THE BESSEL-STRUVE INTERTWINING OPERATOR ON \( \mathbb{C} \) AND MEAN-PERIODIC FUNCTIONS

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We give a description of all transmutation operators from the Bessel-Struve operator to the second-derivative operator. Next we define and characterize the mean-periodic functions on the space \( \mathcal{H} \) of entire functions and we characterize the continuous linear mappings from \( \mathcal{H} \) into itself which commute with Bessel-Struve operator.

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1. Introduction. Let \( A \) and \( B \) be two differential operators on a linear space \( X \). We say that \( \chi \) is a transmutation operator of \( A \) into \( B \) if \( \chi \) is an isomorphism from \( X \) into itself such that \( A\chi = \chi B \). This notion was introduced by Delsarte in [2] and some generalization and applications were given in [1, 3, 7, 10].

In the case where \( A \) and \( B \) are two differential operators having the same order and without any singularity on the complex plane, acting on the space of entire functions on \( \mathbb{C} \) denoted here by \( \mathcal{H} \), Delsarte showed in [3] the existence of a transmutation operator between \( A \) and \( B \) and gave some applications on the theory of mean-periodic functions on \( \mathbb{C} \).

In this paper, we consider the operator \( \ell_\alpha \), \( \alpha > -1/2 \), on \( \mathbb{C} \), given by

\[
\ell_\alpha f(z) = \frac{d^2 f}{dz^2}(z) + \frac{2\alpha + 1}{z} \left[ \frac{df}{dz}(z) - \frac{df}{dz}(0) \right],
\]

(1.1)

where \( f \) is an entire function on \( \mathbb{C} \). We call this operator Bessel-Struve operator on \( \mathbb{C} \).

The Bessel-Struve kernel \( S_\alpha(\lambda \cdot) \), \( \lambda \in \mathbb{C} \), which is the unique solution of the initial value problem \( \ell_\alpha u(z) = \lambda^2 u(z) \) with the initial conditions \( u(0) = 1 \) and \( u'(0) = \lambda \Gamma(\alpha + 1)/\sqrt{\pi} \Gamma(\alpha + 3/2) \), is given by

\[
S_\alpha(\lambda z) = j_\alpha(i\lambda z) - ih_\alpha(i\lambda z) \quad \forall z \in \mathbb{C},
\]

(1.2)

where \( j_\alpha \) and \( h_\alpha \) are the normalized Bessel and Struve functions (see [4]).

Moreover, the Bessel-Struve kernel is a holomorphic function on \( \mathbb{C} \times \mathbb{C} \) and it can be expanded in a power series in the form

\[
S_\alpha(\lambda z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^n}{c_n(\alpha)}, \quad c_n(\alpha) = \frac{\sqrt{\pi} n!\Gamma(n/2 + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma((n + 1)/2)}.
\]

(1.3)
The Bessel-Struve intertwining operator $\chi_\alpha$ is defined from the space $\mathcal{H}$ into itself by

$$\chi_\alpha f(z) = \sum_{n=0}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^n}{c_n(\alpha)} \quad \forall f \in \mathcal{H}, \quad z \in \mathbb{C}. \quad (1.4)$$

The dual intertwining operator $\dagger \chi_\alpha$ of $\chi_\alpha$ is defined on $\mathcal{H}'$ (the dual space of $\mathcal{H}$) by

$$\langle \dagger \chi_\alpha T, g \rangle = \langle T, \chi_\alpha g \rangle \quad \forall g \in \mathcal{H}, \quad T \in \mathcal{H}'. \quad (1.5)$$

The Bessel-Struve transform $\mathcal{F}_\alpha$ is defined on $\mathcal{H}'$ by

$$\mathcal{F}_\alpha(T)(\lambda) = \langle T, S_\alpha(-i\lambda \cdot) \rangle \quad \forall \lambda \in \mathbb{C}. \quad (1.6)$$

We use the transmutation operator $\chi_\alpha$ to define the Bessel-Struve translation operators $\tau_z, z \in \mathbb{C},$ associated with $\ell_\alpha,$ and the Bessel-Struve convolution on $\mathcal{H}$ and $\mathcal{H}'$. A function $f$ in $\mathcal{H}$ is said to be mean periodic if the closed subspace $\Omega(f)$ generated by $\tau_z f, z \in \mathbb{C},$ satisfies $\Omega(f) \neq \mathcal{H}$.

The objective of this paper is to characterize every transmutation operator of $\ell_\alpha$ into the second derivative operator from $\mathcal{H}$ into itself. Next, we study the mean-periodic functions associated with the Bessel-Struve operator and we characterize the continuous linear mappings from $\mathcal{H}$ into itself which commute with $\ell_\alpha$.

We point out that the harmonic analysis associated with differential and differential-difference operators allows many applications as the study of integral representations (see [9]), Plancherel, and reconstruction formulas and other applications as the use of wavelets packets in the inversion of transmutation operators for the J. L. Lions operator and the Dunkl operator (see [5, 6]).

The content of this paper is as follows.

In Section 2, we prove that the Bessel-Struve intertwining operator $\chi_\alpha$ is a topological isomorphism from $\mathcal{H}$ into itself satisfying

$$\forall f \in \mathcal{H}, \quad \ell_\alpha \chi_\alpha f = \chi_\alpha \frac{d^2}{dz^2} f,$$

$$\chi_\alpha f(0) = f(0), \quad (\chi_\alpha f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \quad (1.7)$$

Using this operator and its dual, we study the harmonic analysis associated with the operator $\ell_\alpha$ (Bessel-Struve transform, Bessel-Struve translation operators, and Bessel-Struve convolution). Next, we determine all transmutation operators $W$ from the Bessel-Struve operator $\ell_\alpha$ to the second derivative operator $d^2/dz^2$.

