IJMMS 2004:32, 1671–1677
PII. S0161171204306514
http://ijmms.hindawi.com

ON BIRATIONAL MONOMIAL TRANSFORMATIONS OF PLANE

ANATOLY B. KORCHAGIN

Received 20 June 2003 and in revised form 7 October 2003

We study birational monomial transformations of the form $\varphi(x:y:z)=(\varepsilon_1 x^{\alpha_1}y^{\beta_1}z^{\gamma_1} : \varepsilon_2 x^{\alpha_2}y^{\beta_2}z^{\gamma_2} : x^{\alpha_3}y^{\beta_3}z^{\gamma_3})$, where $\varepsilon_1,\varepsilon_2\in\{-1,1\}$. These transformations form a group. We describe this group in terms of generators and relations and, for every such transformation φ, we prove a formula, which represents the transformation φ as a product of generators of the group. To prove this formula, we use birationally equivalent polynomials $Ax+By+C$ and $Ax^p+By^q+Cx^ry^s$. If φ is the transformation which carries one polynomial onto another, then the integral powers of generators in the product, which represents the transformation φ, can be calculated by the expansion of p/q in the continued fraction.

2000 Mathematics Subject Classification: 14E07, 14L30.

1. Introduction. Birational monomial transformations of the projective plane have already found a lot of applications. For example, such transformations are actively used for construction of real algebraic curves and surfaces (see, e.g., [1, 4, 5, 6, 8, 9]). We think that formula (3.6) will be helpful for description of construction of algebraic objects.

In Section 2 we give a little exposition on projective polynomials in three variables. In Section 3 we describe the birational monomial group in terms of its generators and relations and give the statement of a theorem of decomposition of birational monomial transformations. In Section 4 we give the proof of the theorem.

2. Preliminaries. A nonzero homogeneous polynomial of degree n in three variables, x, y, z, is the expression

$$f(x,y,z)=\sum_{\omega_1+\omega_2+\omega_3=n} f_\omega x^{\omega_1}y^{\omega_2}z^{\omega_3},\quad \omega=(\omega_1,\omega_2,\omega_3). \tag{2.1}$$

The convex hull of the set $\{(\omega_1,\omega_2)\in\mathbb{R}^2 \mid f_\omega\neq 0\}$ is called the *Newton polygon* of the polynomial $f(x,y,z)$ and is denoted as $N(f)$. The plane with coordinates (ω_1,ω_2) is called the *plane of Newton's polygons*.

Every polynomial $f(x,y,1)$ can be represented in the form $f(x,y,1)=x^iy^jf^{\hat{}}(x,y,1)$, where i and j are nonnegative integers, and the polynomial $f^{\hat{}}(x,y,1)$ has no factors x and y. If φ is a transformation, then clearly $(f^{\hat{}}\circ\varphi)^{\hat{}}=(f\circ\varphi)^{\hat{}}$. It is also clear that the Newton polygon $N(f^{\hat{}})$ can be obtained from the Newton polygon $N(f)$ by translation in the plane of Newton's polygons by the vector $(-i,-j)$.

3. The birational monomial group. Let $(x:y:z)$ be homogeneous point coordinates in the projective plane $\mathbb{K}P^2$ over a field $\mathbb{K}$ and let (x,y) be affine coordinates in the affine chart $\mathbb{K}^2=\mathbb{K}P^2\setminus\{z=0\}$. A projective transformation φ is defined by the formula $\varphi(x:y:z)=(\varphi_1(x,y,z):\varphi_2(x,y,z):\varphi_3(x,y,z))$, where φ_1, φ_2, φ_3 are homogeneous polynomials of the same degree, assumed to have no common factors. For the transformation $\varphi(x:y:z)$, we define its natural restriction $\varphi(x,y)$ to the affine chart $\mathbb{K}^2=\mathbb{K}P^2\setminus\{z=0\}$ by the formula $\varphi(x,y)=(\varphi_1(x,y,1)/\varphi_3(x,y,1),\varphi_2(x,y,1)/\varphi_3(x,y,1))$.

