SPACES OF $D_{L^p}$ TYPE AND A CONVOLUTION PRODUCT ASSOCIATED WITH THE SPHERICAL MEAN OPERATOR

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We define and study the spaces $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p \leq \infty$, that are of $D_{L^p}$ type. Using the harmonic analysis associated with the spherical mean operator, we give a new characterization of the dual space $\mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n)$ and describe its bounded subsets. Next, we define a convolution product in $\mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \times M_r(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq r \leq p < \infty$, and prove some new results.

1. Introduction

The spherical mean operator $\mathcal{R}$ is defined, for a function $f$ on $\mathbb{R}^{n+1}$, even with respect to the first variable, by

$$\mathcal{R}(f)(r,x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta, \xi), \quad (r,x) \in \mathbb{R} \times \mathbb{R}^n, \quad (1.1)$$

where $S^n$ is the unit sphere $\{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n : \eta^2 + \|\xi\|^2 = 1\}$ in $\mathbb{R}^{n+1}$ and $\sigma_n$ is the surface measure on $S^n$ normalized to have total measure one.

This operator plays an important role and has many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data (see [7, 8]), or in the linearized inverse scattering problem in acoustics [6]. In [10], the authors associate to the operator $\mathcal{R}$ a Fourier transform and a convolution product and have established many results of harmonic analysis (inversion formula, Paley-Wiener and Plancherel theorems, etc.).

In [11], the authors define and study Weyl transforms related to the mean operator $\mathcal{R}$ and have proved that these operators are compact. The spaces $D_{L^p}$, $1 \leq p \leq \infty$, have been studied by many authors [1, 2, 4, 5, 12, 13]. In this work, we introduce the function spaces $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq p \leq \infty$, similar to $D_{L^p}$, but replace the usual derivatives by the operator

$$L = I + \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^2, \quad (1.2)$$
where \( l \) is the Bessel operator defined on \([0, +\infty[\) by

\[
  l = \left( \frac{\partial}{\partial r} \right)^2 + \frac{n}{r} \frac{\partial}{\partial r}.
\]

The main result of this paper gives a new characterization of the dual space \( \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \) of the space \( \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \) and a description of its bounded subsets. More precisely, in Section 2, we recall some harmonic results related to a convolution product and the Fourier transform connected with the spherical mean operator, that we use in the following sections.

In the Section 3, we define the space \( \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \), \( 1 \leq p \leq \infty \), to be the space of measurable functions \( f \) on \([0, +\infty[ \times \mathbb{R}^n \) such that for all \( k \in \mathbb{N} \), \( L^k f \) belongs to the space \( L^p(d\nu) \) (the space of functions of \( p \)th power integrable on \([0, +\infty[ \times \mathbb{R}^n \) with respect to the measure \( r^n dr \otimes dx \)). We give some properties of this space, in particular we prove that it is a Frechet space.

Section 4 is consecrated to the study of the dual space \( \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \). We give a nice description of the elements of this space and we characterize its bounded subsets.

In the last section, we define and study a convolution product in \( \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \times M_r(\mathbb{R} \times \mathbb{R}^n) \), \( 1 \leq r \leq p < \infty \), where \( M_r(\mathbb{R} \times \mathbb{R}^n) \) is the closure of the Schwartz space \( S^* \) in \( \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \).

### 2. Spherical mean operator

In this section, we define and recall some properties of the spherical mean operator. For more details see [3, 6, 10, 11]. We denote by

(A) \( \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n) \) the space of infinitely differentiable functions on \( \mathbb{R} \times \mathbb{R}^n \), even with respect to the first variable,

(B) \( S^n \) the unit sphere in \( \mathbb{R} \times \mathbb{R}^n \),

\[
S^n = \{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n; \eta^2 + ||\xi||^2 = 1\}, \quad (2.1)
\]

where for \( \xi = (\xi_1, \ldots, \xi_n) \), we have \( ||\xi||^2 = \xi_1^2 + \cdots + \xi_n^2 \),

(C) \( d\sigma \) the normalized surface measure on \( S^n \).

**Definition 2.1.** The spherical mean operator is defined on \( \mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n) \) by

\[
\forall (r, x) \in [0, +\infty[ \times \mathbb{R}^n, \quad \mathcal{R}_n f (r, x) = \int_{S^n} f (r\eta, x + r\xi) d\sigma_n(\eta, \xi).
\]

For \( (\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n \), we put

\[
\forall (r, x) \in [0, +\infty[ \times \mathbb{R}^n, \quad \varphi_{\mu, \lambda} (r, x) = \mathcal{R}_n \left( \cos(\mu \cdot) e^{-i(\lambda \cdot)} \right) (r, x).
\]

We have

\[
\varphi_{\mu, \lambda} (r, x) = j_{(n-1)/2} \left( r \sqrt{\mu^2 + \lambda^2} \right) e^{-i(\lambda \cdot x)}, \quad (2.4)
\]
where $j_{(n-1)/2}$ is the normalized Bessel function defined by

$$j_{(n-1)/2}(x) = 2^{(n-1)/2} \frac{\Gamma(n+1) J_{(n-1)/2}(z)}{2 z^{(n-1)/2}} = \frac{\Gamma(n+1)}{2} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma((2k+1+n)/2)} \left(\frac{z}{2}\right)^{2k}$$

with $J_{(n-1)/2}$ the Bessel function of first kind and index $(n-1)/2$ [9, 15], and if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we put $\lambda^2 = \lambda_1^2 + \cdots + \lambda_n^2$ and $\langle \lambda/x \rangle = \lambda_1 x_1 + \cdots + \lambda_n x_n$.

The normalized Bessel function $j_{(n-1)/2}$ has the following Mehler integral representation:

$$\forall r \in \mathbb{R}, \quad j_{(n-1)/2}(r) = \frac{2 \Gamma((n+1)/2)}{\sqrt{\pi} \Gamma(n/2)} \int_0^1 (1-t^2)^{(n-1)/2-1} \cos(tr) dt,$$ (2.6)

and therefore

$$\forall k \in \mathbb{N}, \quad \forall r \in \mathbb{R}, \quad |j_{(n-1)/2}^{(k)}(r)| \leq 1.$$ (2.7)

Moreover, for all $\lambda \in \mathbb{C}$, the function

$$r \mapsto j_{(n-1)/2}(\lambda r)$$ (2.8)

is the unique solution of the differential equation

$$lu(r) = -\lambda^2 u(r),$$

$$u(0) = 1, \quad u'(0) = 0,$$ (2.9)

where $l$ is the Bessel operator defined on $]0, +\infty[$ by (1.3).

On the other hand, the function $\varphi_{\mu, \lambda}$ is the unique solution of the system

$$D_j v(r, x) = -i\lambda_j v(r, x), \quad j = 1, 2, \ldots, n,$$

$$(l-\Delta) v(r, x) = -\mu^2 v(r, x),$$

$$v(0, 0) = 1; \quad \frac{\partial v}{\partial r}(0, x) = 0 \forall x \in \mathbb{R}^n,$$ (2.10)

where $D_j = \partial/\partial x_j$, and $\Delta$ is the Laplacian operator on $\mathbb{R}^n$:

$$\Delta = \sum_{j=1}^{n} D_j^2.$$ (2.11)

Now let $\Gamma$ be the set

$$\Gamma = \mathbb{R} \times \mathbb{R}^n \cup \{(it, x); (t, x) \in \mathbb{R} \times \mathbb{R}^n, |t| \leq \|x\|\}. \quad (2.12)$$
We denote by (see [11]) functions associated with the spherical mean operator. For this, we use the product formula for the translation operator associated with the spherical mean operator.

\[ \varphi_{\mu,\lambda}(r, x) = \varphi_{\mu}(r) \varphi_{\lambda}(x) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi} \Gamma(n/2)} \int_0^{\pi} \varphi_{\mu}(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y) \times (\sin \theta)^{n-1} \theta. \]

We define for all \((r, x) \in \mathbb{R} \times \mathbb{R}^n,\)
\[
\varphi_{\mu,\lambda}(r, x)\varphi_{\mu,\lambda}(s, y) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi} \Gamma(n/2)} \int_0^{\pi} \varphi_{\mu,\lambda}(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y) \times (\sin \theta)^{n-1} \theta.
\]

We denote by (see [11])

(A) \(d\nu(r, x)\) the measure defined on \([0, +\infty] \times \mathbb{R}^n\) by
\[
d\nu(r, x) = k_n r^n dr \otimes dx
\]
with
\[
k_n = \frac{1}{2^{(n-1)/2} \Gamma((n+1)/2) (2\pi)^{n/2}}.
\]

(B) \(L^p(d\nu), 1 \leq p \leq +\infty,\) the space of measurable functions on \([0, +\infty] \times \mathbb{R}^n,\) satisfying
\[
\|f\|_{p,\nu} = \left( \int_{\mathbb{R}^n} \int_0^{\infty} |f(r, x)|^p d\nu(r, x) \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty,
\]
\[
\|f\|_{\infty,\nu} = \operatorname{ess sup}_{(r, x) \in [0, +\infty] \times \mathbb{R}^n} |f(r, x)| < \infty, \quad p = +\infty;
\]