In Section 3, we study the mean-periodic functions associated with $\ell_\alpha$. Next, we give the central result of the paper, which characterizes the continuous linear mappings from $\mathcal{H}$ into itself which commute with $\ell_\alpha$. 

2. Bessel-Struve transmutation operators. In this section, we consider the normalized Bessel and Struve functions which allow to define the Bessel-Struve kernel. Next, we define the Bessel-Struve intertwining operator $\chi_{\alpha}$ and its dual $t\chi_{\alpha}$; after that, we study the harmonic analysis associated with the operator $\ell_{\alpha}$. The aim of this section is to characterize every transmutation operator of $\ell_{\alpha}$ into $d^2/dz^2$ from $H_5$ into itself.

Let $\alpha > -1/2$. The normalized Bessel function $j_{\alpha}$ is the kernel defined on $\mathbb{C}$ by

$$j_{\alpha}(z) = 2^{\alpha}\Gamma(\alpha + 1)\frac{J_{\alpha}(z)}{z^{\alpha}} = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n + \alpha + 1)},$$  \hspace{1cm} (2.1)$$

where $J_{\alpha}$ is the Bessel function of order $\alpha$ (see [4, 12]).

The normalized Struve function $h_{\alpha}$ is the kernel defined on $\mathbb{C}$ by

$$h_{\alpha}(z) = 2^{\alpha}\Gamma(\alpha + 1)\frac{H_{\alpha}(z)}{z^{\alpha}} = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n (z/2)^{2n+1}}{(n + 3/2) \Gamma(n + 3/2) \Gamma(n + \alpha + 3/2)},$$  \hspace{1cm} (2.2)$$

where $H_{\alpha}$ is the Struve function of order $\alpha$ (see [4, 12]).

This function has the following Poisson integral representation:

$$h_{\alpha}(z) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^1 (1 - t^2)^{\alpha - 1/2} \sin(zt) \, dt. \hspace{1cm} (2.3)$$

The function $z \rightarrow h_{\alpha}(i\lambda z)$, $\lambda, z \in \mathbb{C}$, is the unique solution of the differential equation

$$\ell_{\alpha}u(z) = \lambda^2 u(z), \quad u(0) = 0, \quad u'(0) = \frac{\lambda \Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 3/2)}. \hspace{1cm} (2.4)$$

The functions $h_{\alpha}$ and $j_{\alpha}$ are related by the formula

$$h_{\alpha}(z) = \frac{\Gamma(\alpha + 1)z}{\sqrt{\pi} \Gamma(\alpha + 3/2)} \int_0^{\pi/2} j_{\alpha+1/2}(z \sin \varphi) \sin \varphi \, d\varphi. \hspace{1cm} (2.5)$$

The Bessel-Struve kernel is the function $S_{\alpha}$ defined on $\mathbb{C}$ by

$$S_{\alpha}(z) = j_{\alpha}(iz) - ih_{\alpha}(iz). \hspace{1cm} (2.6)$$

This kernel can be expanded in a power series in the form

$$S_{\alpha}(z) = \sum_{n=0}^{+\infty} z^n c_n(\alpha), \quad c_n(\alpha) = \frac{\sqrt{\pi} n! \Gamma(n/2 + \alpha + 1)}{\Gamma(\alpha + 1) \Gamma((n + 1)/2)}, \hspace{1cm} (2.7)$$

and has the following integral representation:

$$S_{\alpha}(z) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^1 (1 - t^2)^{\alpha - 1/2} \exp(zt) \, dt. \hspace{1cm} (2.8)$$
The function $z \rightarrow S_{\alpha}(\lambda z)$, $\lambda \in \mathbb{C}$, is the unique solution of the differential equation
\[
\ell_{\alpha}u(z) = \lambda^2 u(z),
\]
\[
u(0) = 1, \quad u'(0) = \frac{\lambda \Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 3/2)}.
\] (2.9)

**Notations.**

(i) We denote by $\mathcal{H}$, the space of entire functions on $\mathbb{C}$, with the topology of the uniform convergence on compact subsets of $\mathbb{C}$. Thus $\mathcal{H}$ is a Fréchet space.

(ii) We denote by $\mathcal{H}'$, the dual space of $\mathcal{H}$.

**Proposition 2.1.** The operator $\chi_{\alpha}$ defined by
\[
\chi_{\alpha}f(z) = \sum_{n=0}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^n}{c_n(\alpha)}, \quad \forall f \in \mathcal{H}, \ z \in \mathbb{C},
\] (2.10)
is an isomorphism from $\mathcal{H}$ into itself satisfying the transmutation relation
\[
\forall f \in \mathcal{H}, \quad \ell_{\alpha}\chi_{\alpha}f = \chi_{\alpha}d^2f, \quad \chi_{\alpha}f(0) = f(0), \quad (\chi_{\alpha}f)'(0) = \frac{f'(0)}{c_1(\alpha)}.
\] (2.11)

The inverse of $\chi_{\alpha}$ is given by
\[
\chi_{\alpha}^{-1}(f)(z) = \sum_{n=0}^{+\infty} \ell_{\alpha}^n(f)(0) \frac{z^{2n}}{(2n)!} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_{\alpha}^n f)}{dz}(0) \frac{z^{2n+1}}{(2n+1)!}, \quad \forall f \in \mathcal{H}, \ z \in \mathbb{C}.
\] (2.12)

**Proof.** First we prove that the image of the function $f$ in $\mathcal{H}$ by $\chi_{\alpha}$ is an entire function, and that $\chi_{\alpha}$ is a continuous linear operator.