Let $\mathrm{id}:\mathbb{K}P^2\to\mathbb{K}P^2$ be the identity map. If φ is a birational transformation, then we denote as usual

$$\varphi^0=\mathrm{id},\qquad \underbrace{\varphi\circ\cdots\circ\varphi}_{n\text{ times}}=\varphi^n,\qquad \underbrace{\varphi^{-1}\circ\cdots\circ\varphi^{-1}}_{n\text{ times}}=\varphi^{-n}. \tag{3.1}$$

Let $r_1,r_2,r_3:\mathbb{K}P^2\to\mathbb{K}P^2$ be maps defined by formulas $r_1(x:y:z)=((-x):y:z)$, $r_2(x:y:z)=(x:(-y):z)$, and $r_3(x:y:z)=(x:y:(-z))$. The set of maps $R=\{\mathrm{id},r_1,r_2,r_1\circ r_2\}$ with the operation of composition of the maps, with generators r_1 and r_2, and with relations

$$r_1^2=r_2^2=\mathrm{id},\qquad r_1\circ r_2=r_2\circ r_1, \tag{3.2}$$

is a group isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$. Note that $r_3=r_1\circ r_2$.

Let $s_1,s_2,s_3:\mathbb{K}P^2\to\mathbb{K}P^2$ be maps defined by formulas $s_1(x:y:z)=(x:z:y)$, $s_2(x:y:z)=(z:y:x)$, and $s_3(x:y:z)=(y:x:z)$. The set of maps $S=\{\mathrm{id},s_1\circ s_2,s_2\circ s_1,s_1,s_2,s_1\circ s_2\circ s_1\}$ with the operation of composition of the maps, with generators s_1 and s_2, and with relations

$$s_1^2=s_2^2=\mathrm{id},\qquad s_1\circ s_2\circ s_1=s_2\circ s_1\circ s_2, \tag{3.3}$$

is a group isomorphic to the symmetric group S_3. Note that $s_3=s_1\circ s_2\circ s_1$.

Let hy be the birational transformation defined by the formula $hy(x:y:z)=(x^2:yz:xz)$, whose inverse transformation is $hy^{-1}(x:y:z)=(xz:xy:z^2)$. Due to Newton, the transformation hy is called a *hyperbolism*. The set $H=\{\dots,hy^{-2},hy^{-1},\mathrm{id},hy,hy^2,\dots\}$ of integral powers of hy is a free group isomorphic to $\mathbb{Z}$.

Let $G=R*S*H$ be the free product of groups R, S, and H. This means that the set of generators of G is the union of the generators of R, S, and H, and the set of relations of G is the union of the relations of R, S, and H.

Definition 3.1. The factor group $G/\mathcal{R}$ with generators r_1, r_2, s_1, s_2, hy, where $\mathcal{R}$ is the system of relations

$$\mathcal{R}:\begin{cases} r_1\circ s_1=s_1\circ r_1,\\ r_2\circ s_1=s_1\circ r_2\circ r_1,\\ r_1\circ s_2=s_2\circ r_1\circ r_2,\\ r_2\circ s_2=s_2\circ r_2,\\ r_1\circ hy=hy\circ r_1\circ r_2,\\ r_2\circ hy=hy\circ r_2,\\ s_1\circ hy=hy\circ s_1\circ s_2\circ hy\circ s_1,\\ hy\circ s_2\circ hy=s_2,\end{cases} \tag{3.4}$$

is called the *group of birational monomial transformations* of $\mathbb{K}P^2$ and denoted by $T(\mathbb{K}P^2)$.

The group of birational monomial transformations $T(\mathbb{K}P^2)$ is a subgroup of the Cremona group $\mathrm{Cr}(\mathbb{K}P^2)$ (see [2, 3]).

Below in this paper, the word "transformation" without an adjective always means a "birational monomial transformation."

Every transformation $\varphi \in T(\mathbb{K}P^2)$ can be represented as a composition $\varphi_1 \circ \cdots \circ \varphi_s$, where each of $\varphi_1, \dots, \varphi_s$ is a positive integral power of one of the generators of the group $T(\mathbb{K}P^2)$, because $r_1^{-1} = r_1$, $r_2^{-1} = r_2$, $s_1^{-1} = s_1$, $s_2^{-1} = s_2$, and $hy^{-n} = s_2 \circ hy^n \circ s_2$. Every transformation φ can be represented in the form

$$\varphi(x:y:z) = \left(\varepsilon_1 x^{\alpha_1} y^{\beta_1} z^{\gamma_1} : \varepsilon_2 x^{\alpha_2} y^{\beta_2} z^{\gamma_2} : x^{\alpha_3} y^{\beta_3} z^{\gamma_3}\right), \tag{3.5}$$