(C) \(d\gamma(\mu, \lambda)\) the measure defined on the set \(\Gamma\) by
\[
\int_{\Gamma} f(\mu, \lambda) d\gamma(\mu, \lambda) = k_n \left\{ \int_{\mathbb{R}^n} \int_0^{\infty} f(\mu, \lambda)(\mu^2 + ||\lambda||^2)^{(n-1)/2} \mu d\mu d\lambda \right.
\]
\[
+ \int_{\mathbb{R}^n} \int_0^{||\lambda||} f(\mu, \lambda)(||\lambda||^2 - \mu^2)^{(n-1)/2} \mu d\mu d\lambda \left. \right\}.
\]

(D) \(L^p(d\gamma), 1 \leq p \leq +\infty,\) the space of measurable functions on \(\Gamma,\) satisfying
\[
\|f\|_{p,\gamma} = \left( \int_{\Gamma} |f(\mu, \lambda)|^p d\gamma(\mu, \lambda) \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty,
\]
\[
\|f\|_{\infty,\gamma} = \operatorname{ess sup}_{(\mu, \lambda) \in \Gamma} |f(\mu, \lambda)| < \infty, \quad p = +\infty.
\]

**Definition 2.2.** (i) The translation operator associated with the spherical mean operator is defined on \(L^1(d\nu)\) by for all \((r, x), (s, y) \in [0, +\infty] \times \mathbb{R}^n,\)
\[
\tau_{(r, x)} f(s, y) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi} \Gamma(n/2)} \int_0^{\pi} f\left(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y\right)(\sin \theta)^{n-1} d\theta.
\]
(ii) A convolution product associated with the spherical mean operator of \( f, g \in L^1(d\nu) \) is defined by for all \((r, x) \in [0, +\infty[ \times \mathbb{R}^n\),

\[
f \ast g(r, x) = \int_{\mathbb{R}^n} \int_0^\infty f(s, y) \tau_{(r,-x)} \hat{g}(s, y) d\nu(s, y),
\]

(2.21)

where

\[
\hat{g}(r, x) = g(r, -x).
\]

(2.22)

We have the following properties.

(A) \( \tau_{(r,x)} \varphi_{\mu,\lambda}(s, y) = \varphi_{\mu,\lambda}(r, x) \varphi_{\mu,\lambda}(s, y) \).

(B) If \( f \in L^p(d\nu) \), \( 1 \leq p \leq +\infty \), then for all \((s, y) \in [0, +\infty[ \times \mathbb{R}^n\), the function \( \tau_{(s,y)} f \in L^p(d\nu) \), and we have

\[
\|\tau_{(s,y)} f\|_{p,\nu} \leq \| f\|_{p,\nu}.
\]

(2.23)

(C) Let \( 1 \leq p, q, r \leq +\infty \) such that \( 1/r = 1/p + 1/q - 1 \), then for all \( f \in L^p(d\nu) \) and all \( g \in L^q(d\nu) \), the function \( f \ast g \in L^r(d\nu) \), and we have

\[
\| f \ast g\|_{r,\nu} \leq \| f\|_{p,\nu} \| g\|_{q,\nu}.
\]

(2.24)

**Definition 2.3.** The Fourier transform associated with the spherical mean operator is defined on \( L^1(d\nu) \) by

\[
\forall (\mu, \lambda) \in \Gamma, \quad \hat{F} f(\mu, \lambda) = \int_{\mathbb{R}^n} \int_0^\infty f(r, x) \varphi_{\mu,\lambda}(r, x) d\nu(r, x).
\]

(2.25)

We have the following properties.

(A) For all \((\mu, \lambda) \in \Gamma\),

\[
\hat{F} f(\mu, \lambda) = B_0 \hat{F} f(\mu, \lambda),
\]

(2.26)

where for all \((\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n\),

\[
\hat{F} f(\mu, \lambda) = \int_{\mathbb{R}^n} \int_0^\infty f(r, x) j_{(n-1)/2}(r\mu) e^{-i(\lambda \cdot x)} d\nu(r, x),
\]

(2.27)

\[
\forall (\mu, \lambda) \in \Gamma, \quad B f(\mu, \lambda) = f \left( \sqrt{\mu^2 + \lambda^2}, \lambda \right).
\]

(B) For \( f \in L^1(d\nu) \) such that \( \hat{F} f \in L^1(dy) \), we have the inversion formula for \( \hat{F} \): for almost every \((r, x) \in [0, +\infty[ \times \mathbb{R}^n\),

\[
f(r, x) = \iint_{\Gamma} \hat{F} f(\mu, \lambda) \overline{\varphi_{\mu,\lambda}(r, x)} d\gamma(\mu, \lambda).
\]

(2.28)
(C) Let $f$ be in $L^1(d\nu)$. For all $(s, y) \in [0, +\infty[ \times \mathbb{R}^n$, we have
\[ \forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}(\tau_{s, -y})f(\mu, \lambda) = \varphi_{\mu, \lambda}(s, y)\mathcal{F}f(\mu, \lambda). \tag{2.29} \]

(D) For $f, g \in L^1(d\nu)$, we have
\[ \forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}(f \ast g)(\mu, \lambda) = \mathcal{F}f(\mu, \lambda)\mathcal{F}g(\mu, \lambda). \tag{2.30} \]

(E) For all $p \in [1, +\infty]$ and $f \in L^p(d\nu),$
\[ Bf \in L^p(dy), \quad \|Bf\|_{p,y} = \|f\|_{p,\nu}. \tag{2.31} \]

In particular, the mapping $B$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(dy)$. The mapping $\mathcal{F}$ is also an isometric isomorphism from $L^2(d\nu)$ onto itself. Consequently, the Fourier transform $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(dy)$.

Thus,
\[ \forall f \in L^2(d\nu), \quad \mathcal{F}f \in L^2(dy), \quad \|\mathcal{F}f\|_{2,y} = \|f\|_{2,\nu}. \tag{2.32} \]

**Proposition 2.4 (see[11]).** Let $f$ be in $L^p(d\nu)$, with $p \in [1, 2]$. Then $\mathcal{F}f \in L^{p'}(dy)$, with $1/p + 1/p' = 1$, and
\[ \|\mathcal{F}f\|_{p',y} \leq \|f\|_{p,\nu}. \tag{2.33} \]

We denote by

(A) $S_\alpha(\mathbb{R} \times \mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, rapidly decreasing together with all their derivatives;

(B) $S_\alpha(\Gamma)$ the space of infinitely differentiable functions on $\Gamma$, even with respect to the first variable, rapidly decreasing together with all their derivatives; that means for all $k_1, k_2 \in \mathbb{N}$, for all $\alpha \in \mathbb{N}^n$,
\[ \sup \left\{ (1 + |\mu|^2 + \|\lambda\|^2)^{k_1} \left( \frac{\partial}{\partial \mu} \right)^{k_2} D_\alpha^f \varphi(\mu, \lambda) \right\} ; (\mu, \lambda) \in \Gamma \right\} < +\infty, \tag{2.34} \]

where
\[ \frac{\partial f}{\partial \mu}(\mu, \lambda) = \begin{cases} \frac{\partial}{\partial r}(f(r, \lambda)) & \text{if } \mu = r \in \mathbb{R}, \\ \frac{1}{i} \frac{\partial}{\partial t}(f(it, \lambda)) & \text{if } \mu = it, \ |t| \leq \|\lambda\|, \end{cases} \tag{2.35} \]
\[ D_\alpha^f = \left( \frac{\partial}{\partial \lambda_1} \right)^{a_1} \left( \frac{\partial}{\partial \lambda_2} \right)^{a_2} \cdots \left( \frac{\partial}{\partial \lambda_n} \right)^{a_n}, \]

(see [10]);
(C) $S'_* (\mathbb{R} \times \mathbb{R}^n)$ and $S'_* (\Gamma)$ are, respectively, the dual spaces of $S_* (\mathbb{R} \times \mathbb{R}^n)$ and $S_* (\Gamma)$. Each of these spaces is equipped with its usual topology.

**Remark 2.5.** From [10], the Fourier transform $\mathcal{F}$ is a topological isomorphism from $S_* (\mathbb{R} \times \mathbb{R}^n)$ onto $S_* (\Gamma)$. The inverse mapping is given by for all $(r, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\mathcal{F}^{-1} f (r, x) = \int_{\Gamma} f (\mu, \lambda) \varphi_{\mu, \lambda} (r, x) d\gamma (\mu, \lambda).$$  

(2.36)

**Definition 2.6.** The Fourier transform $\mathcal{F}$ is defined on $S'_* (\mathbb{R} \times \mathbb{R}^n)$ by

$$\forall T \in S'_* (\mathbb{R} \times \mathbb{R}^n), \quad \langle \mathcal{F} (T), \varphi \rangle = \langle T, \mathcal{F}^{-1} (\varphi) \rangle, \quad \varphi \in S_* (\Gamma).$$  

(2.37)

Since the Fourier transform $\mathcal{F}$ is an isomorphism from $S_* (\mathbb{R} \times \mathbb{R}^n)$ onto $S_* (\Gamma)$, we deduce that $\mathcal{F}$ is also an isomorphism from $S'_* (\mathbb{R} \times \mathbb{R}^n)$ onto $S'_* (\Gamma)$.