Since $f$ is an entire function, from the Cauchy integral formula, we have
\[
\forall n \in \mathbb{N}, \quad \frac{d^n f}{dz^n}(0) = \frac{n!}{2i\pi} \int_{C_R} \frac{f(w)}{w^{n+1}} dw,
\] (2.13)
where $C_R$ is a circle with center 0 and radius $R > 0$. Hence there exists a positive constant $M$ such that
\[
\forall n \in \mathbb{N}, \quad \left| \frac{d^n f}{dz^n}(0) \right| \frac{1}{c_n(\alpha)} \leq MR^{-n} \| f \|_R,
\] (2.14)
where
\[
\| f \|_R = \max_{|z| \leq R} |f(z)|.
\] (2.15)

As $R$ is arbitrary, the radius of convergence of the power series in (2.10) is infinite. Thus $\chi_{\alpha}(f)$ is an entire function.
Using (2.14), we obtain

\[ \forall f \in \mathcal{H}, \quad \|\chi_{\alpha}(f)\|_R \leq 2M\|f\|_2R. \]  

(2.16)

Thus \( \chi_{\alpha} \) defines a continuous linear mapping from \( \mathcal{H} \) into itself. Furthermore, using the fact that

\[ \forall n \geq 2, \quad \ell_{\alpha}(z^n) = \frac{c_n(\alpha)}{c_{n-2}(\alpha)} z^{n-2}, \]  

(2.17)

we get

\[ \forall z \in \mathbb{C}, \quad \ell_{\alpha}\chi_{\alpha}f(z) = \sum_{n=2}^{+\infty} \frac{d^n f}{dz^n}(0) \frac{z^{n-2}}{c_{n-2}(\alpha)} = \sum_{n=0}^{+\infty} \frac{d^{n+2} f}{dz^{n+2}}(0) \frac{z^n}{c_n(\alpha)} = \chi_{\alpha} \frac{d^2}{dz^2} f(z). \]  

(2.18)

It is clear that

\[ \chi_{\alpha}f(0) = f(0), \quad (\chi_{\alpha}f)'(0) = \frac{f'(0)}{c_1(\alpha)}. \]  

(2.19)

Suppose now that \( \chi_{\alpha}f = 0 \) for a certain \( f \in \mathcal{H} \). Then, according to (2.10), \( (d^n f / dz^n)(0) = 0, \ n \in \mathbb{N} \). Hence \( f = 0 \), thus we prove that \( \chi_{\alpha} \) is a one-to-one mapping from \( \mathcal{H} \) into itself.

Now we consider the operator \( \psi \) on \( \mathcal{H} \) defined by

\[ \psi f(z) = \sum_{n=0}^{+\infty} \ell_{\alpha}^{n} f(0) \frac{z^{2n}}{(2n)!} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_{\alpha}^{n} f)}{dz}(0) \frac{z^{2n+1}}{(2n+1)!}, \quad \forall z \in \mathbb{C}. \]  

(2.20)

In the same way as for \( \chi_{\alpha} \) and by a simple calculation, we prove that \( \psi \) is a continuous linear mapping from \( \mathcal{H} \) into itself and

\[ \forall f \in \mathcal{H}, \quad \chi_{\alpha}\psi f = \psi\chi_{\alpha}f = f. \]  

(2.21)

Then \( \chi_{\alpha} \) is a topological isomorphism from \( \mathcal{H} \) into itself.

**Remarks 2.2.** (i) The operator \( \chi_{\alpha} \) which is a transmutation operator from \( \ell_{\alpha} \) into \( d^2 / dz^2 \) on \( \mathcal{H} \) will be called the Bessel-Struve intertwining operator on \( \mathbb{C} \).

(ii) Formula (2.10) means that the Taylor coefficients of the image of an entire function by \( \chi_{\alpha} \) are multiplied by the Taylor coefficients of the Bessel-Struve kernel.

**Corollary 2.3.** (i) For \( \lambda, z \in \mathbb{C} \),

\[ S_{\alpha}(\lambda z) = \chi_{\alpha}(e^{\lambda \cdot })(z). \]  

(2.22)

(ii) Every function \( f \) in \( \mathcal{H} \) can be expanded in a power series:

\[ \forall z \in \mathbb{C}, \quad f(z) = \sum_{n=0}^{+\infty} \ell_{\alpha}^{n} f(0) \frac{z^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_{\alpha}^{n} f)}{dz}(0) \frac{z^{2n+1}}{c_{2n+1}(\alpha)}. \]  

(2.23)
Definition 2.4. The dual intertwining operator $^{t}\chi_{\alpha}$ of $\chi_{\alpha}$ is defined on $\mathcal{H}'$ by

$$\langle^{t}\chi_{\alpha}(T),g\rangle = \langle T,\chi_{\alpha}(g) \rangle \quad \forall g \in \mathcal{H}. \quad (2.24)$$

Remark 2.5. From the properties of the operator $\chi_{\alpha}$, we deduce that the operator $^{t}\chi_{\alpha}$ is an isomorphism from $\mathcal{H}'$ into itself; the inverse operator $(^{t}\chi_{\alpha})^{-1}$ is given by

$$\langle (^{t}\chi_{\alpha})^{-1}(T),g \rangle = \langle T,\chi_{\alpha}^{-1}(g) \rangle \quad \forall g \in \mathcal{H}. \quad (2.25)$$

