

ON COMMON FIXED POINTS, PERIODIC POINTS, AND RECURRENT POINTS OF CONTINUOUS FUNCTIONS

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It is known that two commuting continuous functions on an interval need not have a common fixed point. However, it is not known if such two functions have a common periodic point. We had conjectured that two commuting continuous functions on an interval will typically have disjoint sets of periodic points. In this paper, we first prove that S is a nowhere dense subset of $[0, 1]$ if and only if $\{f \in C([0, 1]) : F_m(f) \cap \bar{S} \neq \emptyset\}$ is a nowhere dense subset of $C([0, 1])$. We also give some results about the common fixed, periodic, and recurrent points of functions. We consider the class of functions f with continuous ω_f studied by Bruckner and Ceder and show that the set of recurrent points of such functions are closed intervals.

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1. Introduction. The subject of commuting continuous functions on an interval has been considered by a group of researchers in the 20th century. In the 1920s, J. F. Ritt in a sequence of papers investigated the algebraic properties of function composition as a binary operation on the set of rational complex functions. In particular, he showed that commuting polynomials always have a common fixed point. In 1954, Eldon Dyer asked whether two commuting continuous functions must have a common fixed point. The same question was asked by A. J. Shields in 1955 and L. Dubins in 1956. For sometimes, this conjecture was considered by a group of people and led them to some partial results. Boyce [4] and Huneke [6] answered this question in negative by the construction of a pair of commuting continuous functions which have no fixed point in common. In [2], we considered a generalization of the common fixed-point problem and conjectured that in general two commuting continuous self-maps of the intervals do not have a common periodic point. To settle the conjecture, some partial results that gave more information about the structure of such pairs were obtained. However, in constructing such examples, we were convinced that the construction of such pair is extremely difficult, even though it seems that most pairs should not have any periodic points in common. Thus, in the Twenty-First Summer Symposium in Real Analysis, we posed

Conjecture 1.1 and raised the possibility of using a Baire-category argument to settle the problem. It is also not difficult to show that commuting continuous self-maps of the intervals must have common recurrent points. However, this is not true in general for the case of metric spaces. In [1], we also constructed an example of a pair of commuting continuous self-maps of a compact metric space with no common periodic points, which also provided an answer to a question raised in [2].

CONJECTURE 1.1. *Two commuting continuous functions on an interval will typically have disjoint sets of periodic points.*

Steele [7] considered the above conjecture and investigated the likelihood of such pair f and g when $P(f)$ is first category and pointed out the difficulties that one might face using a Baire-category argument.

In this paper, we first provide a natural setting for the study of this problem and give some results including a generalization of Steele's main theorem. We also consider the class of functions with continuous $\omega_f(x)$ studied by Bruckner and Ceder [5] and show that the set of recurrent points of such functions is a closed interval. We begin with some preliminaries.

The class of continuous self-maps of a closed interval I is denoted by $C(I, I)$. For $f \in C(I, I)$ and any integer $n \geq 1$, f^n denotes the n th iterate of f . The orbit of x under f (i.e., the set $\{f^k(x) : k \geq 0\}$) is denoted by $O(f, x)$. The set of cluster points of $O(f, x)$ is denoted by $\omega(f, x)$. We simply use $\omega(x)$ instead of $\omega(f, x)$ when there is no room for confusion. We also use $\omega_f(x)$ for the function $x \rightarrow \omega(f, x)$. A point x is in $R(f)$ if $x \in \omega(f, x)$. Let $F_n(f) = \{x \in I : f^n(x) = x\}$. We call $F(f) = F_1(f)$ the set of fixed points of f , $P(f) = \bigcup_{n=1}^{\infty} F_n(f)$ the set of periodic points of f , and $R(f)$ the set of recurrent points of f , respectively. A subset Y of I is called invariant under g if $g(Y) \subseteq Y$. A closed, invariant, nonempty subset of I is called minimal if it contains no proper subset that is also closed, invariant, and nonempty. Every closed, invariant, and nonempty subset of I contains a minimal set. If Y is a minimal set, then $Y \subseteq R(f)$, and if it is not the orbit of a periodic point, then it is a perfect set. A minimal set is also nowhere dense.