### 3. The space $\mathcal{M}_p (\mathbb{R} \times \mathbb{R}^n)$

We denote by

(A) $L$ the partial differential operator defined by

$$L = -\left( \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} \right) - \sum_{j=0}^{n} \frac{\partial^2}{\partial x_j^2};$$  

(3.1)

(B) for $f \in L^p (d\nu)$, $p \in [1, \infty]$, $T_f$ is the element of $S'_* (\mathbb{R} \times \mathbb{R}^n)$ defined by

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} \int_{0}^{\infty} f (r, x) \varphi (r, x) d\nu (r, x), \quad \varphi \in S_* (\mathbb{R} \times \mathbb{R}^n);$$  

(3.2)

(C) for $g \in L^p (d\gamma)$, $p \in [1, \infty]$, $T_g$ is the element of $S'_* (\Gamma)$ defined by

$$\langle T_g, \psi \rangle = \int_{\Gamma} g (\mu, \lambda) \psi (\mu, \lambda) d\gamma (\mu, \lambda), \quad \psi \in S_* (\Gamma).$$  

(3.3)

From Proposition 2.4 and Remark 2.5, we deduce that for all $f \in L^p (d\nu)$, $1 \leq p \leq 2$, $\mathcal{F} f$ belongs to the space $L^p (d\gamma)$ and we have

$$\mathcal{F} (T_f) = T_{\mathcal{F} (f)}.$$  

(3.4)

**Definition 3.1.** Let $p \in [1, \infty]$. We define $\mathcal{M}_p (\mathbb{R} \times \mathbb{R}^n)$ to be the set of measurable functions $f$ on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, and such that for all $k \in \mathbb{N}$ there exists $g_k \in L^p (d\nu)$ satisfying

$$L^k T_f = T_{g_k}.$$  

(3.5)

The space $\mathcal{M}_p (\mathbb{R} \times \mathbb{R}^n)$ is equipped with the topology generated by the family of norms

$$\gamma_{m, p} (f) = \max_{0 \leq k \leq m} \| g_k \|_{p, p}, \quad m \in \mathbb{N},$$  

(3.6)
where $g_k, k \in \mathbb{N}$, is the function given by the relation (3.5). Let

$$
d_p : M_p(\mathbb{R} \times \mathbb{R}^n) \times M_p(\mathbb{R} \times \mathbb{R}^n) \rightarrow [0, \infty[,
$$

$$
(f, g) \mapsto d_p(f, g) = \sum_{m=0}^{\infty} \frac{y_{m,p}(f - g)}{1 + y_{m,p}(f - g)}.
$$

(3.7)

Then $d_p$ is a distance on $M_p(\mathbb{R} \times \mathbb{R}^n)$. Moreover the sequence $(f_k)_{k \in \mathbb{N}}$ converges to 0 in $(M_p(\mathbb{R} \times \mathbb{R}^n), d_p)$ if and only if

$$
\forall m \in \mathbb{N}, \quad y_{m,p}(f_k) \xrightarrow[k \to \infty]{} 0.
$$

(3.8)

In the following, we will give some properties of the space $M_p(\mathbb{R} \times \mathbb{R}^n)$.

**Proposition 3.2.** $(M_p(\mathbb{R} \times \mathbb{R}^n), d_p)$ is a Frechet space.

**Proof.** Let $(f_m)_{m \in \mathbb{N}}$ be a Cauchy sequence in $(M_p(\mathbb{R} \times \mathbb{R}^n), d_p)$ and let $(g_{m,k})_{m \in \mathbb{N}} \subset L^p(d\nu)$ such that

$$
L^k T f_m = T g_{m,k}, \quad k \in \mathbb{N}.
$$

(3.9)

Then for all $k \in \mathbb{N}$, $(g_{m,k})_{m \in \mathbb{N}}$ is a Cauchy sequence in $L^p(d\nu)$. We put

$$
f = g_0 = \lim_{m \to \infty} f_m,
$$

$$
g_k = \lim_{m \to \infty} g_{m,k}, \quad k \in \mathbb{N}^*,
$$

(3.10)

in $L^p(d\nu)$. Thus

$$
\forall k \in \mathbb{N}, \quad T g_{m,k} \xrightarrow[m \to \infty]{} T g_k,
$$

(3.11)

in $S'_*(\mathbb{R} \times \mathbb{R}^n)$. Since $L^k$ is a continuous operator from $S'_*(\mathbb{R} \times \mathbb{R}^n)$ into itself, we deduce that

$$
L^k T f_m \xrightarrow[m \to \infty]{} L^k T f,
$$

(3.12)

in $S'_*(\mathbb{R} \times \mathbb{R}^n)$.

From relations (3.9) and (3.11), we deduce that

$$
\forall k \in \mathbb{N}, \quad L^k T f = T g_k.
$$

(3.13)

This proves that $f \in M_p(\mathbb{R} \times \mathbb{R}^n)$ and

$$
f_m \xrightarrow[m \to \infty]{} f
$$

(3.14)

in $(M_p(\mathbb{R} \times \mathbb{R}^n), d_p)$. □
Proposition 3.3. Let $p \in [1,2]$ and $f \in M_p(\mathbb{R} \times \mathbb{R}^n)$, then

(i) for all $k \in \mathbb{N}$, the function

$$(\mu,\lambda) \mapsto (1 + \mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(f)(\mu,\lambda)$$

belongs to the space $L^{p'}(dy)$ with $p' = p/(p - 1)$;

(ii) $M_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_c(\mathbb{R} \times \mathbb{R}^n) \subset \mathcal{C}_c(\mathbb{R} \times \mathbb{R}^n)$, where $\mathcal{C}_c(\mathbb{R} \times \mathbb{R}^n)$ is the space of continuous functions on $\mathbb{R} \times \mathbb{R}^n$ even with respect to the first variable.

Proof. (i) Let $f \in M_p(\mathbb{R} \times \mathbb{R}^n), 1 \leq p \leq 2$, and $g_k \in L^p(dy)$ such that

$$L^k T_f = T g_k \quad k \in \mathbb{N}. \quad (3.16)$$

From relation (3.4), we have

$$\mathcal{F}(T g_k) = T \mathcal{F}(g_k), \quad (3.17)$$

which gives

$$\mathcal{F}(L^k T_f) = T \mathcal{F}(g_k). \quad (3.18)$$

On the other hand

$$\mathcal{F}(L^k T_f) = (\mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(T_f) = T(\mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(f), \quad (3.19)$$

hence

$$(\mu^2 + 2\|\lambda\|^2)^k \mathcal{F}(f) = \mathcal{F}(g_k). \quad (3.20)$$

This equality, together with the fact that the function $\mathcal{F}(g_k)$ belongs to the space $L^{p'}(dy)$ implies (i).

(ii) Let $f \in M_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_c(\mathbb{R} \times \mathbb{R}^n)$. From the assertion (i) and relations (2.26) and (2.31), we deduce that for all $k \in \mathbb{N}$, the function

$$(r,x) \mapsto (r^2 + \|x\|^2)^k \tilde{\mathcal{F}}(f)$$

belongs to the space $L^{p'}(dy)$, in particular $\tilde{\mathcal{F}}(f) \in L^1(dy) \cap L^2(dy)$.

On the other hand, the transform $\tilde{\mathcal{F}}$ is an isometric isomorphism from $L^2(dy)$ onto itself, then from the inversion formula for $\tilde{\mathcal{F}}$ and using the continuity of the function $f$, we have for all $(r,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$f(r,x) = \int_{\mathbb{R}^n} \int_0^\infty \tilde{\mathcal{F}} f(\mu,\lambda) j_{n-1}/2(r\mu)e^{i(\lambda/x)} dy(\mu,\lambda). \quad (3.22)$$

Consequently, (ii) follows from relation (2.7) and the fact that for all $k \in \mathbb{N}, \alpha \in \mathbb{N}^n$, the function

$$(\mu,\lambda) \mapsto \mu^k \lambda^\alpha \tilde{\mathcal{F}}(\mu,\lambda) \quad (3.23)$$

belongs to the space $L^1(dy)$.
Proposition 3.4. Let $p \in [1,2]$, then, for all $r \in [2,\infty]$,

\[ \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_b(\mathbb{R} \times \mathbb{R}^n) \subset \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n). \]  

(3.24)

Proof. Let $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \cap \mathcal{C}_b(\mathbb{R} \times \mathbb{R}^n)$, $p \in [1,2]$, $r \geq 2$, and $r' = r/(r - 1)$. From Proposition 3.3, we deduce that $f \in \mathcal{C}_b(\mathbb{R} \times \mathbb{R}^n)$ and for all $k \in \mathbb{N}$, the function (3.21) belongs to the space $L^{p'}(dv)$. By applying Holder’s inequality, it follows that this last function belongs to the space $L^{r'}(dv)$. On the other hand, for all $(r,x) \in \mathbb{R} \times \mathbb{R}^n$,

\[ L^k f(r,x) = \int_{\mathbb{R}^n} \int_0^\infty (\mu^2 + \|\lambda\|^2)^k \tilde{\mathcal{F}}(f)(\mu,\lambda) j_{(n-1)/2}(r \mu) e^{i(\lambda \cdot x)} d\nu(\mu,\lambda) \]

\[ = \tilde{\mathcal{F}}((\mu^2 + \|\lambda\|^2)^k \tilde{\mathcal{F}}(f))(r,x). \]

(3.25)

From Proposition 2.4 and the fact that

\[ \|\tilde{\mathcal{F}}(g)\|_{r',\nu} = \|\tilde{\mathcal{F}}(g)\|_{r',\nu}, \quad g \in L^{r'}(dv), \]

we deduce that, for all $k \in \mathbb{N}$, the function $L^k f$ belongs to the space $L^{r'}(dv)$. □