Notations.
(i) We denote by $\operatorname{Exp}_a(C)$, $a > 0$, the space of functions of exponential type $a$. It is the space of functions $f \in \mathcal{H}$ such that

$$N_a(f) = \sup_{z \in \mathbb{C}} |f(z)| e^{-a|z|} < +\infty. \quad (2.26)$$

(ii) We denote by $\operatorname{Exp}(C)$, the space of functions with exponential type. It is given by

$$\operatorname{Exp}(C) = \bigcup_{a > 0} \operatorname{Exp}_a(C). \quad (2.27)$$

The space $\operatorname{Exp}(C)$ is endowed with the inductive limit topology.
(iii) We denote by $\mathcal{F}$, the classical Fourier transform defined on $\mathcal{H}'$ by

$$\mathcal{F}(T)(\lambda) = \langle T,e^{-i\lambda \cdot} \rangle \quad \forall \lambda \in \mathbb{C}. \quad (2.28)$$

(iv) We denote by $\ast_{\alpha}$, the classical convolution product given by

$$T \ast_{\alpha} f(z) = \langle T_{w}, f(w + z) \rangle \quad \forall T \in \mathcal{H}', f \in \mathcal{H}, z \in \mathbb{C}. \quad (2.29)$$

Definition 2.6. The Bessel-Struve transform $\mathcal{F}_{\alpha}$ of $T \in \mathcal{H}'$ is given by

$$\mathcal{F}_{\alpha}(T)(\lambda) = \langle T,S_{\alpha}(-i\lambda \cdot) \rangle \quad \forall \lambda \in \mathbb{C}. \quad (2.30)$$

Remark 2.7. From Corollary 2.3(i) and Definition 2.4, we obtain

$$\forall T \in \mathcal{H}', \quad \mathcal{F}_{\alpha}(T)(\lambda) = \mathcal{F}_{\alpha}(^{t}\chi_{\alpha}(T))(\lambda). \quad (2.31)$$

Proposition 2.8. The Bessel-Struve transform $\mathcal{F}_{\alpha}$ is a topological isomorphism from $\mathcal{H}'$ into $\operatorname{Exp}(\mathbb{C})$.

Proof. According to [8], the classical Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{H}'$ into $\operatorname{Exp}(\mathbb{C})$. Then the result follows from (2.25) and (2.31). \qed
**Lemma 2.9.** Let $f \in \mathcal{H}$. The Cauchy problem

$$\ell_{\alpha,z} u(z, w) = \ell_{\alpha,w} u(z, w),$$

$$u(0, w) = f(w), \quad \frac{\partial}{\partial z} u(0, w) = f'(w) \quad (2.32)$$

has a unique solution that is an entire function on $\mathbb{C} \times \mathbb{C}$ given by

$$u(z, w) = \chi_{\alpha,z} \chi_{\alpha,w} [X^{-1}_\alpha(f)(z + w)] \quad \forall z, w \in \mathbb{C}. \quad (2.33)$$

**Proof.** From Proposition 2.1, (2.32) is equivalent to the Cauchy problem

$$\frac{\partial^2}{\partial z^2} v(z, w) = \frac{\partial^2}{\partial w^2} v(z, w),$$

$$v(0, w) = X^{-1}_\alpha(f)(w), \quad \frac{\partial}{\partial z} v(0, w) = \frac{d}{dz} (X^{-1}_\alpha(f))(w), \quad (2.34)$$

where

$$v(z, w) = \chi_{\alpha,z} \chi_{\alpha,w} u(z, w). \quad (2.35)$$

But the solution of (2.34) is given by

$$v(z, w) = X^{-1}_\alpha(f)(z + w) \quad \forall z, w \in \mathbb{C}. \quad (2.36) \quad \Box$$

**Definition 2.10.** The Bessel-Struve translation operators $\tau_z$, $z \in \mathbb{C}$, associated with the operator $\ell_{\alpha}$, is defined on $\mathcal{H}$ by

$$\tau_z f(w) = \chi_{\alpha,z} \chi_{\alpha,w} [X^{-1}_\alpha(f)(z + w)] \quad \forall w \in \mathbb{C}. \quad (2.37)$$

The operator $\tau_z$, $z \in \mathbb{C}$, satisfies the following properties.

(i) For all $z \in \mathbb{C}$, the operator $\tau_z$ is linear continuous from $\mathcal{H}$ into itself.

(ii) For all $f \in \mathcal{H}$ and $z, w \in \mathbb{C}$,

$$\tau_z f(w) = \tau_w f(z), \quad \tau_0 f(w) = f(w), \quad \tau_z (\tau_w f) = \tau_w (\tau_z f), \quad \ell_{\alpha} \tau_z f = \tau_z \ell_{\alpha} f. \quad (2.38)$$

(iii) The following product formula holds:

$$\forall z, w \in \mathbb{C}, \quad \tau_z (S_\alpha(\lambda \cdot))(w) = S_\alpha(\lambda w) S_\alpha(\lambda z). \quad (2.39)$$

**Corollary 2.11.** Let $f \in \mathcal{H}$ and $z \in \mathbb{C}$. Then the function $w \to \tau_z f(w)$ can be expanded in the Taylor series:

$$\forall w \in \mathbb{C}, \quad \tau_z f(w) = \sum_{n=0}^{+\infty} \frac{\ell^n_\alpha f(z)}{c_{2n}(\alpha)} \frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell^n_\alpha f)}{dz}(z) \frac{w^{2n+1}}{c_{2n+1}(\alpha)}. \quad (2.40)$$
Proof. For $z, w \in \mathbb{C}$, we have
\[
\tau_z f(w) = \chi_{\alpha, w}^{-1}(f)(z + w).
\] (2.41)

Applying Corollary 2.3(ii) to the function $w \to \tau_z f(w)$, we obtain
\[
\tau_z f(w) = \sum_{n=0}^{\infty} \ell^n_{\alpha}(w) \frac{w^{2n}}{c_{2n}(\alpha)} + c_1(\alpha) \sum_{n=0}^{\infty} \frac{d(\ell^n_{\alpha}(\tau_z f)(\xi))}{dz} \left(0\right) \frac{w^{2n+1}}{c_{2n+1}(\alpha)},
\] (2.42)
which proves the result.

