

A CLASS OF PRINCIPAL IDEAL RINGS ARISING FROM THE CONVERSE OF THE CHINESE REMAINDER THEOREM

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Let R be a (nonzero commutative unital) ring. If I and J are ideals of R such that $R/I \oplus R/J$ is a cyclic R -module, then $I + J = R$. The rings R such that $R/I \oplus R/J$ is a cyclic R -module for all distinct nonzero proper ideals I and J of R are the following three types of principal ideal rings: fields, rings isomorphic to $K \times L$ for the fields K and L , and special principal ideal rings (R, M) such that $M^2 = 0$.

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1. Introduction

All rings considered in this note are nonzero, commutative, and unital. Our starting point is the following formulation of the Chinese remainder theorem (cf. [2, Proposition 1.10(ii)]): if I and J are ideals of a ring R , then the canonical ring homomorphism $R/(I \cap J) \rightarrow R/I \times R/J$, given by $r + I \cap J \mapsto (r + I, r + J)$, is an isomorphism if and only if $I + J = R$. Often, only the “if” assertion is referred to as the Chinese remainder theorem. It seems natural to ask if one can strengthen the “only if” assertion by concluding that $I + J = R$, given only the existence of some (unspecified) ring isomorphism between $R/(I \cap J)$ and $R/I \times R/J$. Proposition 2.3 gives an affirmative answer by establishing even more, namely, that the existence of an R -module isomorphism between $R/I \oplus R/J$ and some cyclic R -module implies that $I + J = R$. Thus, one may say that it is somewhat rare for ideals I and J of a ring R to be such that $R/I \oplus R/J$ is a cyclic R -module. In fact, it follows easily from Proposition 2.3 that there is no ring R such that $R/I \oplus R/J$ is a cyclic R -module for *all* ideals I and J of R . However, putting modest restrictions on the relevant ideals leads us to an interesting class of principal ideal rings. Indeed, in our main result, Theorem 2.6, we show that a ring R is such that $R/I \oplus R/J$ is a cyclic R -module for *all* distinct nonzero ideals I and J of R if and only if R is one of the following three types: a field; a ring isomorphic to $K \times L$ for the fields K and L ; a special principal ideal ring (R, M) such that $M^2 = 0$. Any background material is either standard or explained in passing.

2. Results

Propositions 2.1 and 2.3 are essentially known; their simple proofs are included for the sake of completeness. Recall that if E is an R -module, then the *annihilator of E* is $(0 :_R E) := \{r \in R \mid rx = 0 \text{ for all } x \in E\}$.

PROPOSITION 2.1. *If R is a ring, and I and J are ideals of R such that $R/I \cong R/J$ as R -modules, then $I = J$.*

Proof. It is easy to see that if A is any ideal of R , then $(0 :_R R/A) = A$. As isomorphic R -modules have the same annihilator, we have $I = (0 :_R R/I) = (0 :_R R/J) = J$. \square

The method of proof of Proposition 2.1 leads to the next result, which seems not to have been explicitly recorded earlier.

PROPOSITION 2.2. *If I and J are ideals of a ring R such that $R/I \oplus R/J$ is a cyclic R -module, then $R/I \oplus R/J \cong R/(I \cap J)$.*

Proof. By hypothesis, $R/I \oplus R/J$ is isomorphic as an R -module to R/A for some ideal A of R . As isomorphic R -modules have the same annihilator, we have $A = (0 :_R R/I \oplus R/J) = I \cap J$, and the assertion follows. \square

In view of Proposition 2.2, one could restate the results given below in the style of the Chinese remainder theorem by replacing certain cyclic modules by the corresponding expressions of the form $R/(I \cap J)$. In particular, the next result is essentially known; one needs only to combine Proposition 2.2 with the case $A = R$ of the result of Anderson [1, Proposition 1(3)] that a finitely generated R -module A satisfies the CC1 property (which is defined in [1]) if (and only if) $(0 :_R A)$ is contained in the Jacobson radical of R .

PROPOSITION 2.3. *If R is a ring and I and J are ideals of R such that $R/I \oplus R/J$ is a cyclic R -module, then $I + J = R$.*

Proof. We offer two proofs, the first via tensor products and the second via exterior powers.

The first proof is inspired by that of [1, Proposition 1(2)]. If the assertion fails, Zorn's lemma supplies a maximal ideal M of R such that $I + J \subseteq M$ (cf. [2, Corollary 1.4]). Consider the field $F := R/M$. Note, by Proposition 2.2, that $R/I \oplus R/J \cong R/(I \cap J)$. Applying the functor $F \otimes_R -$, we see that $F \otimes_R (R/I \oplus R/J)$ and $F \otimes_R R/(I \cap J)$ are isomorphic as vector spaces over F and, hence, must have the same vector space dimension over F . However, since $IF = 0 = JF$, we have that

$$F \otimes_R (R/I \oplus R/J) \cong (F \otimes_R R/I) \oplus (F \otimes_R R/J) \cong F/IF \oplus F/JF \cong F \oplus F \quad (2.1)$$

is a two-dimensional vector space over F , while

$$F \otimes_R R/(I \cap J) \cong F/(I \cap J)F = F/0 \cong F \quad (2.2)$$

is a one-dimensional vector space over F , the desired contradiction.