4. The dual space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$

In this section, we will give a new characterization of the dual space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ of $\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$. We recall that for every $f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, the family $\{V_{m,p,\epsilon}(f), \ m \in \mathbb{N}, \ \epsilon > 0\}$ is a basic of neighborhoods of $f$ in $(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \langle \cdot, \cdot \rangle)$, where

\[ V_{m,p,\epsilon}(f) = \{g \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \ \gamma_{m,p}(f-g) < \epsilon\}. \]

(4.1)

In addition, $T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ if and only if there exist $m \in \mathbb{N}$ and $c > 0$ such that

\[ \forall f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \ \|\langle T, f \rangle\| \leq c \gamma_{m,p}(f). \]

(4.2)

For $f \in L^{p'}(dv)$ and $\varphi \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)$, we put

\[ \langle L^k(T_f), \varphi \rangle = \int_{\mathbb{R}^n} \int_0^\infty f(r,x) \psi_k(r,x) d\nu(r,x) \]

(4.3)

with $L^kT_f = T_{\psi_k}$. Then

\[ \|\langle L^k(T_f), \varphi \rangle\| \leq \|f\|_{p',\nu}\|\psi_k\|_{p',\nu} \leq \|f\|_{p',\nu}\gamma_{k,p}(\varphi). \]

(4.4)

This proves that for all $f \in L^{p'}(dv)$ and $k \in \mathbb{N}$, the functional $L^kT_f$ defined by the relation (4.3) belongs to the space $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$.

In the following, we will prove that every element of $\mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)$ is also of this type.
Theorem 4.1. Let \( T \in S'_* (\mathbb{R} \times \mathbb{R}^n) \). Then \( T \in \mathcal{M}'_p (\mathbb{R} \times \mathbb{R}^n) \), 1 \( \leq \) \( p < \infty \), if and only if there exist \( m \in \mathbb{N} \) and \( \{ f_0, \ldots, f_m \} \subset L^p (dv) \) such that

\[
T = \sum_{k=0}^{m} L^k T_{f_k},
\]  
(4.5)

where \( L^k T_{f_k} \) is given by relation (4.3).

Proof. It is clear that if

\[
T = \sum_{k=0}^{m} L^k T_{f_k}, \quad \{ f_0, \ldots, f_m \} \subset L^p (dv),
\]  
(4.6)

then \( T \) belongs to the space \( \mathcal{M}'_p (\mathbb{R} \times \mathbb{R}^n) \).

Conversely, suppose that \( T \in \mathcal{M}'_p (\mathbb{R} \times \mathbb{R}^n) \). From relation (4.2) there exist \( m \in \mathbb{N} \) and \( c > 0 \) such that

\[
\forall \varphi \in \mathcal{M}_p (\mathbb{R} \times \mathbb{R}^n), \quad | \langle T, \varphi \rangle | \leq c \gamma_{m,p} (\varphi).
\]  
(4.7)

Let

\[
(L^p (dv))^{m+1} = \{ (f_0, \ldots, f_m), f_k \in L^p (dv), 0 \leq k \leq m \}
\]  
(4.8)

equipped with the norm

\[
\| (f_0, \ldots, f_m) \|_{(L^p (dv))^{m+1}} = \max_{0 \leq k \leq m} \| f_k \|_{p,v}.
\]  
(4.9)

We consider the mappings

\[
\mathcal{A} : \mathcal{M}_p (\mathbb{R} \times \mathbb{R}^n) \longrightarrow (L^p (dv))^{m+1},
\]

\[
\varphi \mapsto (f_0, g_1, \ldots, g_m),
\]  
(4.10)

where

\[
L^k T_{\varphi} = T_{g_k}, \quad k \geq 1,
\]

\[
\mathcal{B} : \text{Im} (\mathcal{A}) \longrightarrow \mathbb{C},
\]

\[
\mathcal{B} (\mathcal{A} \varphi) = \langle T, \varphi \rangle.
\]  
(4.11)

From relation (4.2) we deduce that

\[
| \mathcal{B} \mathcal{A} (\varphi) | = | \langle T, \varphi \rangle | \leq c \| \mathcal{A} (\varphi) \|_{(L^p (dv))^{m+1}}.
\]  
(4.12)

This means that \( \mathcal{B} \) is a continuous functional on the subspace \( \text{Im} (\mathcal{A}) \) of the space \( (L^p (dv))^{m+1} \). From Hahn-Banach theorems, there exists a continuous extension of \( \mathcal{B} \) to \( (L^p (dv))^{m+1} \), denoted again by \( \mathcal{B} \).
By Riesz’s theorem there exist \((f_0, \ldots, f_m) \in (L^p(d\nu))^{m+1}\) such that for all \((\varphi_0, \ldots, \varphi_m) \in (L^p(d\nu))^{m+1}\),

\[
\mathcal{B}(\varphi_0, \ldots, \varphi_m) = \sum_{k=0}^{m} \int_{\mathbb{R}^n} \int_{0}^{\infty} f_k(r, x) \varphi_k(r, x) d\nu(r, x). \tag{4.13}
\]

By means of relation (4.3), we deduce that for \(\varphi \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)\), we have

\[
\langle T, \varphi \rangle = \sum_{k=0}^{m} \int_{\mathbb{R}^n} \int_{0}^{\infty} f_k(r, x) \varphi_k(r, x) d\nu(r, x) = \sum_{k=0}^{m} \langle L^k T f_k, \varphi \rangle. \tag{4.14}
\]

This completes the proof of Theorem 4.1. \(\square\)

**Proposition 4.2.** Let \(p \geq 2\). Then for all \(T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)\), there exist \(m \in \mathbb{N}\) and \(F \in L^2(d\gamma)\) such that

\[
\mathcal{T}(T) = T(1+\mu^2+2\|\lambda\|^2)^m F. \tag{4.15}
\]

**Proof.** Let \(T \in \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n)\). From Theorem 4.1 there exist \(m \in \mathbb{N}\) and \((f_0, \ldots, f_m) \in (L^p(d\nu))^{m+1}\), \(p' = p/(p-1)\), such that

\[
T = \sum_{k=0}^{m} L^k T f_k. \tag{4.16}
\]

Consequently

\[
\mathcal{T}(T) = \sum_{k=0}^{m} \mathcal{T}(L^k T f_k) = \sum_{k=0}^{m} (\mu^2 + 2\|\lambda\|^2)^k \mathcal{T}(T f_k). \tag{4.17}
\]

By using relation (3.4) we get (4.15), where

\[
F = \sum_{k=0}^{m} \frac{(\mu^2 + 2\|\lambda\|^2)^k}{(1+\mu^2+2\|\lambda\|^2)^m} \mathcal{T}(f_k), \tag{4.18}
\]

which proves the result. \(\square\)

**Proposition 4.3.** Let \(T \in S'_*(\mathbb{R} \times \mathbb{R}^n)\), then \(T \in \mathcal{M}'_2(\mathbb{R} \times \mathbb{R}^n)\) if and only if there exist \(m \in \mathbb{N}\) and \(F \in L^2(d\gamma)\) such that (4.15) holds.

**Proof.** From Proposition 4.2, we deduce that if \(T \in \mathcal{M}'_2(\mathbb{R} \times \mathbb{R}^n)\), then there exist \(m \in \mathbb{N}\) and \(F \in L^2(d\gamma)\) verifying (4.15). Conversely, suppose that (4.15) holds with \(F \in L^2(d\gamma)\). Since \(\mathcal{T}\) is an isometric isomorphism from \(L^2(d\nu)\) onto \(L^2(dy)\), then there exists \(G \in L^2(dy)\) such that \(\mathcal{T}(G) = F\) and from relation (3.4) we have

\[
\mathcal{T}(T G) = T F. \tag{4.19}
\]
Consequently

$$F(T) = F((I + L)^m T_G),$$

(4.20)

thus

$$T = \sum_{k=0}^{m} C_m^k L^k T_G,$$

(4.21)

and Theorem 4.1 implies that \( T \in M'_2(\mathbb{R} \times \mathbb{R}^n) \).

We denote by

(A) \( D_*(\mathbb{R} \times \mathbb{R}^n) \) the space of infinitely differentiable functions on \( \mathbb{R} \times \mathbb{R}^n \), even with respect to the first variable and with compact support, equipped with its usual topology;

(B) for \( a > 0 \), \( D_{*,a}(\mathbb{R} \times \mathbb{R}^n) \) the subspace of \( D_*(\mathbb{R} \times \mathbb{R}^n) \) consisting of function \( f \) such that \( \text{supp} f \subset B(0,a) = \{(r,x) \in \mathbb{R} \times \mathbb{R}^n, r^2 + ||x||^2 \leq a^2\} \);

(C) for \( a > 0 \), \( D'_{*,a}(\mathbb{R} \times \mathbb{R}^n) \) the dual space of \( D_{*,a}(\mathbb{R} \times \mathbb{R}^n) \); 

(D) for \( a > 0 \) and \( m \in \mathbb{N} \), \( W^m_a(\mathbb{R} \times \mathbb{R}^n) \) the space of function \( f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C} \) of class \( C^2m \) on \( \mathbb{R} \times \mathbb{R}^n \), even with respect to the first variable and with support in \( B(0,a) \), normed by

$$N_{\infty,m}^a(f) = \max_{0 \leq k \leq m} ||L^k(f)||_{\infty,y}. \quad (4.22)$$

**Proposition 4.4.** Let \( a > 0 \) and \( m \in \mathbb{N} \). Then there exists \( p_o \in \mathbb{N} \) such that for every \( p \in \mathbb{N} \), \( p \geq p_o \), it is possible to find \( \varphi_p \in W^m_a(\mathbb{R} \times \mathbb{R}^n) \) and \( \psi_p \in D_{*,a}(\mathbb{R} \times \mathbb{R}^n) \) satisfying

$$\delta = (I + L)^p T_{\varphi_p} + T_{\psi_p} \quad (4.23)$$

in \( S'_*(\mathbb{R} \times \mathbb{R}^n) \).