Definition 2.12. (i) The convolution product of two elements $T$ and $K$ in $\mathcal{H}'$ is defined by
\[
\langle T \ast K, f \rangle = \langle T_z, \langle K_w, \tau_z f(w) \rangle \rangle \quad \forall f \in \mathcal{H}.
\] (2.43)

(ii) Let $T \in \mathcal{H}'$ and $f \in \mathcal{H}$. The convolution product of $T$ and $f$ is the function in $\mathcal{H}$ defined by
\[
T \ast f(z) = \langle T_w, \tau_z f(w) \rangle \quad \forall z \in \mathbb{C}.
\] (2.44)

The convolution $\ast$ satisfies the following properties.
(i) Let $T, K \in \mathcal{H}'$ and let $f \in \mathcal{H}$. Then
\[
T \ast (K \ast f) = (T \ast K) \ast f.
\] (2.45)

(ii) Let $T, K \in \mathcal{H}'$. Then
\[
\mathcal{F}_\alpha(T \ast K) = \mathcal{F}_\alpha(T) \mathcal{F}_\alpha(K).
\] (2.46)

Proposition 2.13. Let $T \in \mathcal{H}'$ and let $f \in \mathcal{H}$. Then
\[
(\tau^{\alpha}_T)^{-1}(T \ast \alpha(f)) = \chi_{\alpha}(T \ast_0 f),
\]
\[
(\tau^{\alpha}_T)^{-1}(T \ast_0 \alpha^{-1}(f)) = \chi_{\alpha}^{-1}(T \ast f),
\] (2.47)
where $\ast_0$ is the classical convolution product given by (2.29).

Proof. From Definition 2.12, we have
\[
\forall z \in \mathbb{C}, \quad (\tau^{\alpha}_T)^{-1}(T \ast \alpha(f))(z) = \langle (\tau^{\alpha}_T)^{-1}(T), \tau_z (\alpha(f)) \rangle = \langle (\tau^{\alpha}_T)^{-1}(T), \tau_z (\alpha(f)) \rangle (\xi).
\] (2.48)

But from Definition 2.10, we obtain
\[
\forall \xi \in \mathbb{C}, \quad \chi_{\alpha}^{-1}(T \ast_0 \alpha^{-1}(f))(\xi) = \chi_{\alpha,z}(f)(\xi - z).
\] (2.49)
Thus
\[(t^{\dagger} \chi_\alpha)^{-1}(T) \ast \chi_\alpha(f)(z)\]
\[= \langle T_\xi, \chi_{\alpha,z}(f)(\xi - z) \rangle = \chi_{\alpha,z}(\langle T_\xi, f(\xi - z) \rangle) = \chi_\alpha(T \ast_0 f)(z),\]
which proves the first relation.

For the second relation, we have
\[\forall z \in \mathbb{C}, \quad \chi_\alpha(T) \ast_0 (t^{\dagger} \chi_\alpha)^{-1}(f)(z)\]
\[= \langle t^{\dagger} \chi_\alpha(T)_\xi, \chi_{\alpha}^{-1}(f)(\xi - z) \rangle = \langle T_\xi, \chi_\alpha^{-1}(f)(\xi - z) \rangle.\]

But
\[\forall z, \xi \in \mathbb{C}, \quad \chi_\alpha^{-1}(f)(\xi - z) = \chi_{\alpha,z}^{-1}(\tau_z f)(\xi).\]
So
\[\forall z \in \mathbb{C}, \quad \chi_\alpha(T) \ast (\chi_\alpha)^{-1}(f)(z) = \chi_{\alpha,z}^{-1}(T_\xi, f(\xi)) = \chi_{\alpha,z}^{-1}(T \ast f)(z),\]
which finishes the proof.

Now we are in position to derive the main result of this section.

**Notations.**
(i) We denote \(D = d/dz\).
(ii) We denote by \(\mathcal{G}_{D^2}\), the group of isomorphisms \(Y\) from \(\mathcal{H}\) into itself such that
\[YD^2 = D^2 Y.\]

**Theorem 2.14.** Every transmutation operator \(W\) of \(\ell_\alpha\) into \(D^2\) from \(\mathcal{H}\) into itself is of the form
\[Wf(z) = \left(\langle t^{\dagger} \chi_\alpha \rangle^{-1} T_0 \ast \chi_\alpha(f)(z) + \langle t^{\dagger} \chi_\alpha \rangle^{-1} T_1 \ast \chi_\alpha(f)(-z)\right) \quad \forall z \in \mathbb{C},\]
where \(T_0, T_1 \in \mathcal{H}'\).

**Proof.** It is clear that every transmutation operator \(W\) of \(\ell_\alpha\) into \(D^2\) from \(\mathcal{H}\) into itself is of the form \(W = \chi_\alpha Y\), where \(Y \in \mathcal{G}_{D^2}\). Then according to [3], every element \(Y\) of \(\mathcal{G}_{D^2}\) has the form
\[Yf(z) = T_0 \ast_0 f(z) + T_1 \ast_0 f(-z),\]
where \(T_0, T_1 \in \mathcal{H}'\). Thus, we can write
\[\forall z \in \mathbb{C}, \quad Wf(z) = \chi_\alpha(T_0 \ast_0 f)(z) + \chi_\alpha(T_1 \ast_0 f)(-z).\]
Hence the result follows from Proposition 2.13.
3. Mean-periodic functions and commutators of \( \ell_{\alpha} \)

### 3.1. Mean-periodic functions

**Definition 3.1.** A function \( f \) in \( \mathcal{H} \) is said to be mean periodic if the closed subspace \( \Omega(f) \) generated by \( \tau_{zf}, z \in \mathbb{C} \), satisfies
\[
\Omega(f) \neq \mathcal{H}.
\] (3.1)

From Hahn-Banach theorem, this definition is equivalent to the following.