where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$; α_i, β_i, and γ_i are nonnegative integers; and the monomials $x^{\alpha_1} y^{\beta_1} z^{\gamma_1}$, $x^{\alpha_2} y^{\beta_2} z^{\gamma_2}$, $x^{\alpha_3} y^{\beta_3} z^{\gamma_3}$ have no common factors. We stress this convention, for example, $(hy \circ hy)(x:y:z) = (x^4 : xyz^2 : x^3 z) = (x^3 : yz^2 : x^2 z)$, and accept only the last form. It means that one or two of α_1, α_2, α_3, one or two of β_1, β_2, β_3, one or two of γ_1, γ_2, γ_3 are equal to 0, and $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = \alpha_3 + \beta_3 + \gamma_3$. The integer $\alpha_1 + \beta_1 + \gamma_1$ is a degree of the transformation φ. For example, the degree of the transformations r_1, r_2, s_1, s_2 equals 1, and the degree of hy^n equals $|n| + 1$, where $n \in \mathbb{Z}$.

Denote the element $s_3 \circ hy \circ s_3 \in T(\mathbb{K}P^2)$ as hx. Its inverse is $hx^{-1} = s_2 \circ s_1 \circ hy \circ s_1 \circ s_2$. In homogeneous coordinates it is defined by formulae $hx(x:y:z) = (xz : y^2 : yz)$ and $hx^{-1}(x:y:z) = (xy : yz : z^2)$.

In the following theorem and below, the phrase "a polynomial $f(x,y,1)$ subjected to the transformation φ is carried onto the polynomial $l(x,y,1)$" means that $l(x,y,1) = [(f \circ \varphi^{-1})(x,y,z)|_{z=1}]\hat{}$.

Theorem 3.2. *Let p and q be mutually prime natural integers, $0 < q < p$. Let r and s be integers which satisfy the following conditions:* (1) $0 < r < p$, $0 \le s < q$, (2) $r/p + s/q < 1$, *and* (3) $r \equiv -q^{\phi(p)-1} (\operatorname{mod} p)$ *and* $s \equiv -p^{\phi(q)-1} (\operatorname{mod} q)$, *where $\phi(m)$ is the Euler function. Then every polynomial $f(x,y,1) = Ax^p + By^q + Cx^r y^s$, where at least two of A, B, C are not zero, subjected to the transformation*

$$\varphi = \left(hx^{(1-(-1)^{k+1})/2} \circ hy^{(1-(-1)^k)/2}\right)^{a_k} \circ \cdots \circ hx^{a_4} \circ hy^{a_3} \circ hx^{a_2} \circ hy^{a_1} \tag{3.6}$$

is carried onto the polynomial $l(x,y,1) = Ax + By + C$, where the integers $a_1, a_2, \dots, a_k$ are provided by expansion of p/q in the continued fraction with adjusted last denominator

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{\ddots}{+ \cfrac{1}{a_{k-1} + \cfrac{1}{(a_k + 1)}}}}; \tag{3.7}$$

in other words, $l = (f \circ \varphi^{-1})\hat{}$.

Corollary 3.3. (1) *If the polynomial* $f(x,y,1)$ *subjected to a transformation* ψ *is carried onto the polynomial* $\varepsilon_1 Ax+\varepsilon_2 By+\varepsilon_3 C$, *where* $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\{1,-1\}$, *then* $\psi = r_1^{(1/2)(1-\varepsilon_1)}\circ r_2^{(1/2)(1-\varepsilon_2)}\circ(r_1\circ r_2)^{(1/2)(1-\varepsilon_3)}\circ\varphi$.

(2) *If the polynomial* $f(x,y,1)$ *subjected to a transformation* ψ *is carried onto the polynomial* $A+Bx+Cy$, $Ay+B+Cx$, $Ax+B+Cy$, $A+By+Cx$, *or* $Ay+Bx+C$, *then* $\psi=s_1\circ s_2\circ\varphi$, $\psi=s_2\circ s_1\circ\varphi$, $\psi=s_1\circ\varphi$, $\psi=s_2\circ\varphi$, *or* $\psi=s_1\circ s_2\circ s_1\circ\varphi$, *respectively.*

(3) *If condition (2) of Theorem 3.2 is changed to condition* (2′), $r/p+s/q>1$, *and other conditions and notations are kept, and if the polynomial* $f(x,y,1)$ *subjected to a transformation* ψ *is carried onto the polynomial* $Ax+By+C$, *then* $\psi=\varphi\circ\mathrm{tr}=\mathrm{tr}\circ\varphi$, *where* $\mathrm{tr}=s_1\circ hy^{-1}\circ s_1\circ s_2\circ s_1\circ hy$ *is well-known standard (triangular) quadratic transformation* $\mathrm{tr}(x:y:z)=(yz:xz:xy)$.