**Proof.** Let \( p \geq n + 1 \) and \( g_p \) the function defined by

$$\forall (\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^n, \quad g_p(\mu,\lambda) = \hat{\mathcal{F}} \left( \frac{1}{(1 + r^2 + ||x||^2)^p} \right)(\mu,\lambda). \quad (4.24)$$

Using relation (2.7), we deduce that there exists \( p_0 \in \mathbb{N} \) such that for all \( p \geq p_0 \) the function \( g_p \) is of class \( C^{2m} \) on \( \mathbb{R} \times \mathbb{R}^n \) (e.g., we can choose \( p_0 = 3n + 1 + 2m \)).

Now, we prove that the function \( g_p \) is infinitely differentiable on \( \mathbb{R} \times \mathbb{R}^n \setminus \{(0,\ldots,0)\} \). The function \( g_p \) can be written as

$$g_p(\mu,\lambda) = \frac{1}{2^{n-1/2} \Gamma(n+1/2)} \int_0^\infty \frac{1}{(1 + s^2)^{p}} j_{n-1/2} \left(s \sqrt{\mu^2 + ||\lambda||^2} \right) s^{2n} ds. \quad (4.25)$$
By relation (2.6) and Fubini’s theorem we get
\[
g_p(\mu, \lambda) = \frac{1}{2^{n-1/2}} \sqrt{\pi \Gamma(n)} \int_{-1}^{1} (1 - t^2)^{n-1} \left[ \int_{0}^{\infty} \frac{\cos \left( ts \sqrt{\mu^2 + \|\lambda\|^2} \right)}{(1 + s^2)^p} s^{2n} ds \right] dt
\]
(4.26)
where
\[
h_p(u) = \int_{0}^{\infty} \frac{\cos(su)}{(1 + s^2)^p} s^{2n} ds = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{isu}}{(1 + s^2)^p} s^{2n} ds.
\]
(4.27)

By standard calculus, we have
\[
\int_{0}^{\infty} \frac{\cos(su)}{(1 + s^2)^p} s^{2n} ds = e^{-u}P(u)
\]
(4.28)
with
\[
P(u) = \frac{\pi}{2^{2p-1}} \sum_{k=0}^{p-1} \frac{C_{2p-2-k}^{p-1}}{k!} (2u)^k.
\]
(4.29)

On the other hand, we have
\[
h_p(u) = (-1)^n \left( \frac{d}{du} \right)^{2n} \left( \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{isu}}{(1 + s^2)^p} ds \right),
\]
(4.30)
then, we get
\[
\forall u \geq 0, \quad h_p(u) = Q_p(u)e^{-u},
\]
(4.31)
where \(Q_p\) is a real polynomial. Since \(h_p\) is an even function on \(\mathbb{R}\), then we deduce that
\[
\forall u \in \mathbb{R}, \quad h_p(u) = k_p(|u|),
\]
(4.32)
where \(k_p\) is the infinitely differentiable function defined on \(\mathbb{R}\) by
\[
k_p(u) = Q_p(u)e^{-u}.
\]
(4.33)

Now, the function
\[
u \mapsto F_p(u) = \frac{1}{2^{n-3/2}} \sqrt{\pi \Gamma(n)} \int_{0}^{1} (1 - t^2)^{n-1} k_p(tu) dt
\]
(4.34)
is infinitely differentiable on \(\mathbb{R}\) and we have
\[
g_p(\mu, \lambda) = F_p\left( \sqrt{\mu^2 + \|\lambda\|^2} \right).
\]
(4.35)
This shows that the function \( g_\rho \) is infinitely differentiable on \( \mathbb{R} \times \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \), even with respect to the first variable.

Let \( y \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n) \) such that

\[
\forall (r, x) \in \mathbb{R} \times \mathbb{R}^n, \quad r^2 + x^2 \leq \frac{a^2}{4}, \quad y(r, x) = 1. \tag{4.36}
\]

Since \((I + L)^p T_{g_\rho} = \delta\), we get

\[
y(I + L)^p T_{g_\rho} = (I + L)^p T_{g_\rho} = \delta. \tag{4.37}
\]

On the other hand, by using the fact that the function \( g_\rho \) is infinitely differentiable on \( \mathbb{R} \times \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \), we deduce that the function

\[
\varphi_p(r, x) = (y - 1)(I + L)^p g_\rho + (I + L)^p ((1 - y)g_\rho) \tag{4.38}
\]

belongs to the space \( \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n) \).

Moreover, from relation (4.37), we have

\[
T_{(y - 1)(I + L)^p g_\rho} = (y - 1)(I + L)^p T_{g_\rho} = 0, \tag{4.39}
\]

and this implies by using relation (4.38) that

\[
T_{\varphi_p} = T_{(I + L)^p ((1 - y)g_\rho)} = (I + L)^p T_{((1 - y)g_\rho)}. \tag{4.40}
\]

Hence,

\[
T_{\varphi_p} + (I + L)^p T_{g_\rho} = (I + L)^p T_{g_\rho} = \delta, \tag{4.41}
\]

and this completes the proof of the proposition by taking \( \psi_p = y g_\rho \).

To prove the main result of this section, that is, Theorem 4.7, we will define some new families of norms on the space \( \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n) \). We use these norms to prove that the elements of all bounded subset \( B' \subset \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n) \) can be continuously extended on the space \( \mathcal{W}_a^m(\mathbb{R} \times \mathbb{R}^n) \).

For \( f \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n) \), \( a > 0 \),

(A) \( P_m(f) = \max_{k + |\alpha| \leq m} \| (\partial/\partial r)^k D^\alpha f \|_{\nu, \nu} \),

(B) \( \tilde{P}_m(f) = \max_{k + |\alpha| \leq m} \| l^k D^\alpha f \|_{\nu, \nu} \),

(C) \( N_{p,m}(f) = \max_{0 \leq k \leq m} \| L^k(f) \|_{p, \nu}, \ p \in [1, \infty] \),

where \( l \) is defined by relation (1.3).

**Lemma 4.5.** (i) For all \( m \in \mathbb{N} \), there exists \( c_1 > 0 \) such that

\[
\forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n), \quad P_m(\varphi) \leq c_1 \tilde{P}_m(\varphi). \tag{4.42}
\]

(ii) For all \( m \in \mathbb{N} \), there exist \( c_2 > 0 \) and \( m' \in \mathbb{N} \) such that

\[
\forall \varphi \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n), \quad \tilde{P}_m(\varphi) \leq c_2 N_{p,m'}(\varphi). \tag{4.43}
\]
Spaces of $D_{2p}$ type

Proof. (i) Let $m \in \mathbb{N}$, and $\varphi \in \mathcal{D}_{s,a}(\mathbb{R} \times \mathbb{R}^n)$. By induction on $k$ we have

$$
\left( \frac{\partial}{\partial r} \right)^k D^a \varphi(r,x) = \sum_{s=0}^{k} P_s(r) \left( \frac{\partial}{\partial r^2} \right)^s D^a \varphi(r,x),
$$

where $P_s$ is a real polynomial. On the other hand, and also by induction, we deduce that for all $s \geq 1$,

$$
\left( \frac{\partial}{\partial r^2} \right)^s D^a \varphi(r,x) = \int_0^1 \cdots \int_0^1 \frac{1}{t_1^{n+2(s-1)} \cdots t_s^n} dt_1, \ldots, dt_s.
$$