**Definition 3.2.** A function \( f \) in \( \mathcal{H} \) is said to be mean periodic if there exists \( T \in \mathcal{H}' \setminus \{0\} \) such that
\[
\forall z \in \mathbb{C}, \quad T \ast f(z) = 0.
\] (3.2)

**Definition 3.3.** Let \( \lambda \in \mathbb{C} \) and \( \ell \in \mathbb{N} \). The function \( S_{\alpha,\ell}(\lambda, \cdot) \) is defined by
\[
S_{\alpha,\ell}(\lambda, z) = \frac{d\ell}{d\mu^\ell} S_{\alpha}(\mu z) \bigg|_{\mu=-i\lambda} \quad \forall z \in \mathbb{C}.
\] (3.3)

**Lemma 3.4.** Let \( \lambda \in \mathbb{C} \) and \( \ell \in \mathbb{N} \). Then the function \( S_{\alpha,\ell}(\lambda, \cdot) \) is mean periodic and
\[
\forall z \in \mathbb{C}, \quad S_{\alpha,\ell}(\lambda, z) = \chi_{\alpha}(\xi^\ell \exp(-i\lambda \xi))(z).
\] (3.4)

**Proof.** Let \( \lambda \in \mathbb{C} \) and \( \ell \in \mathbb{N} \). According to Proposition 2.8, there exists \( T \in \mathcal{H}' \setminus \{0\} \) such that
\[
\forall j = 0, \ldots, \ell, \quad \frac{d^j}{d\mu^j} (\mathcal{F}_{\alpha}(T))(\mu) \bigg|_{\mu=\lambda} = 0.
\] (3.5)

Then from the properties of the Bessel-Struve translation for every \( z \in \mathbb{C} \), we can write
\[
(T \ast S_{\alpha,\ell}(\lambda, \cdot))(z) = \bigg<T(w), \frac{d^\ell}{d\mu^\ell} (\tau_w (S_{\alpha}(\mu \cdot))(z)) \bigg|_{\mu=-i\lambda}\bigg>
\]
\[
= \bigg<T(w), \frac{d^\ell}{d\mu^\ell} (S_{\alpha}(\mu z)S_{\alpha}(\mu w)) \bigg|_{\mu=-i\lambda}\bigg>
\]
\[
= \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{d^{\ell-j}}{d\mu^{\ell-j}} (S_{\alpha}(\mu z)) \bigg|_{\mu=-i\lambda} \frac{d^j}{d\mu^j} (\mathcal{F}_{\alpha}(T))(\mu) \bigg|_{\mu=\lambda}
\]
\[
= 0.
\] (3.6)

Thus we prove that \( S_{\alpha,\ell}(\lambda, \cdot) \) is a mean-periodic function. The result follows from (1.3) and (2.10).

Let \( f \in \mathcal{H} \). The following proposition characterizes the functions which belong to \( \Omega(f) \).
**Proposition 3.5.** Let $f \in \mathcal{H}$, $\ell \in \mathbb{N}$, and $\lambda \in \mathbb{C}$. The function $S_{\alpha,j}(\lambda, \cdot)$, $0 \leq j \leq \ell$, belongs to $\Omega(f)$ if and only if for all $T$ in $\mathcal{H}'$ satisfying

$$\forall z \in \mathbb{C}, \quad T \ast f(z) = 0,$$

(3.7)

then

$$\frac{d^j}{d\mu^j}(\ell_\alpha(T))(\mu) \bigg|_{\mu = \lambda} = 0, \quad 0 \leq j \leq \ell.\quad (3.8)$$

**Proof.** If $S_{\alpha,j}(\lambda, \cdot)$, $0 \leq j \leq \ell$, belongs to $\Omega(f)$, then for all $T \in \mathcal{H}'$ satisfying (3.7) we have

$$\langle T, S_{\alpha,j}(\lambda, \cdot) \rangle = 0.\quad (3.9)$$

Then

$$\langle T, S_{\alpha,j}(\lambda, \cdot) \rangle = \frac{d^j}{d\mu^j} \langle T, S_{\alpha}(\mu \cdot) \rangle \bigg|_{\mu = -i\lambda},\quad (3.10)$$

The converse follows from the Hahn-Banach theorem. \qed

**Definition 3.6.** Let $f \in \mathcal{H}$ be a mean-periodic function. The spectrum $\text{Sp}(f)$ of $f$ is the set

$$\text{Sp}(f) = \{(\lambda, \ell), \lambda \in \mathbb{C}, \ell \in \mathbb{N}, S_{\alpha,j}(\lambda, \cdot) \in \Omega(f), \ 0 \leq j \leq \ell\}.\quad (3.11)$$

**Remarks 3.7.** (i) From Proposition 3.5, we have

$$\text{Sp}(f) = \left\{ (\lambda, \ell), \lambda \in \mathbb{C}, \ell \in \mathbb{N}, \frac{d^j}{d\mu^j}(\ell_\alpha(T))(\mu) \bigg|_{\mu = \lambda} = 0, j = 0, 1, \ldots, \ell, T \in (\Omega(f))^\perp \right\}.\quad (3.12)$$

(ii) If $\text{Sp}(f) \neq \emptyset$, we say that $\Omega(f)$ admits a spectral analysis associated with $\ell_\alpha$.