Suppose $\mathcal{H} = \{(f, g) : f, g \in C(I, I)\}$. Define $\rho : \mathcal{H} \rightarrow [0, \infty)$ by $\rho((f_1, g_1), (f_2, g_2)) = \|f_1 - f_2\| + \|g_1 - g_2\|$ where $\|f - g\| = \sup\{|f(x) - g(x)| : x \in I\}$. The open ball about x with radius ϵ is denoted by $B_\epsilon(x)$, and the interior and closure of A are denoted by A° and \bar{A} , respectively. The set of rational numbers is denoted by \mathbb{Q} and the set of natural numbers is denoted by \mathbb{N} . For two compact subsets A and B of I , by $d(A, B)$ and $d_H(A, B)$, we mean the distance of the sets A and B with respect to the ordinary metric and Hausdorff metric, respectively. Even though the results given here are true on any closed interval I , we only prove them for $I = [0, 1]$. In approximating a continuous function with a polynomial, we consider its Bernstein polynomials restricted to I . These polynomials have the interesting property that for $f \in C(I, I)$ all

its Bernstein polynomials are also in $C(I, I)$. In the sequel, we use [Theorem 1.2](#) given below from [3]. The proof of [Theorem 1.3](#) is also trivial, and thus, it is omitted.

THEOREM 1.2. *If $f \in C(I, I)$, then $\overline{P(f)} = \overline{R(f)}$.*

THEOREM 1.3. *Let \mathcal{H} and ρ be as defined, then (\mathcal{H}, ρ) is a compact metric space.*

2. Main results

LEMMA 2.1. *Let $f \in C(I, I)$ and let S be an F_σ subset of I . Then S is of first category if and only if S does not contain an interval.*

PROOF. Suppose S is of first category and $S = \cup_{m=1}^\infty S_m$ with each S_m closed. Then, from the Baire-category theorem, it follows that S does not contain an interval. On the other hand, if S does not contain an interval, then for each $m \geq 1$, S_m is a nowhere dense set, implying that S is a first-category subset of I . □

COROLLARY 2.2. *Let $f \in C(I, I)$. Then $P(f)$ is of first category if and only if $P(f)$ does not contain an interval.*

THEOREM 2.3. *The set $A = \{(f, g) : P(f) \text{ and } P(g) \text{ are both first category}\}$ is a residual subset of \mathcal{H} .*

PROOF. Let $A_1 = \{(f, g) \in \mathcal{H} : P(f) \text{ is not first category}\}$ and let $A_2 = \{(f, g) \in \mathcal{H} : P(g) \text{ is not first category}\}$, then we have $A = \mathcal{H} \setminus (A_1 \cup A_2)$. We show that A is a residual subset of \mathcal{H} by showing that A_1 and A_2 are first-category subsets of \mathcal{H} . We prove our claim for set A_1 ; the case for A_2 similarly follows. Suppose $(f, g) \in A_1$, then $P(f)$ is not of first category in I . Thus it contains an interval, hence we have

$$\begin{aligned} A_1 &= \{(f, g) \in \mathcal{H} : P(f) \text{ is not first category}\} \\ &= \{(f, g) \in \mathcal{H} : P(f) \text{ contains an interval}\} \\ &= \bigcup_{m=1}^\infty \{(f, g) \in \mathcal{H} : F_m(f) \text{ contains an interval}\}. \end{aligned} \tag{2.1}$$

Thus it suffices to show that, for each $m \geq 1$, the set $B_m = \{(f, g) \in \mathcal{H} : F_m(f) \text{ contains an interval}\}$ is a first-category subset of \mathcal{H} . This is achieved if we show that $B_{m,n} = \{(f, g) \in \mathcal{H} : \mathbb{Q}_n \subseteq F_m(f)\}$ is a nowhere dense subset of \mathcal{H} , where $\{\mathbb{Q}_n\}$ is an enumeration of the rational intervals (a, b) with $a, b \in \mathbb{Q} \cap [0, 1]$. Suppose $(f_1, g_1) \in B_{m,n}$ and $\epsilon > 0$, then we have $\mathbb{Q}_n \subseteq F_m(f_1)$. Let P be a polynomial with $\|f_1 - P\| < \epsilon$, then $(P, g_1) \in B_\epsilon((f_1, g_1))$. It is easy to see that $B_{m,n}$ is a closed subset of \mathcal{H} and $F_m(P)$ has finitely many elements for

