MEASURE OF NONCOMPACTNESS OF OPERATORS AND MATRICES ON THE SPACES $c$ AND $c_0$

BRUNO DE MALAFOSSE, EBERHARD MALKOWSKY, AND VLADIMIR RAKOČEVIĆ

Received 16 September 2005; Revised 11 December 2005; Accepted 4 January 2006

In this note, using the Hausdorff measure of noncompactness, necessary and sufficient conditions are formulated for a linear operator and matrices between the spaces $c$ and $c_0$ to be compact. Among other things, some results of Cohen and Dunford are recovered.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

We will write $s$, $c$, and $c_0$, for the set of all complex, convergent, and null sequences, respectively. Let $A = (a_{nm})_{n,m \geq 1}$ be an infinite matrix and consider the sequence $x = (x_n)_{n \geq 1}$. We will define the product

$$Ax = (Ax)_n = (A_n(x))_{n \geq 1} \quad \text{with} \quad A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m, \quad n = 1,2,\ldots, (1)$$

whenever the series are convergent for all $n \geq 1$. For any given subsets $X, Y$ of $s$, we will say that the operator represented by the infinite matrix $A = (a_{nm})_{n,m \geq 1}$ maps $X$ into $Y$ that is $A \in (X, Y)$, if

(i) the series defined by $A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m$ are convergent for all $n \geq 1$ and for all $x \in X$;

(ii) $Ax \in Y$ for all $x \in X$.

If $c \subset c_A = \{x : Ax \in c\}$, $A$ is conservative. Well-known necessary and sufficient conditions for $A$ to be conservative are

$$\|A\| = \sup_{n \geq 1} \sum_{m=1}^{\infty} |a_{nm}| < \infty, (2)$$

$$a_{00} = \lim_{n \to \infty} \sum_{m=1}^{\infty} a_{nm} \quad \text{exists}, (3)$$

$$a_{0m} = \lim_{n \to \infty} a_{nm} \quad \text{exists for } m = 1, 2, \ldots. (4)$$
2 Measure of noncompactness of operators

In this case, (2) is the norm of operator $A$. If $A$ is conservative, then $\chi(A) = \lim_n \sum_{m=1}^{\infty} a_{nm} - \sum_{m=1}^{\infty} a_{0m}$ is called the characteristic of $A$, and in the case $\chi(A) = 0$, $A$ is conull. If $\lim_n (Ax)_n = \lim_n x_n$ for all $x \in c$, then $A$ is called regular. A conservative matrix is regular if and only if $a_{00} = 1$ and $a_{0m} = 0$ for all $m$ [5, 6].

Let $B(c)$ be the set of all bounded linear operators on $c$. It is well known (see [6, Theorem 4.51-D]) that each bounded linear operator $A$ on $c$ into $c$ determines and is determined by a matrix of scalars $a_{nm}$, $n = 1, 2, \ldots$, $m = 0, 1, 2, \ldots$, $y = Ax$, is defined by

$$y_n = a_{n0}x_0 + \sum_{m=1}^{\infty} a_{nm}x_m, \quad n = 1, 2, \ldots,$$

where $x = (x_n)$ in $c$, and $\lim_n x_n = x_0$. In this case, the norm of $A$ is defined by

$$\|A\| = \sup_{n \geq 1} \sum_{m=0}^{\infty} |a_{nm}|,$$

and for $A \in B(c)$, the additional conditions are (4) and

$$\lim_{n \to \infty} \sum_{m=0}^{\infty} a_{nm} = a_{00}.$$

Let $X$, $Y$ be Banach spaces, and let $B(X, Y)$ be the set of all linear bounded operators from $X$ to $Y$. If $Q$ is a bounded subset of $X$, then the Hausdorff measure of noncompactness of $Q$ is denoted by $q(Q)$, and

$$q(Q) = \inf \{ \epsilon > 0 : Q \text{ has a finite } \epsilon - \text{net in } X \}. \quad (8)$$

The function $q$ is called the Hausdorff measure of noncompactness (ball measure of noncompactness); it was introduced by Gohberg et al. [4], later studied by Goldenstein and Markus in 1968, Istrătescu in 1972, and others. Let us recall that the notation of the measure of noncompactness has proved useful results in several areas of functional analysis, operator theory, fixed point theory, differential equations, and so forth (see, e.g., [1, 2, 4]). Let us recall that $q(Q) = 0$ if and only if $Q$ is a totally bounded set. For $A \in B(X, Y)$, the Hausdorff measure of noncompactness of $A$, denoted by $\|A\|_q$, is defined by $\|A\|_q = \inf_{Q} q(AB_1)$, where $B_1 = \{ x \in X : \|x\| \leq 1 \}$ is the unit ball in $X$. Hence, $A$ is compact if and only if $\|A\|_q = 0$.

Let us recall that if $X$ is a Banach space with a Schauder basis $\{v_1, v_2, \ldots\}$, $Q$ a bounded subset of $X$, $P_n : X \to X$ the projector onto the linear span of $\{v_1, v_2, \ldots, v_n\}$, and $\mu(Q) = \limsup_{n \to \infty} \sup_{x \in Q} \| (I - P_n)x \|$, then the following inequality holds:

$$\frac{1}{a} \mu(Q) \leq q(Q) \leq \inf_{n} \sup_{x \in Q} \| (I - P_n)x \| \leq \mu(Q), \quad (9)$$

where $a = \limsup_{n \to \infty} \| I - P_n \|$ [1, 2, 4].