**Proposition 3.8.** Let $f \in \mathcal{H}$. Denote by $S(f)$ the closed subspace of $\mathcal{H}$ generated by \{D^k\ell_\alpha^n f\}_{n \in \mathbb{N}; k = 0, 1}$. Then $\Omega(f) = S(f)$.

**Proof.** According to Corollary 2.11, we have, for every $g \in \mathcal{H}$,

$$Dg = \lim_{w \to 0} \frac{1}{w} [\tau_wg - g],\quad (3.13)$$

$$\ell_\alpha g = \lim_{w \to 0} \frac{c_2(\alpha)}{w^2} [\tau_wg - g - wDg],\quad (3.14)$$

$$D\ell_\alpha g = \lim_{w \to 0} \frac{c_3(\alpha)}{c_1(\alpha)w^2} \left[ \tau_wg - g - w^2 - \frac{w^2}{c_2(\alpha)}\ell_\alpha g \right] \quad (3.15)$$

in the sense of the convergence in $\mathcal{H}$. 
Suppose that $g \in \Omega(f)$. Then, for every $w \in \mathbb{C}$, $\tau_w g \in \Omega(f)$. Hence we conclude that for $k = 0, 1$, $D^k \ell_{\alpha} g \in \Omega(f)$. By induction, we can prove that, for every $n \in \mathbb{N}$ and $k = 0, 1$, $D^k \ell_{\alpha}^n f \in \Omega(f)$. In particular, for every $n \in \mathbb{N}$ and $k = 0, 1$, $D^k \ell_{\alpha}^n f \in \Omega(f)$. Thus we conclude that $\Omega(f) \subset \Omega(f)$.

Let now $g \in S(f)$. Using once more Corollary 2.11, we prove that, for every $w \in \mathbb{C}$, $\tau_w g \in S(f)$. In particular, for every $w \in \mathbb{C}$, $\tau_w f \in S(f)$. Hence, $\Omega(f) = S(f)$.

**Corollary 3.9.** Let $f \in \mathcal{H}$. Then $f$ is a mean periodic if and only if $S(f) \neq \mathcal{H}$.

**Corollary 3.10.** Let $f \in \mathcal{H}$. Then $f$ is a mean-periodic function if and only if $\chi_{\alpha}^{-1}(f)$ is a classical mean-periodic function.

**Theorem 3.11.** Let $f \in \mathcal{H}$. Then $f$ is a mean-periodic function if and only if $f$ is a limit of finite linear combination of the functions $S_{\alpha,j}(\lambda, \cdot)$, $0 \leq j \leq \ell$, such that $(\lambda, \ell) \in \text{Sp}(f)$.

**Proof.** To see this property, we can use Lemma 3.4 and a celebrated result about classical mean-periodic functions established in [11, page 926].

**Corollary 3.12.** Every mean-periodic function such that $\text{Sp}(f) = \emptyset$ is zero.

3.2. The commutator of $\ell_{\alpha}$

**Notations.**

(i) We denote by $\mathcal{G}_{\alpha}$, the group of isomorphisms $Y$ of $\mathcal{H}$ into itself such that

$$Y \ell_{\alpha} = \ell_{\alpha} Y; \quad (3.16)$$

(ii) We denote by $\mathcal{G}_{\alpha}(f)$ (resp., $\mathcal{G}_{D^2}(f)$), the closed subspaces of $\mathcal{H}$ generated by $Yf$, $Y \in \mathcal{G}_{\alpha}$, (resp., $\mathcal{G}_{D^2}$).

**Proposition 3.13.** (i) The group $\mathcal{G}_{\alpha}$ is isomorphic to $\mathcal{G}_{D^2}$.

(ii) $\forall f \in \mathcal{H}$, $\mathcal{G}_{\alpha}(f) = \chi_{\alpha} \mathcal{G}_{D^2}(\chi_{\alpha}^{-1}(f))$. (3.17)

**Proposition 3.14.** The set of functions $f$ in $\mathcal{H}$ satisfying

$$\mathcal{G}_{\alpha}(f) \neq \mathcal{H} \quad (3.18)$$

with the set of mean-periodic functions is identified.

**Proof.** From Proposition 3.13, $f \in \mathcal{H}$ satisfies (3.18) if and only if $\chi_{\alpha}^{-1}(f)$ satisfies

$$\mathcal{G}_{D^2}(\chi_{\alpha}^{-1}(f)) \neq \mathcal{H}. \quad (3.19)$$

But these functions are classical mean-periodic functions. The result follows from Proposition 3.13.

Now we are able to state the main result of this paper.
Theorem 3.15. Let $L$ be a continuous linear mapping from $\mathcal{H}$ into itself. The following statements are equivalent.

(i) $L$ commutes with Bessel-Struve translation operators $\tau_z$, $z \in \mathbb{C}$, on $\mathcal{H}$, that is, $\tau_zL = L\tau_z$, $z \in \mathbb{C}$, on $\mathcal{H}$.

(ii) $L$ commutes with the Bessel-Struve operator $\ell_\alpha$ on $\mathcal{H}$, that is, $\ell_\alpha L = L\ell_\alpha$ on $\mathcal{H}$.

(iii) There exists a unique element $T$ in $\mathcal{H}'$ such that $Lf = T * f$, $f \in \mathcal{H}$.

