We develop an extension of the classical Bell polynomials introducing the Laguerre-type version of this well-known mathematical tool. The Laguerre-type Bell polynomials are useful in order to compute the $n$th Laguerre-type derivatives of a composite function. Incidentally, we generalize a result considered by L. Carlitz in order to obtain explicit relationships between Bessel functions and generalized hypergeometric functions.

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1. Introduction

The Bell polynomials [1] appear in different frameworks. They are often used in combinatorial analysis [20], and even in statistics [14], although without explicit references. Moreover these polynomials have been applied even in many other contexts, such as the Blissard problem (see [20, page 46]), the representation of Lucas polynomials of the first and second kinds [4, 9], the representation formulas of Newton sum rules for polynomials’ zeros [12, 13], the recurrence relations for a class of Freud-type polynomials [3], the representation of symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas [15]. Consequently they were also used [6] in order to find reduction formulas for the orthogonal invariants of a strictly positive compact operator, deriving in a simple way the so-called Robert formulas [21].

Some generalized forms of Bell polynomials already appeared in literature (see, e.g., [11, 17, 19]). A generalization of the Bell polynomials suitable for the differentiation of multivariable composite functions can also be found in [18]. Lastly, in [2], the so-called multidimensional Bell polynomials of higher order were introduced, which are suitable for representing the derivative of a composite function of several (say $m$) variables $f(\phi^{(1)}(t), \phi^{(2)}(t), \ldots, \phi^{(m)}(t))$, where $\phi^{(i)}(t) = \phi^{(i,1)}(\phi^{(i,2)}(\cdots \phi^{(i,r)}(t))$, $(i=1,2,\ldots,m)$.

In this article we find explicit representation formulas for the $n$th Laguerre-type derivatives of a composite function. The case of the first Laguerre derivative $DxD$, $D := d/dx$ is essentially related to an article by Carlitz [5], originated by a preceding paper by Lardner [16] in which the powers $(DxD)^n$ of this derivative appear.
2 Laguerre-type Bell polynomials

2. Recalling the Bell polynomials

We recall that the Bell polynomials are a classical mathematical tool for representing the \( n \text{th} \) derivative of a composite function. In fact by considering the composite function \( \Phi(t) := f(g(t)) \) of functions \( x = g(t) \) and \( y = f(x) \) defined in suitable intervals of the real axis and \( n \) times differentiable with respect to the relevant independent variables and by using the following notations:

\[
\Phi_h := D^h_t \Phi(t), \quad f_h := D^h_x f(x)|_{x=g(t)}, \quad g_h := D^h_t g(t),
\]

(2.1)

they are defined as follows:

\[
Y_n([f,g]_n) := \Phi_n. \tag{2.2}
\]

For example one has

\[
Y_1([f,g]_1) = f_1 g_1, \\
Y_2([f,g]_2) = f_1 g_2 + f_2 g_1^2, \\
Y_3([f,g]_3) = f_1 g_3 + f_2 (3g_2 g_1) + f_3 g_1^3.
\]

(2.3)

Further examples can be found in [20, page 49]. Inductively, we can write

\[
Y_n([f,g]_n) = \sum_{k=1}^{n} A_{n,k}(g_1,g_2,\ldots,g_n) f_k, \tag{2.4}
\]

where the coefficient \( A_{n,k} \), for any \( k = 1,\ldots,n \), is a polynomial in \( g_1,g_2,\ldots,g_n \), homogeneous of degree \( k \) and isobaric of weight \( n \) (i.e., it is a linear combination of monomials 

\[
g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}
\]

whose weight is constantly given by \( k_1 + 2k_2 + \cdots + nk_n = n \).

For them the following result holds true.

**Proposition 2.1.** The Bell polynomials satisfy the recurrence relation

\[
Y_0([f,g]_0) := f_1, \quad Y_{n+1}([f,g]_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}([f,g]_{n-k}) g_{k+1}, \tag{2.5}
\]
where
\[(f_1, g_{n-k}) := (f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}).\]  
(2.6)

An explicit expression for the Bell polynomials is also given by the Faà di Bruno formula [10]:
\[\Phi_n = Y_n([f, g]_n) = \sum_{\pi(n)} \frac{n!}{j_1! j_2! \cdots j_n!} f_{j_1} g_{1!}^{j_1} f_{j_2} g_{2!}^{j_2} \cdots f_{j_n} g_{n!}^{j_n},\]  
(2.7)

where the sum runs over all partitions \(\pi(n)\) of the integer \(n\), that is, \(n = j_1 + 2j_2 + \cdots + nj_n\). In (2.7) \(j_h\) denotes the number of parts of size \(h\), and \(j = j_1 + j_2 + \cdots + j_n\) denotes the number of parts of the considered partition. A proof of the Faà di Bruno formula can be found in [20]. In [22] the proof is based on the umbral calculus (see [23] and the references therein).