For the second proof of Proposition 2.3, note (once again by Proposition 2.2) that $R/I \oplus R/J \cong R/(I \cap J)$. Applying the second exterior power functor, we have that $\wedge_R^2(R/I \oplus R/J)$ and $\wedge_R^2(R/(I \cap J))$ are isomorphic as R -modules. Now, since $\wedge_R^2 R/I = 0 = \wedge_R^2 R/J$, we see that

$$\wedge_R^2(R/I \oplus R/J) \cong \oplus_{i=0}^2 ((\wedge_R^i R/I) \otimes_R (\wedge_R^{2-i} R/J)) \cong 0 \oplus (R/I \otimes_R R/J) \oplus 0 \quad (2.3)$$

is isomorphic to $R/(I + J)$, while $\wedge_R^2(R/(I \cap J)) = 0$. Therefore, $R/(I + J) = 0$, whence $I + J = R$, thus completing the second proof. \square

The next result gives one sense in which it is rare that a direct sum of cyclic modules is itself cyclic.

COROLLARY 2.4. *If I is a proper ideal of a ring R , then there exists a maximal ideal M of R such that $R/I \oplus R/M$ is not a cyclic R -module.*

Proof. Since I is a proper ideal of R , we can choose a maximal ideal M of R such that $I \subseteq M$ (cf. [2, Corollary 1.4]). As $I + M = M \neq R$, the assertion now follows from Proposition 2.3. \square

Remark 2.5. (a) Corollary 2.4 is best possible, in the sense that there may be only one maximal ideal M of R which contains the given ideal I . A concrete example where this occurs is provided by taking $R := K \times L$, where K and L are fields, and $I := K \oplus 0$.

(b) Generalizing the construction in (a), consider $R := F_1 \times \cdots \times F_n$, where F_1, \dots, F_n are finitely many fields with $n \geq 3$, and $I := F_1 \oplus 0 \oplus \cdots \oplus 0$. Then there exist exactly $n - 1$ maximal ideals M of R such that $I \subseteq M$. Rings of the above form $F_1 \times \cdots \times F_n$ will arise naturally in the proof of Theorem 2.6.

Recall from [3] that a ring R is called a *principal ideal ring* if each ideal of R is a principal ideal of R ; and a *special principal ideal ring* (SPIR) if, in addition, R is quasilocal and its unique maximal ideal is nilpotent. According to a fundamental structure theorem [3, Theorem 33, page 245], principal ideal rings are the rings that are (isomorphic to) products of finitely many principal ideal domains and finitely many SPIRs. (As usual, “finitely many” includes the possibility of “at most one.”) If the maximal ideal of an SPIR has index of nilpotency $e (\geq 1)$, it is convenient to say that R is an SPIR of *exponent* e . For instance, if X is an indeterminate over a field F and k is a positive integer, then $F[X]/(X^k)$ is an SPIR of exponent k . One may view fields as the SPIRs of exponent 1.

We next present our main result.

THEOREM 2.6. *Let R be a ring. Then the following conditions are equivalent:*

- (1) $R/I \oplus R/J$ is a cyclic R -module for all distinct nonzero proper ideals I and J of R ;
- (2) $I + J = R$ for all distinct nonzero proper ideals I and J of R ;
- (3) R is one of the following three types of (principal ideal) rings: (i) fields; (ii) rings isomorphic to $K \times L$, for some fields K and L ; and (iii) special principal ideal rings of exponent 2.

Proof. (1) \Rightarrow (2) by Proposition 2.3, while (2) \Rightarrow (1) by the version of the Chinese remainder theorem [2, Proposition 1.10(ii)] that was recalled in Section 1.

4 A class of principal ideal rings

(3) \Rightarrow (2). Assume that (3) holds. One may verify directly (or cite the above-mentioned structure theorem [3, Theorem 33, page 245]) to show that R is a principal ideal ring. In each of the cases (i)–(iii), it is straightforward to verify (2) after first listing all the nonzero proper ideals of R . Indeed, in case (i), there are no such ideals (and so (2) is satisfied vacuously). In case (ii), with R identified as $K \times L$, the only nonzero proper ideals of R are $K \oplus 0$ and $0 \oplus L$ (and the sum of these two ideals is indeed R , as required in (2)). Finally, to handle case (iii), first recall from [3, page 245] that if M is the unique maximal ideal of a special principal ideal ring R of exponent e , then the only ideals of R are the ideals M^i , for $i = 0, 1, \dots, e$. In particular, in case (iii), $e = 2$, and so M is the only nonzero proper ideal of R (whence (2) is satisfied vacuously).