Remark 3.4. There is only one more possible case: when $p=q=1$, which does not satisfy the theorem. In this case either $r=s=0$, and the polynomial $Ax+By+C$ is carried onto itself by the identity transformation: $\varphi=\mathrm{id}$, or $r=s=1$, and the polynomial $Ax+By+Cxy$ is carried onto the polynomial $Ax+By+C$ by the transformation $\varphi=s_3\circ\mathrm{tr}=s_3\circ s_1\circ hy^{-1}\circ s_1\circ s_2\circ s_1\circ hy$.

4. Proof of the theorem.

A birational monomial transformation ψ maps a polynomial f onto a polynomial $(f\circ\psi^{-1})\hat{}$. We find the connection between $N(f\circ\psi^{-1})$ and $N(f)$.

Every transformation ψ^{-1} written in the form (3.5) induces a generic linear mapping $A(\psi):\mathbb{R}^2\to\mathbb{R}^2$ of the plane of Newton's polygons, which can be defined on any monomial. Namely, if g is a monomial, say $g(x,y,z)=x^{\omega_1}y^{\omega_2}z^{\omega_3}$, then

$$(g\circ\psi^{-1})(x,y,z)=x^{\alpha_1\omega_1+\alpha_2\omega_2+\alpha_3\omega_3}y^{\beta_1\omega_1+\beta_2\omega_2+\beta_3\omega_3}z^{\gamma_1\omega_1+\gamma_2\omega_2+\gamma_3\omega_3},\tag{4.1}$$

thus, the linear mapping $A(\psi)$ is defined by the matrix

$$A_\psi=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\\ \beta_1&\beta_2&\beta_3\\ \gamma_1&\gamma_2&\gamma_3\end{pmatrix}.\tag{4.2}$$

And thus, $N(f\circ\psi^{-1})=A_\psi(N(f))$ for every polynomial f.

Remark that the generators of the birational monomial group have matrices

$$A_{\mathrm{id}}=A_{r_1}=A_{r_2}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$
$$A_{s_1}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad A_{s_2}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\quad A_{hy}=\begin{pmatrix}2&0&1\\0&1&0\\0&1&1\end{pmatrix}.\tag{4.3}$$

The set of matrices $A(T(\mathbb{K}P^2))=\{A_\varphi\mid\varphi\in T(\mathbb{K}P^2)\}$ is a subset in the linear group $GL(3,\mathbb{R})$ of 3×3-matrices. The operation $\diamond$ defined by the formula $A_\varphi\diamond A_\psi=A_{\varphi\circ\psi}$ converts the set $A(T(\mathbb{K}P^2))$ into a group, which is natural ($\varphi\mapsto A_\varphi$) homomorphic image of $T(\mathbb{K}P^2)$.

It is clear that every birational monomial transformation φ induces one-to-one correspondence between monomials of the polynomials f and $(f\circ\varphi^{-1})\hat{}$. Thus, to represent a transformation φ as a composition of generators of birational monomial group, it is enough to study the action of the transformation φ on a polynomial f whose Newton's polygon $N(f)$ is a triangle with the area $1/2$.

We consider a polynomial $f(x,y,1) = Ax^p + By^q + Cx^r y^s$, with $ABC \neq 0$, where p and q are mutually prime integers, $0 < q < p$, and r and s are integers, which satisfy the following conditions: (1) $0 < r < p$, $0 \le s < q$, (2) $r/p + s/q < 1$, and (3) $r \equiv -q^{\phi(p)-1} (\bmod\, p)$ and $s \equiv -p^{\phi(q)-1} (\bmod\, q)$, where $\phi(m)$ is the Euler function. The Newton polygon $N(f)$ is the triangle with integer vertices $(p,0)$, $(0,q)$, (r,s) which has no other integer points belonging to its interior and boundary but its three vertices. According to the Pick theorem [7], the area of such a triangle equals $1/2$. The genus of any curve $f(x,y,1) = Ax^p + By^q + Cx^r y^s = 0$ with such properties is zero and thus, all such curves are birationally equivalent.

Note that $hy^{-a}(x:y:1) = (x:x^a y:1)$ and $hx^{-a}(x:y:1) = (xy^a:y:1)$. We evaluate $(f\circ\varphi^{-1})(x,y,1)$ as follows.

