

A COMMUTATIVITY-OR-FINITENESS CONDITION FOR RINGS

ABRAHAM A. KLEIN and HOWARD E. BELL

Received 20 May 2004

We show that a ring with only finitely many noncentral subrings must be either commutative or finite.

2000 Mathematics Subject Classification: 16U80, 16P99.

1. Preliminaries. In [1] the following theorem was proved.

THEOREM 1.1. *If R is a periodic ring having only finitely many noncentral subrings of zero divisors, then R is either finite or commutative.*

In view of this result, it was conjectured that any ring with only finitely many noncentral subrings is either finite or commutative. It is our principal goal to prove this conjecture; and in the process we provide a new proof of [Theorem 1.1](#).

For any ring R , the symbols N , D , Z , and $\wp(R)$ will denote, respectively, the set of nilpotent elements, the set of zero divisors, the center, and the prime radical; and R will be called reduced if $N = \{0\}$.

We will require the following three lemmas.

LEMMA 1.2 [5, Corollary 5]. *If R is any ring in which N is finite, then $R/\wp(R) = B \oplus C$, where B is reduced and C is a direct sum of finitely many total matrix rings over finite fields.*

LEMMA 1.3. *If R is any ring in which $\wp(R) \subseteq Z$, then zero divisors in $R/\wp(R)$ may be lifted to zero divisors in R .*

PROOF. We show that if $\bar{x} = x + \wp(R)$ is a zero divisor in $\bar{R} = R/\wp(R)$, then x is a zero divisor in R . Clearly, this is the case for $\bar{x} = \bar{0}$, so we assume that $\bar{x} \neq \bar{0}$ and that $\bar{y} \in \bar{R} \setminus \{\bar{0}\}$ is such that $\bar{x}\bar{y} = \bar{0}$. Then, $xy \in \wp(R)$, and there exists n such that $(xy)^n = 0$. Moreover, since $xyr \in \wp(R)$ for all $r \in R$, we have $xy \in Z$ and $xyr \in Z$ for all $r \in R$.

Since $y \notin \wp(R)$, y is not strongly nilpotent, therefore there exist $r_1, r_2, \dots, r_{n-1} \in R$ such that $yr_1yr_2 \cdots yr_{n-1}y \neq 0$. Since xy and all xyr_i are central, we get $x^n yr_1yr_2 \cdots yr_{n-1}y = xyr_1x^{n-1}yr_2 \cdots y = xyr_1xyr_2x^{n-2} \cdots y = xyr_1xyr_2 \cdots xyr_{n-1}xy = (xy)^n r_1r_2 \cdots r_{n-1} = 0$; hence x^n is a zero divisor in R , and so is x . \square

LEMMA 1.4 [3]. *If R is a periodic ring with $N \subseteq Z$, then R is commutative.*

2. The main theorems. We begin with results on rings having only finitely many noncentral subrings of zero divisors, which we will call D -nearly central rings (DNC-rings).

LEMMA 2.1. *If R is any DNC-ring, then $N \subseteq Z$ or N is finite.*

PROOF. Assume $N \setminus Z \neq \emptyset$ and let $a \in N \setminus Z$. By [1, Lemma 3], a has finite additive order and hence a generates a finite noncentral subring of zero divisors. Thus, $N \setminus Z$ is finite; and since $u + N \cap Z \subseteq N \setminus Z$ for all $u \in N \setminus Z$, we see that $N \cap Z$ is also finite. \square

LEMMA 2.2. *Let R be an infinite DNC-ring with N finite. Then every nil ideal of R is central.*

PROOF. Suppose I is a noncentral nil ideal. Since I is finite, the two-sided annihilator $A(I)$ is of finite index in R , hence is infinite. Moreover, for any subring S of $A(I)$, $S + I$ is a noncentral subring of zero divisors of R ; thus, there are only finitely many subrings $S + I$ with S a subring of $A(I)$. It follows at once that $A(I)$ has only finitely many finite subrings, since otherwise there exist infinitely many distinct finite subrings S_1, S_2, \dots , such that $S_1 + I = S_2 + I = \dots = S_n + I = \dots$ is a finite ring with infinitely many subrings.