a polynomial P , thus $F_m(P)$ does not contain \mathbb{Q}_n . This implies that $\overline{B_{m,n}} = B_{m,n}$ has no interior, hence $B_{m,n}$ is a nowhere dense subset of \mathcal{H} . \square

LEMMA 2.4. *Let $S \subset [0, 1]$ be a nowhere dense set and let m be a positive integer. Then the set $A_m = \{f \in C([0, 1]) : F_m(f) \cap \overline{S} \neq \emptyset\}$ is a nowhere dense subset of $C([0, 1])$.*

PROOF. First we show that A_m is a closed subset of $C([0, 1])$. For this, let $\{g_n\} \subseteq A_m$ be a sequence and $\{g_n\}$ converges uniformly to g ; we show that $g \in A_m$. Since $F_m(g_n) \cap \overline{S} \neq \emptyset$, take $y_n \in F_m(g_n) \cap \overline{S}$. Without loss of generality, we may assume $\lim_{n \rightarrow \infty} y_n = y_0$, thus $y_0 \in \overline{S}$. Let $\epsilon > 0$ be arbitrary. Choose k large enough so that $|g^m(y_k) - g^m(y_0)| < \epsilon$, $\|g_k^m - g^m\| < \epsilon$, and $|y_k - y_0| < \epsilon$. Then we have

$$\begin{aligned} |g^m(y_0) - y_0| &\leq |g^m(y_0) - g^m(y_k) + g^m(y_k) - g_k^m(y_k) + g_k^m(y_k) - y_0| \\ &\leq |g^m(y_0) - g^m(y_k)| + |g^m(y_k) - g_k^m(y_k)| + |g_k^m(y_k) - y_0| \\ &\leq |g^m(y_0) - g^m(y_k)| + \|g^m - g_k^m\| + |y_k - y_0| \\ &\leq 3\epsilon. \end{aligned} \tag{2.2}$$

Since ϵ was arbitrary, we have $g^m(y_0) = y_0$, implying that $y_0 \in F_m(g) \cap \overline{S}$, hence $g \in A_m$. To show that A_m is nowhere dense subset of $C([0, 1])$, we show that for each $f \in A_m$ and $\epsilon > 0$, there exists a function $h \in C([0, 1])$ so that $\|h - f\| < \epsilon$ and $F_m(h) \cap \overline{S} = \emptyset$. To this end, let P be a Bernstein polynomial of f so that $\|f - P\| < \epsilon/4m$ and let $E_1 = F_1(P) \cap \overline{S} = \{z_1, z_2, \dots, z_k\}$. Due to the uniform continuity of P , for $\epsilon > 0$, there exists $\delta_k > 0$ so that for $t_1, t_2 \in [0, 1]$, with $|t_1 - t_2| < 4\delta_k$, we have $|P(t_1) - P(t_2)| \leq \epsilon/4mk$. Let $\delta'_k = (1/4) \min\{|z_i - z_j| \text{ for } i \neq j, \delta_k, \epsilon/4\}$. For each $z_i \in E_1$, we have one of the following cases.

- CASE 1.** For all $x \in (z_i - \delta'_k, z_i) \cup (z_i, z_i + \delta'_k)$, $P(x) > x$.
- CASE 2.** For all $x \in (z_i - \delta'_k, z_i) \cup (z_i, z_i + \delta'_k)$, $P(x) < x$.
- CASE 3.** For all $x \in (z_i - \delta'_k, z_i)$, $P(x) > x$, and for all $x \in (z_i, z_i + \delta'_k)$, $P(x) < x$.
- CASE 4.** For all $x \in (z_i - \delta'_k, z_i)$, $P(x) < x$, and for all $x \in (z_i, z_i + \delta'_k)$, $P(x) > x$.