The first step.

$$\begin{aligned}(f\circ hy^{-a_1})(x,y,1) = f(x,x^{a_1}y,1) &= x^{a_1q}\left(Ax^{b_1}+By^q+Cx^{c+a_1d-a_1q}y^d\right)\\ &= x^{u_1}y^{v_1}\left(Ax^{b_1}+By^q+Cx^{c_1}y^{d_1}\right),\end{aligned} \tag{4.4}$$

where $u_1 = a_1q$, $v_1 = 0$, $c_1 = c + a_1d - a_1q$, and $d_1 = d$.

The second step.

$$\begin{aligned}(f\circ hy^{-a_1}\circ hx^{-a_2})(x,y,1) &= (f\circ hy^{-a_1})(xy^{a_2},y,1)\\ &= x^{u_1}y^{a_2u_1+v_1+a_2b_1}\left(Ax^{b_1}+By^{b_2}+Cx^{c_1}y^{a_2c_1+d_1-a_2b_1}\right)\\ &= x^{u_2}y^{v_2}\left(Ax^{b_1}+By^{b_2}+Cx^{c_2}y^{d_2}\right),\end{aligned} \tag{4.5}$$

where $u_2 = u_1$, $v_2 = a_2u_1 + v_1 + a_2b_1$, $c_2 = c_1$, and $d_2 = a_2c_1 + d_1 - a_2b_1$.

The third step.

$$\begin{aligned}(f\circ hy^{-a_1}\circ hx^{-a_2}\circ hy^{-a_3})(x,y,1) &= (f\circ hy^{-a_1}\circ hx^{-a_2})(x,x^{a_3}y,1)\\ &= x^{u_3}y^{v_3}\left(Ax^{b_3}+By^{b_2}+Cx^{c_3}y^{d_3}\right),\end{aligned} \tag{4.6}$$

where $u_3 = u_2 + a_2v_2 + a_3b_2$, $v_3 = v_2$, $c_3 = c_2 + a_3d_2 - a_3b_2$, and $d_3 = d_2$.

The fourth step.

$$\begin{aligned}&(f\circ hy^{-a_1}\circ hx^{-a_2}\circ hy^{-a_3}\circ hx^{-a_4})(x,y,1)\\ &\quad= (f\circ hy^{-a_1}\circ hx^{-a_2}\circ hy^{-a_3})(xy^{a_4},y,1)\\ &\quad= x^{u_1}y^{a_2u_1+v_1+a_2b_1}\left(Ax^{b_1}+By^{b_2}+Cx^{c_1}y^{a_2c_1+d_1-a_2b_1}\right)\\ &\quad= x^{u_2}y^{v_2}\left(Ax^{b_1}+By^{b_2}+Cx^{c_2}y^{d_2}\right),\end{aligned} \tag{4.7}$$

where $u_2 = u_1$, $v_2 = a_2u_1 + v_1 + a_2b_1$, $c_2 = c_1$, and $d_2 = a_2c_1 + d_1 - a_2b_1$. We then proceed until the $(k-1)$th step.

The $(k-1)$*th step*. We have two cases.

The first case: k *is even*.

$$\begin{aligned}
&(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hx^{-a_{k-2}}\circ hy^{-a_{k-1}})(x,y,1)\\
&\quad=(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hx^{-a_{k-2}})(x,x^{a_{k-1}}y,1)\\
&\quad=x^{u_{k-1}}y^{v_{k-1}}(Ax+By^{b_{k-2}}+Cx^{c_{k-1}}y^{d_{k-1}}),
\end{aligned}\tag{4.8}$$

where $u_{k-1}=u_{k-2}+a_{k-1}v_{k-2}+a_{k-1}b_{k-2}$, $v_{k-1}=v_{k-2}$, $c_{k-1}=c_{k-2}+a_{k-1}d_{k-2}-a_{k-1}b_{k-2}$, and $d_{k-1}=d_{k-2}$.

The second case: k *is odd*.

$$\begin{aligned}
&(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hy^{-a_{k-2}}\circ hx^{-a_{k-1}})(x,y,1)\\
&\quad=(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hy^{-a_{k-2}})(xy^{a_{k-1}},y,1)\\
&\quad=x^{u_{k-1}}y^{v_{k-1}}(Ax^{b_{k-2}}+By+Cx^{c_{k-1}}y^{d_{k-1}}),
\end{aligned}\tag{4.9}$$

where $u_{k-1}=u_{k-2}$, $v_{k-1}=a_{k-1}u_{k-2}+v_{k-2}+a_{k-1}b_{k-2}$, $c_{k-1}=c_{k-2}$, and $d_{k-1}=a_{k-1}c_{k-2}+d_{k-2}-a_{k-1}b_{k-2}$.

