We present a backward biorthogonalization technique for giving an orthogonal projection of a biorthogonal expansion onto a smaller subspace, reducing the dimension of the initial space by dropping \( d \) basis functions. We also determine which basis functions should be dropped to minimize the \( L^2 \) distance between a given function and its projection. This generalizes some recent results of Rebollo-Neira.

In [3], Rebollo-Neira gives a backward biorthogonalization technique for projecting a biorthogonal expansion onto a subspace, reducing the dimension \( N \) of the initial space by dropping \( d = 1 \) basis function. In this note, we generalize this method to reduce the space by an arbitrary number \( d \) of basis functions, \( d < N \). Proposition 3.4 in [3] indicates which single basis function is to be removed in order to minimize the \( L^2 \) distance between a function \( f \) and its orthogonal projection into the reduced space. We will also generalize this result in Proposition 7. If more than one basis function is to be dropped, Rebollo-Neira recommends iterating the \( d = 1 \) process. We show via Example 8 that in some circumstances iterating the \( d = 1 \) process \( k \) times leads to results inferior to using Proposition 7 and dropping \( k \) basis functions simultaneously.

We begin with a Hilbert space \( H \) and an \( N \)-dimensional subspace \( V \). Assume biorthogonal bases of \( V \) given by \( \{ x'_i \}_{i=1}^N \) and \( \{ x_i \}_{i=1}^N \) such that \( \langle x'_i, x_j \rangle = \delta_{ij} \). Now drop \( d \) basis elements from each set, without loss of generality the first \( d \) elements for notational purposes, and form the reduced subspaces \( \tilde{V} = \text{span} \{ x_i \}_{i=d+1}^N \) and \( \tilde{V}' = \text{span} \{ x'_i \}_{i=1}^d \). We wish to modify the \( x'_i \) so that the projection from \( V \) to \( \tilde{V} \) is orthogonal. We next recursively construct the sequence \( \{ v'_i \}_{i=1}^d \subset \tilde{V}' \) by

\[
\begin{align*}
v'_1 &= x'_1, \\
v'_i &= x'_i - \sum_{\ell=1}^{i-1} \frac{\langle x'_i, v'_\ell \rangle}{\langle v'_\ell, v'_\ell \rangle} v'_\ell, \quad i \leq d.
\end{align*}
\]
2 A note on comprehensive backward biorthogonalization

We observe that the set \{v_j^i\}_{i=1}^d forms an orthogonal basis of \(\tilde{V}'\) by construction. We then construct the sequence \(\{\tilde{x}_i^i\}_{i=d+1}^N\) by

\[
\tilde{x}_i^i = x_i^i - \sum_{\ell=1}^d \frac{\langle x_i^i, v_j^\ell \rangle}{\langle v_j^\ell, v_j^\ell \rangle} v_j^\ell
\]

and set \(U = \text{span}\{\tilde{x}_i^i\}_{i=d+1}^N\). We will see that this formula generalizes the dual modification of [3, Theorem 3.1] for \(d \geq 1\). Note that each \(\tilde{x}_i^i\) is created to be orthogonal to \(\tilde{V}'\) by subtracting from \(x_i^i\) its projection onto \(\tilde{V}'\).

**Proposition 1.** The spaces \(U\) and \(\tilde{V}'\) are orthogonal complements in \(V\), \(V = \tilde{V} \oplus \tilde{V}'\).

**Proof.** Choose \(i, j\) such that \(j \leq d < i\) and use the definition of \(\tilde{x}_i^i\) and the orthogonality of \(\{v_j^i\}\),

\[
\langle \tilde{x}_i^i, v_j^i \rangle = \langle x_i^i, v_j^i \rangle - \sum_{\ell=1}^d \frac{\langle x_i^i, v_j^\ell \rangle}{\langle v_j^\ell, v_j^\ell \rangle} \langle v_j^\ell, v_j^i \rangle = \langle x_i^i, v_j^i \rangle - \langle x_i^i, v_j^i \rangle = 0.
\]

Thus \(U\) and \(\tilde{V}'\) are orthogonal subspaces of \(V\), and their dimensions add to \(N\). \(\square\)

We next verify that \(U\) and \(\tilde{V}\) are actually the same space.