We only prove Cases 1 and 3. Cases 2 and 4 are proved similarly.

In **Case 1**, choose x_1 and x_2 so that $z_1 - \delta'_k < x_1 < z_1 < x_2 < z_1 + \delta'_k$ and take $P_1(x) = P(x)$ for $x \leq x_1$ or $x \geq x_2$, being linear on the interval $[x_1, x_2]$.

In **Case 3**, choose x_1, x_2 , and x_3 so that $z_1 - \delta'_k < x_3 < z_1 < x_2 < x_1 < z_1 + \delta'_k$, $P(x_3) - \epsilon/8mk < (x_1 + x_2)/2 < P(x_3) + \epsilon/8mk$, and $[x_2, x_1] \cap \overline{S} = \emptyset$. In this case, take $P_1(x) = P(x)$ for $x \leq x_3$ or $x \geq x_1$, $P_1((x_1 + x_2)/2) = (x_1 + x_2)/2$, being linear on the intervals $[x_3, (x_1 + x_2)/2]$ and $[(x_1 + x_2)/2, x_1]$. Therefore we have $F_1(P_1) \cap \overline{S} = F_1(P) \setminus \{z_1\}$ and $\|P_1 - P\| \leq \epsilon/4mk$. Repeating this process for each $z_i \in E_1$, $i \geq 2$, we can choose piecewise polynomials P_i in $C([0, 1])$ so that $\|P_i - P_{i-1}\| \leq \epsilon/4mk$ and $F_1(P_i) \cap \overline{S} = F_1(P) \setminus \{z_1, z_2, z_3, \dots, z_i\}$. By taking

$\mathbb{Q}_1 = P_k$, we have $\|\mathbb{Q}_1 - f\| \leq \|\mathbb{Q}_1 - P_{k-1}\| + \|P_{k-1} - P_{k-2}\| + \dots + \|P_1 - P\| + \|P - f\| \leq k(\epsilon/4mk) + \epsilon/4m = \epsilon/2m$ and $F_1(\mathbb{Q}_1) \cap \bar{S} = \emptyset$. Since \mathbb{Q}_1 is a continuous piecewise polynomial, $F_2(\mathbb{Q}_1)$ has finitely many elements. Thus, by replacing the role of P with \mathbb{Q}_1 and the role of $F_1(P)$ with $F_2(\mathbb{Q}_1)$, similarly, we may choose a piecewise polynomial \mathbb{Q}_2 in $C([0, 1])$ so that $\|\mathbb{Q}_2 - \mathbb{Q}_1\| \leq 3\epsilon/4m$ and $F_2(\mathbb{Q}_2) \cap \bar{S} = \emptyset$. Continuing this process, we may choose piecewise polynomials $\{\mathbb{Q}_i\}_{i=1}^m \subset C([0, 1])$ so that $\|f - \mathbb{Q}_1\| \leq 3\epsilon/4m$, $\|\mathbb{Q}_i - \mathbb{Q}_{i-1}\| \leq 3\epsilon/4m$ for $2 \leq i \leq m$, and $F_i(\mathbb{Q}_i) \cap \bar{S} = \emptyset$ for $1 \leq i \leq m$. Thus we have $\|\mathbb{Q}_m - f\| \leq \|\mathbb{Q}_m - \mathbb{Q}_{m-1}\| + \|\mathbb{Q}_{m-1} - \mathbb{Q}_{m-2}\| + \dots + \|\mathbb{Q}_1 - f\| \leq 3\epsilon/4m + 3\epsilon/4m + \dots + 3\epsilon/4m + 3\epsilon/4m \leq m(3\epsilon/4m) = 3\epsilon/4 < \epsilon$, and $F_m(\mathbb{Q}_m) \cap \bar{S} = \emptyset$. This implies that $\mathbb{Q}_m \notin A_m$ and hence A_m is a nowhere dense subset of $C([0, 1])$. \square

LEMMA 2.5. *Let $S \subseteq [0, 1]$ and let m be a positive integer. If the set $A_m = \{f \in C([0, 1]) : F_m(f) \cap \bar{S} \neq \emptyset\}$ is a nowhere dense subset of $C([0, 1])$, then S is a nowhere dense subset of $[0, 1]$.*