Now, we can state the following main result.
Theorem 1. Let $A \in B(c)$, let $\alpha_{00}$ be as in (7), and let $a_{0n}, n = 1, 2, \ldots$, be as in (4). Then

\[
\frac{1}{2} \limsup_{n \to \infty} \left( |a_{n0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right) \\
\leq \|A\|_q \leq \limsup_{n \to \infty} \left( |a_{n0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right).
\]  

(10)

Proof. Suppose that $x \in c$, $\lim m x_m = x_0$ and $y = Ax$. Now $y_n = a_{n0}x_0 + \sum_{m=1}^{\infty} a_{nm}x_m$, $n = 1, 2, \ldots$, and $\lim_n y_n = y_0$. By [6, page 219, (4.51-11)], (5) and (7),

\[
y_0 = x_0 \alpha_{00} + \sum_{m=1}^{\infty} (x_m - x_0) a_{0m} = x_0 \left( \alpha_{00} - \sum_{m=1}^{\infty} a_{0m} \right) + \sum_{m=1}^{\infty} x_m a_{0m}.
\]  

(11)

The elements $e = (1, 1, 1, \ldots)$ and $e_i = \{\delta_{ij}\}, i = 1, 2, \ldots$ form the Schauder basis in $c$. Let $P_n : c \hookrightarrow c$ be the projector defined by

\[
P_n(x) = x_0 e + \sum_{i=1}^{n} (x_i - x_0) e_i, \quad n = 1, 2, \ldots.
\]  

(12)

It is easy that $\|I - P_n\| = 2$, and by (9) we have

\[
\sup_{\|x\| \leq 1} \|(I - P_n)Ax\| = \sup_{\|x\| \leq 1, k \geq n+1} \|y_k - y_0\| \\
= \sup_{\|x\| \leq 1, k \geq n+1} \left| x_0 \left( a_{k0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m} \right) + \sum_{m=1}^{\infty} (a_{km} - a_{0m}) x_m \right| \\
= \sup_{k \geq n+1} \left( |a_{k0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{km} - a_{0m}| \right).
\]  

(13)

Now, by (9) and (13), we obtain (10).

Corollary 2. Let $A \in B(c)$. Then $A$ is compact if and only if

\[
\lim_{n \to \infty} \left( |a_{n0} - \alpha_{00} + \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right) = 0.
\]  

(14)

Let us recall that if $A \in B(c)$, $y = Ax$, then $y_0 = x_0$ for every choice of $x$ if and only if $\alpha_{00} = 1$ and $a_{01} = a_{02} = \cdots = 0$ (see, e.g., [6]); in this case $A$ is called regular. Now, by Corollary 2, we have the next well-known result of Cohen and Dunford [3, Corollary 3].

Corollary 3. Let $A \in B(c)$ be regular transformation. Then $A$ is compact if and only if

\[
\lim_{n \to \infty} \left( |a_{n0} - 1| + \sum_{m=1}^{\infty} |a_{nm}| \right) = 0.
\]  

(15)
Corollary 4. Let $A \in (c, c)$, let $a_{00}$ be as in (3), and let $a_{0m}, n = 1, 2, \ldots$, be as in (4). Then

$$\frac{1}{2} \limsup_{n \to \infty} \left( |a_{00} - \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right)$$

$$\leq \|A\|_q \leq \limsup_{n \to \infty} \left( |a_{00} - \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right),$$

and $A$ is compact if and only if

$$\lim_{n \to \infty} \left( |a_{00} - \sum_{m=1}^{\infty} a_{0m}| + \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \right) = 0.$$  (17)

Let us remark that Corollary 4 implies that compact conservative matrix is conull.

If we recall the characterizations of the sets $(c, c_0)$ and $(c_0, c_0)$ [5, 6], and remark that in this case the projector $P_n(x) = (x_1, x_2, \ldots, x_n, 0, \ldots)$ maps $c_0$ into $c_0$, and $\|I - P_n\| = 1$, then by the proof of Theorem 1, we have the next result.

Corollary 5. If $A \in (c, c_0)$ or $A \in (c_0, c_0)$, then

$$\|A\|_q = \limsup_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm}|,$$  (18)

and $A$ is compact if and only if

$$\lim_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm}| = 0.$$  (19)

Corollary 6. If $A \in (c_0, c)$, then

$$\frac{1}{2} \limsup_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| \leq \|A\|_q \leq \limsup_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm} - a_{0m}|,$$  (20)

and $A$ is compact if and only if

$$\lim_{n \to \infty} \sum_{m=1}^{\infty} |a_{nm} - a_{0m}| = 0.$$  (21)

Acknowledgment

The authors are thankful to the referee for her/his useful suggestions and comments.

References


Bruno De Malafosse: LMAH Université du Havre, IUT du Havre, BP 4006, 76610 Le Havre, France

E-mail address: bdemalaf@wanadoo.fr

Eberhard Malkowsky: Mathematisches Institut, Universität Giessen, Arndtstrasse 6, D-35392 Giessen, Germany

E-mail addresses: eberhard.malkowsky@math.uni-giessen.de; ema@bankerinter.net

Vladimir Rakočević: Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia and Montenegro

E-mail address: vrakoc@bankerinter.net
Special Issue on
Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/jamds/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>June 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>September 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>December 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be

Hindawi Publishing Corporation
http://www.hindawi.com