(2) \Rightarrow (3). Assume that (2) holds. Without loss of generality, R is not a field. Hence, each maximal ideal of R is nonzero. Now, if $I \subseteq J$ are nonzero proper ideals of R , then $I + J = J \neq R$, and so it follows from (2) that $I = J$. Therefore, each nonzero proper ideal of R is a simple R -module (that is, a simple ideal of R) and, hence, a cyclic R -module. It follows that R is a principal ideal ring and that each maximal ideal of R is a simple R -module.

We pause to show that if M is a maximal ideal of R , then R_M inherits condition (2) from R . (The interested reader is invited to show directly that R_M inherits condition (1) from R .) To see this, let I and J be distinct nonzero proper ideals of R_M . Since each ideal of the localization R_M is extended from R (see the proof of [2, Proposition 3.11(i)]), there exist ideals I_1 and J_1 of R such that $I = I_1 R_M$ and $J = J_1 R_M$. Necessarily, I_1 and J_1 are distinct and nonzero. Therefore, since R satisfies (2), we have $I_1 + J_1 = R$, whence $I + J = I_1 R_M + J_1 R_M = (I_1 + J_1) R_M = R_M$; that is, R_M satisfies (2), as asserted.

Since R is a principal ideal ring, the fundamental structure theorem allows us to express R as the following internal direct product of rings (which is unique apart from the ordering of the factors):

$$R = D_1 \times \cdots \times D_n \times S_1 \times \cdots \times S_k, \quad (2.4)$$

where n and k are nonnegative integers, D_1, \dots, D_n are principal ideal domains (possibly fields), and S_1, \dots, S_k are special principal ideal rings of exponent at least 2. We claim that each D_i is a field and that each S_j has exponent 2.

Suppose that the claim fails because some D_i is not a field. Choose a (nonzero) maximal ideal N_i of D_i . Observe that the localization $T := (D_i)_{N_i}$ is isomorphic to the localization of R at the maximal ideal of R induced by N_i . (The underlying fact that we are using about localizations is if A and B are rings and P is a prime ideal of A , then the localizations $(A \times B)_{P \times B}$ and A_P are isomorphic as rings.) As we have seen that each localization of R at one of its maximal ideals inherits condition (2), it follows that T satisfies condition (2). However, T is an integral domain which is not a field, and so we can choose a nonzero nonunit t of T . Then Tt^2 and Tt are distinct nonzero ideals of T whose sum is $Tt \neq T$, thus contradicting that T satisfies (2). This proves that each D_i is a field.

We next show that each (S_j, M_j) has exponent 2. Indeed, we see as above that S_j inherits condition (2) from R (since $S_j = (S_j)_{M_j}$ is isomorphic to the localization of R at the maximal ideal induced by M_j). Therefore, by the final comment in the first paragraph of this part of the proof, M_j is a simple S_j -module. Since M_j is the only maximal

ideal of S_j , it follows that $M_j \cong S_j/M_j$ as S_j -modules. Equating annihilators, we find that $(0 :_{S_j} M_j) = M_j$, and so $(M_j)^2 = 0$. In particular, S_j is of exponent 2, thus proving the claim.

We can rewrite $R = F_1 \times \cdots \times F_n \times S_1 \times \cdots \times S_k$, where n and k are nonnegative integers, each F_i is a field, and each S_j is a special principal ideal ring of exponent 2. A case analysis reveals that $n + k \leq 2$. For instance, to rule out the possibility that $n \geq 3$, observe that (2) would be violated by taking $I := F_1 \oplus 0 \oplus 0 \oplus \cdots$ and $J := 0 \oplus F_2 \oplus 0 \oplus \cdots$, since $(0, 0, 1, \dots) \notin I + J$. The other obstructions to $n + k \leq 2$ are dispatched similarly, with minor changes in notation. Thus, the number of factors, $n + k$, is either 1 or 2.

It remains only to rule out the possibility that R takes either the form $F \times S$ with F a field and (S, M) a SPIR of exponent 2 or the form $S_1 \times S_2$ (with (S_j, M_j) being SPIRs of exponent 2). The first of these forms would violate (2) by taking $I := F \oplus 0$ and $J := 0 \oplus M$, for then $(0, 1) \notin I + J$; and, similarly, the second of these forms would violate (2) by taking $I := M_1 \oplus 0$ and $J := 0 \oplus M_2$. The proof is complete. \square

Note that the class \mathcal{C} of rings R characterized in Theorem 2.6 has some properties that are reminiscent of properties of the class of all principal ideal rings. For instance, \mathcal{C} is stable under the formation of nonzero homomorphic images and nonzero rings of fractions (and so, in particular, is stable under localization). In addition, \mathcal{C} is not stable under the adjunction of indeterminates (since each ring in \mathcal{C} has Krull dimension 0). However, unlike the class of all principal ideal rings, \mathcal{C} is not stable under finite products.

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