3. Laguerre-type derivatives

The Laguerre-type derivatives were introduced in [7, 8] in connection with a differential isomorphism denoted by the symbol \(\mathcal{F} := \mathcal{F}_x\), acting onto the space \(\mathcal{A} := \mathcal{A}_x\) of analytic functions of the \(x\) variable by means of the correspondence
\[D := \frac{d}{dx} \quad \hat{D}_L := DxD; \quad x \cdot \rightarrow \hat{D}_x^{-1},\]  
(3.1)

where
\[\hat{D}_x^{-1} f(x) := \int_0^x f(\xi)d\xi,\]  
\[\hat{D}_x^{-n} f(x) := \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} f(\xi)d\xi,\]  
(3.2)

so that
\[\mathcal{F}_x(x^n) = \hat{D}_x^{-n}(1) := \frac{1}{(n-1)!} \int_0^x (x - \xi)^{n-1} d\xi = \frac{x^n}{n!}.\]  
(3.3)

According to this isomorphism, the exponential operator \(e^x\) is transformed into the first Laguerre-type exponential \(e_1(x) := \sum_{k=0}^{\infty} x^k/(k!)^2\) which is an eigenfunction of the Laguerre derivative operator \(\hat{D}_L := DxD\). We have, in fact,
\[\mathcal{F}_x(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{F}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x),\]  
(3.4)

\[\hat{D}_Le_1(ax) = ae_1(ax), \quad \forall a \in \mathbb{C}.\]
4 Laguerre-type Bell polynomials

This result can be generalized by considering the $r$th Laguerre-type exponential $e_{r}(x) := \sum_{k=0}^{\infty} x^k/(k!)^{r+1}$, the $r$th Laguerre-type derivative operator $D_{rL} := DxDxD \cdots DxD$ (containing $r + 1$ ordinary derivatives), and the iterated isomorphism $\mathcal{T}^r$, since

$$\mathcal{T}^r_x(e^x) = \sum_{k=0}^{\infty} \mathcal{T}^r_x(x^k)/(k!)^{r+1} = e_{r}(x),$$

(3.5)

$$\hat{D}_{rL} e_{r}(ax) = ae_{r}(ax), \quad \forall a \in \mathbb{C}.$$ 

Remark 3.1. The above results show that, for every positive integer $r$, we can define a Laguerre-type exponential function $e_{r}(x)$, satisfying an eigenfunction property, which is an analog of the elementary property of the exponential. This function reduces to the exponential function when $r = 0$, so that we can put by definition

$$e_0(x) := e^x, \quad \hat{D}_{0L} := D.$$ 

(3.6)

Obviously, $\hat{D}_{1L} := \hat{D}_L$.

For this reason we will refer to such functions as $L$-exponential functions, or shortly $L$-exponentials.

4. Laguerre-type Bell polynomials

The problem of constructing Bell polynomials can be extended in the natural way to the case of the Laguerre-type derivatives.

To this aim, by using notations in (2.1), we introduce the following definition.

Definition 4.1. The $n$th Laguerre-type Bell polynomial, denoted by $rLY_n(x; [f, g]_n)$, represents the $n$th $r$-Laguerre-type derivative of the composite function $f(g(t))$.

We will show that $rLY_n$ can be expressed as a polynomial in the independent variable $x$, depending on $f_1, g_1; f_2, g_2; \ldots; f_n, g_n$ in terms of the classical Bell polynomials.

We start noting that, according to a general result due to Viskov [24], the Laguerre derivative satisfy

$$(D_L)^n = (DxD)^n = D^n x^n D^n,$$

(4.1)

and furthermore, for any order $r$, it turns out that

$$(D_{rL})^n = (DxDxD \cdots DxD)^n = D^n x^n D^n x^n \cdots D^n x^n D^n.$$ 

(4.2)

According to the above equations, the proof of Carlitz [5] can be reduced to a simple application of the Leibnitz rule, since

$$(DxD)^n = D^n (x^n D^n) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k} x^n D^{n+k}$$

$$= \sum_{k=0}^{n} \left( \frac{n}{k} \right)^2 (n-k)! x^k D^{n+k} = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} x^k D^{n+k}. $$

(4.3)
Therefore, the following representation formula for the Laguerre-type Bell polynomials, denoted by $L Y_n$, holds true.

**Theorem 4.2.** The $L Y_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation

$$L Y_n(x; [f,g]_n) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} x^k Y_{n+k} ([f,g]_{n+k}). \quad (4.4)$$

The above results can be easily generalized, since

$$(D_{2L})^n = (DxDx)^n = D^n x^n (D^n x^n D^n)$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \frac{n!}{k_1! (k_1 + k_2)!} \left( \binom{n}{k_1} \binom{n}{k_2} \right) x^{k_1+k_2} D^{n+k_1+k_2}. \quad (4.5)$$

**5. The general case**

The following result follows by induction.

**Theorem 5.1.** The powers of the $r$th Laguerre-type derivative operator $D_{rL} := DxDx \cdots DxD$ (containing $r + 1$ ordinary derivatives) can be expanded in the form

$$(D_{rL})^n = (DxDx \cdots DxD)^n = D^n x^n D^n x^n \cdots D^n x^n D^n$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_r=0}^{n} \frac{n!}{k_1! (k_1 + k_2)! \cdots (k_1 + k_2 + \cdots + k_r)!} \left( \binom{n}{k_1} \binom{n}{k_2} \cdots \binom{n}{k_r} \right) x^{k_1+k_2+\cdots+k_r} D^{n+k_1+k_2+\cdots+k_r}. \quad (5.1)$$

Therefore, for the $r$th Laguerre-type Bell polynomials denoted by $rL Y_n$, the following result holds true.

**Theorem 5.2.** The $rL Y_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation

$$rL Y_n(x; [f,g]_n) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_r=0}^{n} \frac{n!}{k_1! (k_1 + k_2)! \cdots (k_1 + k_2 + \cdots + k_r)!} \left( \binom{n}{k_1} \binom{n}{k_2} \cdots \binom{n}{k_r} \right) x^{k_1+k_2+\cdots+k_r} Y_{n+k_1+k_2+\cdots+k_r} ([f,g]_{n+k_1+k_2+\cdots+k_r}). \quad (5.2)$$
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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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