From relations (4.44) and (4.45), it follows that there exists $c_{a,m} > 0$ satisfying

$$
P_m(\varphi) \leq c_{a,m} \tilde{P}_m(\varphi).
$$

(ii) Let $p \in [1, \infty]$, $m \in \mathbb{N}$, and $m_1 \in \mathbb{N}$ such that

$$
\left\| \frac{1}{1+\mu^2 + \|\lambda\|^2} \right\|_{1,\nu}^{m_1} < \infty,
$$

then, for all $(k,\alpha) \in \mathbb{N} \times \mathbb{N}^n$, $k + |\alpha| \leq m$, we have

$$
\left\| kD^a \varphi \right\|_{\infty,\nu} = \left\| \tilde{\mathcal{F}}^{-1} (\tilde{\mathcal{F}} (kD^a \varphi)) \right\|_{\infty,\nu}
\leq \left\| \tilde{\mathcal{F}} (kD^a \varphi) \right\|_{1,\nu}
\leq \left\| \mu^2 \lambda^a \tilde{\mathcal{F}}(\varphi) \right\|_{1,\nu}
\leq \left\| (1+\mu^2 + \|\lambda\|^2)^m \tilde{\mathcal{F}}(\varphi) \right\|_{1,\nu}
= \left\| \frac{1}{1+\mu^2 + \|\lambda\|^2} \tilde{\mathcal{F}}( (I+L)^{m+m_1} \varphi) \right\|_{1,\nu}
\leq \left\| \tilde{\mathcal{F}} ( (I+L)^{m+m_1} \varphi) \right\|_{\infty,\nu}
\leq \left\| (I+L)^{m+m_1} \varphi \right\|_{1,\nu},
$$

and by Holder’s inequality, we get

$$
\left\| kD^a \varphi \right\|_{\infty,\nu} \leq \left\| \frac{1}{(1+\mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1,\nu} \left( \nu(B(0,a)) \right)^{1/p'} \left\| (I+L)^{m+m_1} \varphi \right\|_{p,\nu}
\leq \left\| \frac{1}{(1+\mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1,\nu} \left( \nu(B(0,a)) \right)^{1/p'} 2^{m+m_1} N_{p,m+m_1}(\varphi),
$$

which implies that

$$
\tilde{P}_m(\varphi) \leq 2^{m+m_1} \left( \nu(B(0,a)) \right)^{1/p'} \left\| \frac{1}{(1+\mu^2 + \|\lambda\|^2)^{m_1}} \right\|_{1,\nu} N_{p,m+m_1}(\varphi),
$$

and the proof of the lemma is complete. □
Theorem 4.6. Let $a > 0$ and $B'$ a weakly* bounded set of $D'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$. Then, there exists $m \in \mathbb{N}$ such that the elements of $B'$ can be continuously extended to $\mathcal{W}^m_a(\mathbb{R} \times \mathbb{R}^n)$. Moreover, the family of these extensions is equicontinuous.

Proof. Let $p \in [1, \infty]$. Since $B'$ is weakly* bounded in $D'_{*,a}(\mathbb{R} \times \mathbb{R}^n)$, then from [14] and Lemma 4.5 there exist a positive constant $c$ and $m \in \mathbb{N}$ such that for all $T \in B'$, for all $\varphi \in D_{*,a}(\mathbb{R} \times \mathbb{R}^n)$,

$$| \langle T, \varphi \rangle | \leq cN_{p,m}(\varphi). \quad (4.51)$$

We consider the mappings

$$A : \mathcal{W}^m_a(\mathbb{R} \times \mathbb{R}^n) \longrightarrow (L^p(d\nu))^{m+1},$$

$$\varphi \longrightarrow (L^k\varphi)_{0 \leq k \leq m}, \quad (4.52)$$

and for all $T \in B'$,

$$L_T : A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n)) \longrightarrow \mathbb{C},$$

$$\langle L_T, A\varphi \rangle = \langle T, \varphi \rangle. \quad (4.53)$$

From relation (4.51), we deduce that for all $\varphi \in D_{*,a}(\mathbb{R} \times \mathbb{R}^n)$,

$$| \langle L_T, A\varphi \rangle | \leq c\|A\varphi\|_{(L^p(d\nu))^{m+1}}. \quad (4.54)$$

This means that $L_T$ is a continuous functional on the subspace $A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n))$ of the space $(L^p(d\nu))^{m+1}$ and that for all $T \in B'$,

$$\|L_T\|_{A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n))} = \sup_{\|A\varphi\|_{(L^p(d\nu))^{m+1}} \leq 1} | \langle L_T, A\varphi \rangle | \leq c. \quad (4.55)$$

From the Hahn-Banach theorems, $L_T$ can be continuously extended on $(L^p(d\nu))^{m+1}$, denoted again by $L_T$. Furthermore, for all $T \in B'$,

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \sup_{\|\psi\|_{(L^p(d\nu))^{m+1}} \leq 1} | \langle L_T, \psi \rangle | = \|L_T\|_{A(D_{*,a}(\mathbb{R} \times \mathbb{R}^n))} \leq c. \quad (4.56)$$

Now, from the Riez theorem, there exists $(f_{T,k})_{0 \leq k \leq m} \subset L^p(d\nu)$ such that for all $\psi = (\psi_0, \ldots, \psi_m) \in (L^p(d\nu))^{m+1}$,

$$\langle L_T, \psi \rangle = \sum_{k=0}^m \int_{\mathbb{R}^n} \int_0^\infty f_{T,k}(r,x) \psi_k(r,x) d\nu \quad (4.57)$$

with

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \max_{0 \leq k \leq m} \|f_{T,k}\|_{L^p(d\nu)}. \quad (4.58)$$

Thus, from (4.56) it follows that for all $T \in B'$, for all $k \in \mathbb{N}$, $0 \leq k \leq m$,

$$\|f_{T,k}\|_{L^p(d\nu)} \leq c. \quad (4.59)$$
In particular, for \( \varphi \in \mathcal{W}_d^m(\mathbb{R} \times \mathbb{R}^n) \) we have

\[
\langle L_T, A \varphi \rangle = \sum_{k=0}^{m} \int_{\mathbb{R}} \int_{0}^{\infty} f_{T,k}(r,x)L^k(\varphi)(r,x)d\nu(r,x). \tag{4.60}
\]

Using H"older's inequality and relation (4.59), we get for all \( T \in B' \), for all \( \varphi \in \mathcal{W}_d^m(\mathbb{R} \times \mathbb{R}^n) \),

\[
\left| \langle L_T, A \varphi \rangle \right| \leq (m+1)c\left[ \nu(B(0,a)) \right]^{1/p}N_{\infty,m}(\varphi). \tag{4.61}
\]

This shows that the mapping \( L_T \circ A \) is a continuous extension of \( T \) on \( \mathcal{W}_d^m(\mathbb{R} \times \mathbb{R}^n) \) and that the family \( \{ L_T \circ A \}_{T \in B'} \) is equicontinuous, when applied to \( \mathcal{W}_d^m(\mathbb{R} \times \mathbb{R}^n) \). This completes the proof of Theorem 4.6. \( \square \)

In the following, we will give a new characterization of the space \( \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \).

**Theorem 4.7.** Let \( T \in S'_*(\mathbb{R} \times \mathbb{R}^n) \), \( p \in [1, \infty[ \), \( p' = p/(p-1) \). Then \( T \in \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \) if and only if for every \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \), the function \( T \ast \varphi \) belongs to the space \( L^{p'}(d\nu) \), where

\[
T \ast \varphi(r,x) = \langle T, \tau_{(r,-x)}\hat{\varphi} \rangle. \tag{4.62}
\]

**Proof.** Let \( T \in \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \). From Theorem 4.1, there exist \( m \in \mathbb{N} \) and \( f_0, \ldots, f_m \in L^{p'}(d\nu) \) such that

\[
T = \sum_{k=0}^{m} L^kT_{f_k}, \tag{4.63}
\]

in \( \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \). Thus, for every \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \),

\[
T \ast \varphi = \sum_{k=0}^{m} T_{f_k} \ast L^k \varphi = \sum_{k=0}^{m} f_k \ast L^k \varphi. \tag{4.64}
\]

Since, for all \( k \in \mathbb{N} \), \( 0 \leq k \leq m \), \( f_k \in L^{p'}(d\nu) \) and \( L^k \varphi \in L^{1}(d\nu) \), then from inequality (2.24), we deduce that \( f_k \ast L^k \varphi \in L^{p'}(d\nu) \). This implies that the function \( T \ast \varphi \) belongs to the space \( L^{p'}(d\nu) \).

Conversely, let \( T \in S'_*(\mathbb{R} \times \mathbb{R}^n) \) such that for every \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \) the function \( T \ast \varphi \) belongs to the space \( L^{p'}(d\nu) \). For \( \varphi, \psi \) in \( \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \), we have

\[
\langle T_{T \ast \varphi}, \psi \rangle = \langle T_{\varphi}, \psi \ast \hat{\varphi} \rangle = \langle T, \psi \ast \hat{\varphi} \rangle = \langle T_{T \ast \varphi}, \varphi \rangle. \tag{4.65}
\]

From H"older's inequality and using the hypothesis, we obtain

\[
\left| \langle T_{T \ast \varphi}, \psi \rangle \right| \leq \| T \ast \varphi \|_{p',\nu} \| \varphi \|_{p,\nu}, \tag{4.66}
\]

In particular, for \( \varphi \in \mathcal{W}_d^m(\mathbb{R} \times \mathbb{R}^n) \) we have
from which we deduce that the set
\[ B' = \{ T_{T_*\varphi}, \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n); \| \varphi \|_{p, \nu} \leq 1 \} \]  
(4.67)
is bounded in \( \mathcal{D}_*'(\mathbb{R} \times \mathbb{R}^n) \).