**Lemma 2.** The spaces \(U\) and \(\tilde{V}\) are orthogonal complements in \(V\), and \(U = \tilde{V}\).

**Proof.** By (1), we can write \(v_j^i = \sum_{n=1}^{j} a_n x_n^i\) for some constants \(a_n\), so the original biorthogonality condition \(\langle x_i^i, x_j \rangle = \delta_{ij}\) says that, for \(j < i\), \(\langle v_j^i, x_i \rangle = \sum_{n=1}^{j} a_n \langle x_n^i, x_i \rangle = 0\). Thus \(\tilde{V}\) and \(\tilde{V}'\) are orthogonal subspaces of \(V\), and their dimensions add to \(N\). By the previous proposition, \(U = \tilde{V}\). \(\square\)

Next we give the desired biorthogonal bases of the reduced subspace \(\tilde{V}\).

**Proposition 3.** The reduced spaces \(U\) and \(\tilde{V}\) are identical and have biorthogonal bases \(\{\tilde{x}_i^i\}_{i=d+1}^N\) and \(\{x_j\}_{j=d+1}^N\).

**Proof.** Using Lemma 2 and (2), we have for \(i, j > d \geq \ell\),

\[
\langle \tilde{x}_i^i, x_j \rangle = \langle x_i^i, x_j \rangle - \sum_{\ell=1}^d \frac{\langle x_i^i, v_j^\ell \rangle}{\langle v_j^\ell, v_j^\ell \rangle} \langle v_j^\ell, x_j \rangle = \delta_{ij} - \sum_{\ell=1}^d \frac{\langle x_i^i, v_j^\ell \rangle}{\langle v_j^\ell, v_j^\ell \rangle} \cdot 0 = \delta_{ij}.
\]

\(\square\)

In order to give an explicit method for determining which basis functions to drop to minimize the residual, we give a formula for the projection operator.

**Proposition 4.** The orthogonal projection of \(V\) onto \(\tilde{V}\) is \(P(\cdot) = \sum_{i=d+1}^N \tilde{x}_i^i(\cdot)x_i\).

**Proof.** By Proposition 3, \(P(w) = w\) for all \(w \in \tilde{V}\) and \(\text{Range}(P) = \tilde{V}\). From Propositions 1 and 3, \(\tilde{V}'\) is the null space of \(P\), and \(\text{Range}(P)\) and \(\tilde{V}' = \text{Null}(P)\) are orthogonal, so \(P\) is an orthogonal projection. \(\square\)
The following generalizes [3, Corollary 3.2] to give the coefficients of $P(f)$ for the case $d \geq 1$.

**Theorem 5.** If $f = \sum_{i=1}^{N} c_i x_i$, where $c_i = \langle x_i', f \rangle$, then

$$P(f) = \sum_{i=d+1}^{N} c'_i x_i,$$

where $c'_i = c_i - \sum_{\ell=1}^{d} \langle x'_i, v'_\ell \rangle \langle v'_\ell, f \rangle.$

**Proof.** We calculate, using (2),

$$P(f) = \sum_{i=d+1}^{N} \tilde{x}'_i(f)x_i = \sum_{i=d+1}^{N} \left( \langle x'_i, f \rangle - \sum_{\ell=1}^{d} \langle x'_i, v'_\ell \rangle \langle v'_\ell, f \rangle \right)x_i.$$

so $P(f) = \sum_{i=d+1}^{N} c'_i x_i$, where

$$c'_i = c_i - \sum_{\ell=1}^{d} \langle x'_i, v'_\ell \rangle \langle v'_\ell, f \rangle.$$

The following generalizes [3, Corollary 3.3] for the case $d \geq 1$.

