{dt}\|\nabla u(t)\|_p^p + C_1\||\nabla u|^{(p+m)/2}\|_{1,2}^2 \leq C_1p^3(1+\|\nabla u(t)\|_p^p) + C_1p\|\nabla u\|_p^{p-1}.\tag{3.9}$$

Then, setting

$$p_1 = m, \quad p_n = 2p_{n-1} - m, \quad \theta_n = 2N\left(1 - p_{n-1}p_n^{-1}\right)(N+2)^{-1}, \quad n = 2,3,\ldots, \tag{3.10}$$

we have, by a variant of the Gagliardo-Nirenberg inequality, again

$$\|\nabla u\|_{p_n} \le C^{2/(p_n+m)} \|\nabla u\|_{p_{n-1}}^{1-\theta_n} \left|\left| |\nabla u|^{(p_n+m)/2} \right|\right|_{1,2}^{2\theta_n/(p_n+m)}. \tag{3.11}$$

Therefore, from (3.9) and (3.11) (set $p = p_n$),

$$\begin{aligned} &\frac{d}{dt}\left|\left|\nabla u(t)\right|\right|_{p_n}^{p_n} + C_1 C^{-2/\theta_n} \|\nabla u\|_{p_{n-1}}^{(p_n+m)(\theta_n-1)/\theta_n} \|\nabla u\|_{p_n}^{(p_n+m)/\theta_n} \\ &\quad \le C_1 p_n^3\left(1 + \left|\left|\nabla u(t)\right|\right|_{p_n}^{p_n}\right) + C_1 p_n \|\nabla u\|_{p_n}^{p_n-1}. \end{aligned} \tag{3.12}$$

Applying Lemma 1.6, by the same argument as in Proposition 2.1, we can obtain (3.1). □

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