Now, using Theorem 4.6, it follows that for all \( a > 0 \) there exists \( m \in \mathbb{N} \) such that for all \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n), \| \varphi \|_{p, \nu} \leq 1 \), the mapping \( T_{T_*\varphi} \) can be continuously extended on the space \( W_a^m(\mathbb{R} \times \mathbb{R}^n) \) and the family of these extensions is equicontinuous, which means that there exists \( c > 0 \) such that for all \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n), \| \varphi \|_{p, \nu} \leq 1 \), for all \( \psi \in W_a^m(\mathbb{R} \times \mathbb{R}^n) \),
\[ |\langle T_{T_*\varphi}, \psi \rangle| \leq cN_{\infty,m}(\psi). \]  
(4.68)

This involves that for all \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n), \psi \in W_a^m(\mathbb{R} \times \mathbb{R}^n) \),
\[ |\langle T_{T_*\varphi}, \psi \rangle| \leq cN_{\infty,m}(\psi)\| \varphi \|_{p, \nu}. \]  
(4.69)

On the other hand, we have for all \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n), \psi \in W_a^m(\mathbb{R} \times \mathbb{R}^n) \),
\[ \langle T_{T_*\varphi}, \psi \rangle = \langle T^* T_{T}, \varphi \rangle, \]  
(4.70)
where for all \( \varphi \in S_*(\mathbb{R} \times \mathbb{R}^n), \)
\[ \langle T^* T_{T}, \varphi \rangle = \langle T, T^* T \varphi \rangle = \langle T, \psi \ast \varphi \rangle. \]  
(4.71)
Relations (4.69) and (4.70) lead to for all \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \),
\[ |\langle T^* T, \varphi \rangle| \leq cN_{\infty,m}(\psi)\| \varphi \|_{p, \nu}. \]  
(4.72)
This last inequality shows that the functional \( T^* T_{T} \) can be continuously extended on the space \( L^p(d\nu) \) and from Riez’s theorem, there exists \( g \in L_p'(d\nu) \) such that
\[ T^* T_{T} = T_g. \]  
(4.73)
Furthermore, from Proposition 4.4, there exist \( s \in \mathbb{N}, \psi_s \in W_a^m(\mathbb{R} \times \mathbb{R}^n), \) and \( \varphi_s \in \mathcal{D}_{*,a}(\mathbb{R} \times \mathbb{R}^n) \) satisfying
\[ \delta = (I + L)^s T_{\psi_s} + T_{\varphi_s}, \]  
(4.74)
then
\[ T = (I + L)^s (T^* T_{\psi}) + T^* T_{\varphi} = (I + L)^s (T^* T_{\psi}) + T^* T_{\psi_s}. \]  
(4.75)
We complete the proof by using the hypothesis, relation (4.73), and Theorem 4.1. \( \square \)

In the following, we will give a characterization of the bounded sets in \( M_p'(\mathbb{R} \times \mathbb{R}^n) \).
**Theorem 4.8.** Let \( p \in [1, \infty[\) and let \( B' \) be a subset of \( \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) \). The following assertions are equivalent:

(i) \( B' \) is weakly bounded in \( \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) \),

(ii) there exist \( c > 0 \) and \( m \in \mathbb{N} \) such that for every \( T \in B' \), it is possible to find \( f_0, T, \ldots, f_m, T \subset L^{p'}(d\nu) \) satisfying

\[
T = \sum_{k=0}^{m} L^k T_f \quad \text{with} \quad \max_{0 \leq k \leq m} \| f_k \|_{p', \nu} \leq c, \quad (4.76)
\]

(iii) for every \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \), the set \( \{ T \ast \varphi \} _{T \in B'} \) is bounded in \( L^{p'}(d\nu) \).

**Proof.** (1) Suppose that \( B' \) is weakly\(^*\) bounded in \( \mathcal{M}'_p(\mathbb{R} \times \mathbb{R}^n) \), then from [14] \( B' \) is equicontinuous. There exist \( c > 0 \) and \( m \in \mathbb{N} \) such that

\[
\forall T \in B', \forall f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n), \quad | \langle T, f \rangle | \leq cy_m(p) f. \quad (4.77)
\]

As in the proof of Theorem 4.6, we consider the mappings

\[
A : \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \rightarrow (L^p(d\nu))^{m+1},
\]

\[
f \mapsto (f, g_1, \ldots, g_m) \quad (4.78)
\]

with

\[
L^k T_f = T_{g_k}, \quad 0 \leq k \leq m, \quad (4.79)
\]

and for all \( T \in B' \),

\[
L_T : A(\mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n)) \rightarrow \mathbb{C},
\]

\[
\langle L_T, A(f) \rangle = \langle T, f \rangle. \quad (4.80)
\]

Then, relation (4.77) implies that for all \( \varphi \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \),

\[
| L_T(A\varphi) | \leq c \| A\varphi \|_{(L^p(d\nu))^{m+1}}. \quad (4.81)
\]

Using Hahn-Banach’s theorem and Riez’s theorem, we deduce that \( L_T \) can be continuously extended on \( (L^p(d\nu))^{m+1} \), denoted again by \( L_T \), and that there exists \( (f_{T, k})_{0 \leq k \leq m} \subset L^{p'}(d\nu) \) verifying for all \( \psi = (\psi_0, \ldots, \psi_m) \in (L^p(d\nu))^{m+1} \),

\[
\langle L_T, \psi \rangle = \sum_{k=0}^{m} \int_{\mathbb{R}^n} \int_{0}^{\infty} f_{T, k}(r, x) \psi_k(r, x) d\nu(r, x) \quad (4.82)
\]

with

\[
\| L_T \|_{(L^p(d\nu))^{m+1}} = \max_{0 \leq k \leq m} \| f_{T, k} \|_{p', \nu} \leq c. \quad (4.83)
\]
In particular, if $\psi = A(f)$, $f \in M_p(\mathbb{R} \times \mathbb{R}^n)$,

$$\langle LT, A(f) \rangle = \langle T, f \rangle = \sum_{k=0}^{m} \langle L_k f T, f \rangle.$$  \hspace{1cm} (4.84)

This proves that (i) $\Rightarrow$ (ii).

(2) Suppose that there exist $c > 0$ and $m \in \mathbb{N}$ such that for every $T \in B'$ we can find $f_0, T, \ldots, f_m, T \subset L_{p'}(dv)$ satisfying

$$T = \sum_{k=0}^{m} L_k f T, \quad \max_{0 \leq k \leq m} \| f_{T,k} \|_{p',v} \leq c.$$  \hspace{1cm} (4.85)

Then for all $f \in M_p(\mathbb{R} \times \mathbb{R}^n)$, for all $T \in B'$,

$$\langle T, f \rangle = \sum_{k=0}^{m} \int_{\mathbb{R}^n} \int_{0}^{\infty} f_{T,k}(r,x) g_k(r,x) dv(r,x),$$  \hspace{1cm} (4.86)

consequently, for all $T \in B'$, for all $f \in M_p(\mathbb{R} \times \mathbb{R}^n)$,

$$\| \langle T, f \rangle \| \leq (m+1) c \gamma m, p(f).$$  \hspace{1cm} (4.87)

which means that the set $B'$ is weakly* bounded in $M_p'(\mathbb{R} \times \mathbb{R}^n)$ and proves that (ii) $\Rightarrow$ (i).

(3) Suppose that (ii) holds. Let $\varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n)$, then from Theorem 4.7 we know that for all $T \in B'$, the function $T * \varphi$ belongs to the space $L_{p'}(dv)$. But

$$T * \varphi = \sum_{k=0}^{m} T f_k * L_k \varphi,$$  \hspace{1cm} (4.88)

consequently, for all $T \in B'$,

$$\| T * \varphi \|_{p',v} \leq (m+1) c \gamma m, p(\varphi).$$  \hspace{1cm} (4.89)

This shows that the set $\{ T * \varphi \}_{T \in B'}$ is bounded in $L_{p'}(dv)$ and therefore (ii) involves (iii).

(4) Suppose that (iii) holds. Let $T \in B'$; for all $\varphi, \psi \in D_*(\mathbb{R} \times \mathbb{R}^n)$, we have

$$\| \langle T * \varphi, \psi \rangle \| = \| \langle T_{* \psi}, \varphi \rangle \| \leq \| T * \psi \|_{p',v} \| \varphi \|_{p,v},$$  \hspace{1cm} (4.90)

from which we deduce that the set

$$B' = \{ T_{* \varphi}, T \in B', \varphi \in D_*(\mathbb{R} \times \mathbb{R}^n); \| \varphi \|_{p,v} \leq 1 \}$$  \hspace{1cm} (4.91)

is bounded in $D_*(\mathbb{R} \times \mathbb{R}^n)$.