**Corollary 6.** If $f = \sum_{i=1}^{N} c_i x_i$, where $c_i = \langle x'_i, f \rangle$, then

$$\|f\|^2 = \|P(f)\|^2 + \sum_{i=1}^{d} \frac{1}{\|v'_i\|^2} \left| \sum_{k=1}^{i} c_k \langle v'_i, x_k \rangle \right|^2.$$

**Proof.** Since $V = \tilde{V} \oplus \tilde{V}'$, we can write $f = P(f) \oplus \text{proj}_{\tilde{V}'}(f)$, where $\text{proj}_{\tilde{V}'}(f) = \sum_{i=1}^{d} \langle v'_i/\|v'_i\|, f \rangle \langle v'_i/\|v'_i\| \rangle$ is the projection of $f$ onto $\tilde{V}'$ using the orthogonal basis $\{v'_i\}$. Thus by Parseval and then Lemma 2, we have

$$\|f\|^2 = \|P(f)\|^2 + \sum_{i=1}^{d} \left( \frac{\langle v'_i/\|v'_i\|, f \rangle \langle v'_i/\|v'_i\| \rangle}{\|v'_i\|^2} \right)^2 = \|P(f)\|^2 + \sum_{i=1}^{d} \frac{1}{\|v'_i\|^2} \left| \langle v'_i, f \rangle \right|^2 \quad (9)$$

$$= \|P(f)\|^2 + \sum_{i=1}^{d} \frac{1}{\|v'_i\|^2} \left| \sum_{k=1}^{i} c_k \langle v'_i, x_k \rangle \right|^2.$$

Next we generalize [3, Proposition 3.4] for the case $d \geq 1$.

**Proposition 7.** By reindexing the original $x_i$ and $x'_i$ to examine all possible $\binom{N}{d}$ combinations of $d$ components dropped from the original basis of $V$ and to minimize the $L^2$ distance between $f$ and $P(f)$, choose the set of $d$ basis elements $x_i$ that minimizes

$$\sum_{i=1}^{d} \frac{1}{\|v'_i\|^2} \left| \sum_{k=1}^{i} c_k \langle v'_i, x_k \rangle \right|^2 \quad (10)$$

We now give an example demonstrating that iterating the process $k$ times with $d = 1$ may give a projection considerably farther from the original $f$ than reducing by $k = d$ basis functions simultaneously.
Example 8. For simplicity, we consider a function $f(t)$ in the four-dimensional subspace $V$ with basis functions generated from cardinal spline wavelets. Let $B_3(x)$ be the standard quadratic cardinal spline supported on $[-1, 2]$ and let $w(t)$ be the standard associated wavelet for the Riesz basis of $L^2(\mathbb{R})$ generated by $B_3(x)$ as mentioned in [1] or [2]. Let $V = \text{span}\{x_1, x_2, x_3, x_4\}$, where $x_1(t) = B_3(t + 2)/\|B_3\|$, $x_2(t) = B_3(t - 2)/\|B_3\|$, $x_3(t) = (B_3(t - 2) + B_3(t + 2) + 0.2B_3(t))/\|B_3\|$, $x_4 = w(t)$. The function $f$ can be expressed as $f(t) = 0.7x_1(t) + 0.5x_2(t) + 0.4x_3(t) + x_4(t)$. We wish to drop $d = 2$ basis elements and obtain the best two-dimensional approximation to $f$. If we iteratively drop one basis element at a time using Proposition 7 with $d = 1$, then we remove $x_3$ and then $x_2$ leaving projection $P(f) = 0.9x_1 + x_4$ as shown in Figure 1(a) with residual error $\|f - P(f)\|^2 = 0.82$. However, if we simultaneously drop two elements with $d = 2$, then we instead drop $x_1$ and $x_2$ leaving projection $P(f) = 1.1x_3 + x_4$ as shown in Figure 1(b) with residual error $\|f - P(f)\|^2 = 0.03$. As can be seen from these errors and the plots in Figure 1, there is a considerable advantage for $t \geq 1.5$ in removing two basis elements together, rather than dropping them iteratively.
When the value of $\binom{N}{d}$ is large, the computational expense of choosing the optimal set of basis elements to be dropped can be quite large. Investigation of this issue merits further study.

References


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