It is well known that every infinite ring has infinitely many subrings (cf. [6]); hence $A(I)$ has infinitely many infinite subrings, and so does every infinite subring of $A(I)$. Thus $A(I)$ has an infinite strictly decreasing sequence $S_1 \supset S_2 \supset \dots$ of subrings; and for some j , $S_j + I = S_i + I$ for all $i \geq j$. For each $i \geq j$, choose $a_i \in S_i \setminus S_{i+1}$ and write $a_i = b_{i+1} + c_i$, where $b_{i+1} \in S_{i+1}$ and $c_i \in I$. Since I is finite, there exist h, k such that $j \leq h < k$ and $c_h = c_k$. It follows that $a_h = b_{h+1} + a_k - b_{k+1}$ so that $a_h \in S_{h+1}$, a contradiction. Therefore, I must be central. \square

LEMMA 2.3. *If R is any DNC-ring, then either R is finite or $N \subseteq Z$.*

PROOF. Let R be any infinite DNC-ring. By Lemmas 2.1 and 2.2, we may assume that N is finite and $\wp(R) \subseteq Z$; and it follows by Lemma 1.3 that $\bar{R} = R/\wp(R)$ is a DNC-ring. By Lemma 1.2, $\bar{R} = B \oplus C$, where B is an infinite reduced ring and C is a direct sum of finitely many total matrix rings over finite fields. Suppose $C \neq \{0\}$ and contains the total matrix ring K_n over the finite field K . We may assume $n > 1$; otherwise K_n can be incorporated into B . Now B has infinitely many subrings; and for each subring S of B , $S + Ke_{1n}$ is a noncentral subring of zero divisors of \bar{R} . Since \bar{R} is a DNC-ring, we must therefore have $C = \{0\}$ so that \bar{R} is reduced. It follows that in R , $N = \wp(R) \subseteq Z$. \square

REMARK 2.4. Theorem 1.1 follows at once from Lemmas 1.4 and 2.3. Thus, we have produced a proof of Theorem 1.1 which is very different from the one in [1].

We now arrive at our main result.

THEOREM 2.5. *If R is any ring with only finitely many noncentral subrings, then R is finite or commutative.*

PROOF. Suppose R is a counterexample with the minimum possible number of noncentral subrings. Then R is infinite and every proper infinite subring of R is commutative. Since R is a DNC-ring, Lemmas 1.4 and 2.3 show that all finite subrings of R are commutative as well; hence R is a so-called one-step noncommutative ring—a noncommutative ring in which every proper subring is commutative.

Note that if H is any set of pairwise noncommuting elements of R , the centralizers of the elements of H are pairwise distinct noncentral subrings of R . Hence any set of

pairwise noncommuting elements is finite, and by [2, Theorem 2.1], Z has finite index in R . But by a theorem of Ikeda [4], a one-step noncommutative ring with $[R : Z] < \infty$ must be finite. Thus, we have contradicted our assumption that R was a counterexample. \square

ACKNOWLEDGMENT. This work was supported by the Natural Sciences and Engineering Research Council of Canada, Grant 3961.

REFERENCES

- [1] H. E. Bell and F. Guerriero, *Some conditions for finiteness and commutativity of rings*, Int. J. Math. Math. Sci. **13** (1990), no. 3, 535-544.
- [2] H. E. Bell, A. A. Klein, and L. C. Kappe, *An analogue for rings of a group problem of P. Erdős and B. H. Neumann*, Acta Math. Hungar. **77** (1997), no. 1-2, 57-67.
- [3] I. N. Herstein, *A note on rings with central nilpotent elements*, Proc. Amer. Math. Soc. **5** (1954), 620.
- [4] M. Ikeda, *Über die einstufig nichtkommutativen Ringe*, Nagoya Math. J. **27** (1966), 371-379.
- [5] A. A. Klein and H. E. Bell, *Rings with finitely many nilpotent elements*, Comm. Algebra **22** (1994), no. 1, 349-354.
- [6] T. Szele, *On a finiteness criterion for modules*, Publ. Math. Debrecen **3** (1954), 253-256.

Abraham A. Klein: School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

E-mail address: aaklein@post.tau.ac.il

Howard E. Bell: Department of Mathematics, Brock University, St. Catharines, ON, Canada L2S 3A1

E-mail address: hb11@brocku.ca