(iv) There exists a complex Borel regular measure $\gamma$ having compact support on $\mathbb{C}$, for which for all $f \in \mathcal{H}$,

$$L(f)(z) = \int_{\mathbb{C}} (\tau_z f)(w) d\gamma(w) \quad \forall z \in \mathbb{C}. \quad (3.20)$$

(v) There exists $\Psi, \Phi \in \text{Exp}(\mathbb{C})$ such that for all $f \in \mathcal{H}$, $Lf = \Psi(\ell_\alpha)f + D\Phi(\ell_\alpha)f$, where $\Psi(\ell_\alpha)f$ and $D\Phi(\ell_\alpha)f$ are given by

$$\left[\Psi(\ell_\alpha)f\right](z) = \sum_{n=0}^{+\infty} a_{2n} \ell_\alpha^n f(z), \quad \forall z \in \mathbb{C}, \quad (3.21)$$

$$\left[D\Phi(\ell_\alpha)f\right](z) = c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d(\ell_\alpha^n f)}{dz}(z), \quad \forall z \in \mathbb{C},$$

where $\Psi(z) = \sum_{n=0}^{+\infty} a_{2n} z^n$ and $\Phi(z) = c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} z^n$.

Proof. (i)$\Rightarrow$(ii). From (3.13) and (3.14), we have

$$D(Lg) = \lim_{w \to 0} \frac{1}{w} \left[\tau_w Lg - Lg - wDLg\right] = L\left(\lim_{w \to 0} \frac{1}{w} \left[\tau_w g - g\right]\right) = L(Dg),$$

$$\ell_\alpha(Lg) = \lim_{w \to 0} \frac{c_2(\alpha)}{w^2} \left[\tau_w Lg - Lg - wDLg\right] = L\left(\lim_{w \to 0} \frac{c_2(\alpha)}{w^2} \left[\tau_w g - g\right]\right) = L(\ell_\alpha g). \quad (3.22)$$

Hence (i) implies (ii).

(ii)$\Rightarrow$(i). We decide the results from Corollary 2.11.

(i)$\Rightarrow$(iii). Assume that (i) holds. We define the functional $T$ on $\mathcal{H}$ as follows:

$$\langle T, f \rangle = L(f)(0), \quad f \in \mathcal{H}. \quad (3.23)$$

It is clear that $T$ is in $\mathcal{H}'$ and $Lf = T * f$, $f \in \mathcal{H}$.

(iii)$\Rightarrow$(iv). It follows immediately from Hahn-Banach and Riesz representation theorems.
(iv)⇒(v). Suppose that for all \( f \in \mathcal{H} \), we have

\[
\forall z \in \mathbb{C}, \quad L(f)(z) = \int_{\mathbb{C}} \tau_z f(w) d\gamma(w),
\]

where \( \gamma \) is a complex Borel regular measure with compact support. According to Corollary 2.11, we obtain for all \( z \in \mathbb{C} \),

\[
L(f)(z) = +\infty \sum_{n=0}^{+\infty} a_n(z) + c_1(\alpha) \sum_{n=0}^{+\infty} \frac{d(\ell_n f)}{dz}(z) + \mathcal{C} \sum_{n=0}^{+\infty} a_{n+1}^2(z) d\gamma(w).
\]

Hence

\[
L(f) = \Psi(\ell_\alpha) f + D\Phi(\ell_\alpha) f,
\]

where

\[
\Psi(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad \Phi(z) = c_1(\alpha) \sum_{n=0}^{+\infty} a_{n+1}^2 z^n,
\]

with, for every \( n \in \mathbb{N} \),

\[
a_n = \int_{\mathbb{C}} \frac{w^n}{c_n(\alpha)} d\gamma(w).
\]

Since \( \gamma \) has compact support on \( \mathbb{C} \), for certain \( a \) and \( C \), we have

\[
\forall n \in \mathbb{N}, \quad |a_n| \leq C \frac{a^n}{c_n(\alpha)}.
\]

Then we have

\[
\forall z \in \mathbb{C}, \quad \Psi(z) \leq C \sum_{n=0}^{+\infty} \frac{(|z|a)^n}{c_n(\alpha)} = CS(\alpha) |z|^a \leq Ce^{|z|^a}.
\]

Similarly we have

\[
\forall z \in \mathbb{C}, \quad |\Phi(z)| \leq c_1(\alpha) Ce^{|z|^a}.
\]

Thus we have proved that (v) is true.

(v)⇒(i). Suppose now that, for every \( f \in \mathcal{H} \) and \( z \in \mathbb{C} \),

\[
(Lf)(z) = \sum_{n=0}^{+\infty} a_{2n} \ell_n^p f(z) + c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d(\ell_n f)}{dz}(z),
\]

for a certain \( a_k \in \mathbb{C}, k \in \mathbb{N} \), where the series converges in \( \mathcal{H} \).
Hence, if $f \in \mathcal{H}$, since $\tau_z \ell_\alpha f = \ell_\alpha \tau_z f$, $z \in \mathbb{C}$, using (2.38) and the fact that $\tau_z$ is a continuous linear mapping from $\mathcal{H}$ into itself, we obtain for every $z, w \in \mathbb{C}$,

$$
\begin{align*}
\tau_w(Lf)(z) &= \sum_{n=0}^{+\infty} a_{2n} \tau_w \left( \ell^n_\alpha f \right)(z) + c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \tau_w \left( \frac{d}{dz} \ell^n_\alpha f \right)(z) \\
&= \sum_{n=0}^{+\infty} a_{2n} \ell^n_\alpha (\tau_w f)(z) + c_1(\alpha) \sum_{n=0}^{+\infty} a_{2n+1} \frac{d}{dz} \ell^n_\alpha (\tau_w f)(z) \\
&= L(\tau_w f)(z).
\end{align*}
$$

(3.33)

Hence (v) implies (i). \hfill \Box

REFERENCES


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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