PROOF. On the contrary, suppose S is somewhere dense. Then \bar{S} contains a closed interval $[a, b]$ of positive length. Define the function f as $f(x) = x$, for $x \leq a$ or $x \geq b$, $f(a + (b - a)/8) = 1$, $f(a + (b - a)/4) = a + (b - a)/4$, $f(x) = x$, for $a + (b - a)/4 \leq x \leq a + 3(b - a)/8$, $f(a + (b - a)/2) = 0$ and linear elsewhere. Take $\epsilon < (1/2) \min\{1 - (a + (b - a)/4), a + (b - a)/2\}$. It is easy to see that for each $g \in C([0, 1])$ with $\|g - f\| < \epsilon$, there exist $x_1, x_2 \in (a, b)$ so that $g(x_1) > x_1, g(x_2) < x_2$. This implies the existence of $x_3 \in (a, b) \cap F_1(g) \subseteq F_m(g) \cap \bar{S}$, hence $g \in A_m$, contrary to the assumption that A_m is a nowhere dense subset of $C([0, 1])$. \square

By combining Lemmas 2.4 and 2.5, we have the following theorem.

THEOREM 2.6. *Let $S \subseteq [0, 1]$ and let m be a positive integer. Then the set $A_m = \{f \in C([0, 1]) : F_m(f) \cap \bar{S} \neq \emptyset\}$ is a nowhere dense subset of $C([0, 1])$ if and only if S is a nowhere dense subset of $[0, 1]$.*

THEOREM 2.7. *Let $S \subset [0, 1]$ be a first-category set. Then there exists a residual subset M of $C([0, 1])$ so that $P(g) \cap S = \emptyset$ for each $g \in M$.*

PROOF. Let $S = \cup_{n=1}^\infty S_n$, where S_n is nowhere dense for each n . For each positive integer m from Theorem 2.6, we have that the set $A_{m,n} = \{g \in C([0, 1]) : F_m(g) \cap \bar{S}_n \neq \emptyset\}$ is a nowhere dense subset of $C([0, 1])$. Thus $B = \cup_{m=1}^\infty \cup_{n=1}^\infty A_{m,n}$ is a first-category subset of $C([0, 1])$. Let $M = C([0, 1]) \setminus B$, then M is residual and if $g \in M$, then $g \notin A_{m,n}$ for each $m \geq 1$ and $n \geq 1$. Thus we have $F_m(g) \cap \bar{S}_n = \emptyset$ for $m \geq 1$ and $n \geq 1$. This implies that $S \cap P(g) = \emptyset$ for each $g \in M$. \square

THEOREM 2.8. *The function $h : \mathcal{H} \rightarrow [0, \infty)$ defined by $h((f, g)) = d(F(f), F(g))$ is a lower semicontinuous function.*

PROOF. The proof is straightforward and is omitted. \square

COROLLARY 2.9. Let $\mathcal{A} = \{(f, g) \in \mathcal{H} : f \text{ and } g \text{ have a common fixed point}\}$ and let $\mathcal{B} = \{(f, g) \in \mathcal{H} : f \text{ and } g \text{ have a common periodic point}\}$. Then \mathcal{A} and \mathcal{B} are closed and F_σ subsets of \mathcal{H} , respectively.

Now we turn our attention to commuting continuous functions on the interval and prove the following lemma that is used in [Theorem 2.11](#).

LEMMA 2.10. Let $f \in C(I, I)$ where $I = [0, 1]$, and let $a, b \in I$, $a \neq b$. If $\epsilon > 0$ is a real number, then there exists a function $h \in C(I, I)$ such that $h(a) = f(a)$, $h(b) \neq f(b)$, and $\|h - f\| < \epsilon$.