Now, using Theorem 4.6, it follows that for all $a > 0$, there exists $m \in \mathbb{N}$ such that for all $\varphi \in D_*(\mathbb{R} \times \mathbb{R}^n)$, $\| \varphi \|_{p,v} \leq 1$, and $T \in B'$, the mapping $T_{T * \varphi}$ can be continuously extended on the space $W_{ap}^r(\mathbb{R} \times \mathbb{R}^n)$ and the family of these extensions is equicontinuous,
which means that there exists \( c > 0 \) satisfying for all \( T \in B' \), for all \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \); for all \( \psi \in \mathcal{W}^m_\alpha(\mathbb{R} \times \mathbb{R}^n) \), (4.69) holds. On the other hand, for every \( T \in B' \), we have for all \( \varphi \in \mathcal{D}_*(\mathbb{R} \times \mathbb{R}^n) \); for all \( \psi \in \mathcal{W}^m_\alpha(\mathbb{R} \times \mathbb{R}^n) \), (4.70) holds. From relations (4.69) and (4.70), we deduce that the functional \( T \ast T\psi \) can be continuously extended on the space \( L^p(d\nu) \) and from Riez’s theorem, there exist \( g_T, \psi \in \mathcal{L}^p'(d\nu) \) such that

\[
T \ast T\psi = Tg_T,\psi. \tag{4.92}
\]

However, relations (4.69) and (4.70) involve that for all \( T \in B' \),

\[
||g_T,\psi||_{p',\nu} \leq cN_{\infty,m}(\psi). \tag{4.93}
\]

Again by Proposition 4.4, it follows that there exist \( s \in \mathbb{N} \), \( \psi_s \in \mathcal{W}^m_\alpha(\mathbb{R} \times \mathbb{R}^n) \), and \( \varphi_s \in \mathcal{D}_*,\alpha(\mathbb{R} \times \mathbb{R}^n) \) verifying for all \( T \in B' \),

\[
T = T \ast \delta = (I + L)^s(T \ast T\psi_s) + T_T \ast \varphi_s, \tag{4.94}
\]

and by relation (4.92) we get

\[
T = (I + L)^sTg_T,\psi + T_T \ast \varphi_s. \tag{4.95}
\]

Thus, from the hypothesis we obtain,

\[
\forall T \in B', \quad ||T \ast \varphi_s||_{p',\nu} \leq c_s, \tag{4.96}
\]

and using relation (4.93), we have

\[
\forall T \in B', \quad ||g_T,\psi_s||_{p',\nu} \leq cN_{\infty,m}(\varphi_s). \tag{4.97}
\]

This completes the proof. \( \square \)

5. Convolution product on the space \( \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \times \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \)

In this section, we define and study a convolution product on the space \( \mathcal{M}_p'(\mathbb{R} \times \mathbb{R}^n) \times \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \), \( 1 \leq r \leq p < \infty \), where \( \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \) is the closure of the space \( \mathcal{S}_s(\mathbb{R} \times \mathbb{R}^n) \) in \( \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \).

**Proposition 5.1.** Let \( p \in [1, \infty[. \) For every \( (r,x) \in [0, \infty[ \times \mathbb{R}^n \), the operator \( \tau_{(r,x)} \) given by Definition 2.2(i), is a continuous mapping from \( \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \) into itself.

**Proof.** Let \( f \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \) and \( g_k \in \mathcal{L}^p(d\nu) \) such that

\[
T_{g_k} = L^kTf, \quad k \in \mathbb{N}. \tag{5.1}
\]
Then for all $\varphi \in S_*(\mathbb{R} \times \mathbb{R}^n)$,
\begin{equation}
\langle L^k T_{T(\tau, x)} f, \varphi \rangle = \langle T_{T(\tau, x)} \tilde{g}_k, \varphi \rangle.
\end{equation}

Since the operator $\tau_{(r, x)}$ is continuous from $L^p(dy)$ into itself, we deduce that for all $f \in M_p(\mathbb{R} \times \mathbb{R}^n)$ and $(r, x) \in [0, \infty[ \times \mathbb{R}^n$, the function $\tau_{(r, x)} f$ belongs to the space $M_p(\mathbb{R} \times \mathbb{R}^n)$. Moreover,
\begin{equation}
y_{m, p}(\tau_{(r, x)} f) = \max_{0 \leq k \leq m} \| \tau_{(r, x)} \tilde{g}_k \|_{p, y} \leq \max_{0 \leq k \leq m} \| g_k \|_{p, y} = y_{m, p}(f),
\end{equation}
which shows that the operator $\tau_{(r, x)}$ is continuous from $M_p(\mathbb{R} \times \mathbb{R}^n)$ into itself. \hfill \square

**Definition 5.2.** A convolution product of $T \in M'_p(\mathbb{R} \times \mathbb{R}^n)$ and $f \in M_p(\mathbb{R} \times \mathbb{R}^n)$ is defined by for all $(r, x) \in [0, \infty[ \times \mathbb{R}^n$,
\begin{equation}
T * f(r, x) = \langle T, \tau_{(r, x)} \tilde{f} \rangle.
\end{equation}

Let $T \in M'_p(\mathbb{R} \times \mathbb{R}^n)$; $T = \sum_{k=0}^m L^k T_{f_k}$ with $\{ f_k \}_{0 \leq k \leq m} \subset L^{p'}(dv)$ and $\phi \in M_r(\mathbb{R} \times \mathbb{R}^n)$, $1 \leq r \leq p$, then for all $k \in \mathbb{N}$, there exists $\phi_k \in L'(dv)$ such that $T_{\phi_k} = L^k T_{\phi}$. From inequality (2.24), it follows that for $0 \leq k \leq m$, the function $f_k * \phi_k$ belongs to the space $L^q(dv)$ with $1/q = 1/r + 1/p' - 1 = 1/r - 1/p$ and by using the density of $S_*(\mathbb{R} \times \mathbb{R}^n)$ in $M_r(\mathbb{R} \times \mathbb{R}^n)$, we deduce that the expression $\sum_{k=0}^m f_k * \phi_k$ is independent of the sequence $\{ f_k \}_{0 \leq k \leq m}$. Then, we put
\begin{equation}
T \ast \phi = \sum_{k=0}^m f_k \ast \phi_k.
\end{equation}

This allows us to say that
\begin{equation}
M'_p(\mathbb{R} \times \mathbb{R}^n) \ast M_r(\mathbb{R} \times \mathbb{R}^n) \subset L^q(dv).
\end{equation}

**Lemma 5.3.** Let $1 \leq r \leq p < \infty$, $T \in M'_p(\mathbb{R} \times \mathbb{R}^n)$, and $\phi \in M_r(\mathbb{R} \times \mathbb{R}^n)$. Then, for all $k \in \mathbb{N}$
\begin{equation}
L^k T_{T \ast \phi} = T T_{\phi_k}
\end{equation}
with $T_{\phi_k} = L^k T_{\phi}$.

**Proof.** If $\phi \in S_*(\mathbb{R} \times \mathbb{R}^n)$, then the function $T \ast \phi$ is infinitely differentiable and we have
\begin{equation}
L^k(T_{T \ast \phi}) = T_{L^k(T \ast \phi)} = T T_{L^k \phi}.
\end{equation}
Therefore, the result follows from the density of $S_*(\mathbb{R} \times \mathbb{R}^n)$ in $M_r(\mathbb{R} \times \mathbb{R}^n)$. \hfill \square

**Proposition 5.4.** Let $1 \leq r \leq p < \infty$ and $q \in [1, \infty]$ such that
\begin{equation}
\frac{1}{q} = \frac{1}{r} - \frac{1}{p}.
\end{equation}
Then for every \( T \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \), the mapping
\[
\phi \mapsto T \ast \phi
\]
is continuous from \( \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \) into \( \mathcal{M}_q(\mathbb{R} \times \mathbb{R}^n) \).

**Proof.** Let \( T \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \); \( T = \sum_{k=0}^{m} L_k T_{f_k} \) with \( \{ f_k \}_{0 \leq k \leq m} \subset L^p(d\nu) \), then for \( \phi \in \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \), \( 1 \leq r \leq p \), and by using relation (5.5), we get
\[
T \ast \phi = \sum_{k=0}^{m} f_k \ast \phi_k, 
\]
where \( \phi_k \in L^r(d\nu) \) and
\[
T_{\phi_k} = L_k T_{\phi}. \tag{5.11}
\]
From Lemma 5.3, we have for all \( s \in \mathbb{N} \), for all \( \phi \in \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \),
\[
L_s T \ast \phi = T \ast \phi_s, \tag{5.12}
\]
Using relation (5.6), we deduce that the function \( T \ast \phi \) belongs to the space \( \mathcal{M}_q(\mathbb{R} \times \mathbb{R}^n) \). On the other hand, from relation (5.12), we obtain
\[
\gamma_{l,q}(T \ast \phi) = \max_{0 \leq s \leq l} \| T \ast \phi_s \|_{q,v}. \tag{5.13}
\]
According to relation (5.12), we have
\[
T \ast \phi_s = \sum_{k=0}^{m} f_k \ast \phi_{k+s}, \tag{5.14}
\]
consequently,
\[
\| T \ast \phi_s \|_{q,v} \leq \sum_{k=0}^{m} \| f_k \|_{p',v} \| \phi_{k+s} \|_{r,v} \leq \left( \sum_{k=0}^{m} \| f_k \|_{p',v} \right) \gamma_{m+p,r}(\phi). \tag{5.15}
\]
Hence
\[
\gamma_{l,q}(T \ast \phi) \leq \left( \sum_{k=0}^{m} \| f_k \|_{p',v} \right) \gamma_{m+p,r}(\phi), \tag{5.16}
\]
which proves the result. \( \square \)

**Definition 5.5.** Let \( 1 \leq p, q, r < \infty \) such that (5.9) holds. A convolution product of \( T \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \) and \( S \in \mathcal{M}_q(\mathbb{R} \times \mathbb{R}^n) \) is defined by for all \( \phi \in \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \),
\[
\langle S \ast T, \phi \rangle = \langle S, T \ast \phi \rangle. \tag{5.17}
\]

From this definition and Proposition 5.4 we deduce the following result.

**Proposition 5.6.** Let \( 1 \leq p, q, r < f \infty \) such that (5.9) holds. Then, for all \( T \in \mathcal{M}_p(\mathbb{R} \times \mathbb{R}^n) \) and \( S \in \mathcal{M}_q(\mathbb{R} \times \mathbb{R}^n) \), the functional \( S \ast T \) is continuous on \( \mathcal{M}_r(\mathbb{R} \times \mathbb{R}^n) \).
References


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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