The k*th step*. We have two cases.

The first case: k *is even*.

$$\begin{aligned}
&(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hy^{-a_{k-1}}\circ hx^{-a_k})(x,y,1)\\
&\quad=(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hy^{-a_{k-1}})(xy^{a_k},y,1)\\
&\quad=x^{u_k}y^{v_k}(Ax+By+Cx^{c_k}y^{d_k}),
\end{aligned}\tag{4.10}$$

where $u_k=u_{k-1}$, $v_k=a_ku_{k-1}+v_{k-1}+a_k$, $c_k=c_{k-1}$, and $d_k=a_kc_{k-1}+d_{k-1}-a_k$.

The second case: k *is odd*.

$$\begin{aligned}
&(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hx^{-a_{k-1}}\circ hy^{-a_k})(x,y,1)\\
&\quad=(f\circ hy^{-a_1}\circ hx^{-a_2}\circ\cdots\circ hx^{-a_{k-2}})(x,x^{a_k}y,1)\\
&\quad=x^{u_k}y^{v_k}(Ax+By+Cx^{c_k}y^{d_k}),
\end{aligned}\tag{4.11}$$

where $u_k=u_{k-1}+a_kv_{k-1}+a_k$, $v_k=v_{k-1}$, $c_k=c_{k-1}+a_kd_{k-1}-a_k$, and $d_k=d_{k-1}$.

This calculation shows that the integers $a_1,a_2,a_3,\dots,a_k$ satisfy the Euclidean algorithm with adjusted last row

$$\begin{aligned}
p&=a_1q+b_1,\\
q&=a_2b_1+b_2,\\
b_1&=a_3b_2+b_3,\\
&\ \vdots\\
b_{k-4}&=a_{k-2}b_{k-3}+b_{k-2},\\
b_{k-3}&=a_{k-1}b_{k-2}+1,\\
b_{k-2}&=a_k+1,
\end{aligned}\tag{4.12}$$

which provides the desired continued fraction.

Acknowledgment. I thank Inna Korchagina (Rutgers University), David Weinberg (Texas Tech University), and Victor Zvonilov (Syktyvkar State University, Russia) for several helpful discussions.

References

[1] B. Chevallier, *Singularités et topologies optimales des hypersurfaces algébriques réelles de petites dimensions*, These d'État, Université Paris VII, Paris, 1996.

[2] L. Godeaux, *Les Transformations Birationnelles du Plan*, Gauthier-Villars, Paris, 1927.

[3] H. P. Hudson, *Cremona Transformation in Plane and Space*, Cambridge University Press, Massachusetts, 1927.

[4] V. M. Kharlamov, *Topological types of nonsingular surfaces of degree* 4 *in* $\mathbf{R}P^3$, Funkcional. Anal. i Priložen. **10** (1976), no. 4, 55–68 (Russian).

[5] ______, *Isotopic types of nonsingular surfaces of degree* 4 *in* $\mathbf{R}P^3$, Funkcional. Anal. i Priložen. **12** (1978), no. 1, 86–87 (Russian).

[6] A. B. Korchagin, *Smoothing of* 6*-fold singular points and constructions of 9th degree M-curves*, Topology of Real Algebraic Varieties and Related Topics, Amer. Math. Soc. Transl. Ser. 2, vol. 173, American Mathematical Society, Rhode Island, 1996, pp. 141–155.

[7] Yu. A. Shashkin, *The Euler Characteristic*, Popular Lectures on Mathematics, vol. 58, Nauka, Moscow, 1984.

[8] O. Ya. Viro, *Gluing algebraic hypersurfaces, removing singularities, and construction of curves*, Proc. International Topological Conference (Leningrad, 1982), Nauka, Moscow, 1983, pp. 149–197.

[9] ______, *Gluing of plane real algebraic curves and constructions of curves of degrees* 6 *and* 7, Topology (Leningrad, 1982), Lecture Notes in Math., vol. 1060, Springer, Berlin, 1984, pp. 187–200.

Anatoly B. Korchagin: Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409-1042, USA

E-mail address: korchag@math.ttu.edu