PROOF. We can assume $\epsilon < 1$. Then, for $x \in I = [0, 1]$, let

$$h(x) = f(x) + (c - f(x))|x - a|c\epsilon, \quad (2.3)$$

with $c = 1/2$ if $f(b) \neq 1/2$ and $c = 1/3$ if $f(b) = 1/2$. Then h is continuous, $h(a) = f(a)$, $h(b) \neq f(b)$, and $|h(x) - f(x)| = |(c - f(x))|x - a|c\epsilon < |x - a|(\epsilon/2) \leq \epsilon/2$ for any $x \in I$. Thus, $\|h - f\| < \epsilon$. Moreover, $h(x) \in [0, 1]$ for $x \in I$. If $f(x) \geq c$, $0 \leq h(x) \leq f(x)$, and $f(x) \leq h(x) \leq 1$ if $f(x) < c$. Thus $h \in C(I, I)$. \square

THEOREM 2.11. Let $\mathcal{X} = \{(f, g) \in \mathcal{H} : f \circ g = g \circ f\}$, then \mathcal{X} is a nonempty, closed, and nowhere dense subset of \mathcal{H} .

PROOF. It is easy to see that $(f, f) \in \mathcal{X}$ when f is the identity function, so \mathcal{X} is nonempty. Suppose $(f_n, g_n) \in \mathcal{X}$ and $(f_n, g_n) \rightarrow (f, g)$. Then we have $f_n \rightarrow f$, $g_n \rightarrow g$, and $f_n \circ g_n = g_n \circ f_n$ for each n . We show that $(f, g) \in \mathcal{X}$. Let $x \in I$ and $\epsilon > 0$. Since f and g are uniformly continuous, we may choose δ , so that $0 < \delta < \epsilon$ and for every $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta$, we have $|f(x_1) - f(x_2)| < \epsilon$, $|g(x_1) - g(x_2)| < \epsilon$. Since $f_n \rightarrow f$ and $g_n \rightarrow g$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\|f_n - f\| < \delta$, $\|g_n - g\| < \delta$. Thus, if $n > N$, $|f(g(x)) - g(f(x))| \leq |f(g(x)) - f(g_n(x))| + |f(g_n(x)) - f_n(g_n(x))| + |g_n(f_n(x)) - g(f_n(x))| + |g(f_n(x)) - g(f(x))| \leq 2\delta + 2\epsilon \leq 4\epsilon$. Since $\epsilon > 0$ and x were arbitrary, we have $f \circ g = g \circ f$, so $(f, g) \in \mathcal{X}$ and \mathcal{X} is closed. Now we show that \mathcal{X} is nowhere dense. Let $\epsilon > 0$, $(f, g) \in \mathcal{X}$. Suppose f or g is not the identity function, hence there exists x_0 so that $f(x_0) \neq x_0$ or $g(x_0) \neq x_0$. Suppose $g(x_0) \neq x_0$, the case where $f(x_0) \neq x_0$ is similar. Let $h_2(x) = g(x)$ for each $x \in I$, and let $h_1 \in C(I, I)$ be a function so that $h_1(x_0) = f(x_0)$, but $h_1(g(x_0)) \neq f(g(x_0))$ and $\|f - h_1\| < \epsilon$. From [Lemma 2.10](#), with $a = x_0$ and $b = g(x_0)$, it follows that such $h_1 \in C(I, I)$ exists. Then we have $(h_1, h_2) \in B_\epsilon((f, g))$, and $h_2(h_1(x_0)) = h_2(f(x_0)) = g(f(x_0)) = f(g(x_0)) \neq h_1(g(x_0)) = h_1(h_2(x_0))$. Thus \mathcal{X} is the union of a nowhere dense set and a singleton $\{(x, x)\}$, thus it is a nowhere dense subset of \mathcal{H} . \square

THEOREM 2.12. Let f and g be two commuting continuous self-maps of the unit interval. Then we have one of the following holds:

- (i) f and g have a common periodic point;
- (ii) for any two positive integers m and n , let $A_{m,n} = \{x : f^n(x) = g^m(x)\}$.
Then for all $x \in A_{m,n}$, $\omega(f, x)$ is a perfect nowhere dense set.

PROOF. The set $A_{m,n}$ is not empty as is shown in [2], however we give the proof for completeness. If $A_{m,n}$ is empty, the continuity of f and g permits us to assume without loss of generality that

$$f^n(x) < g^m(x) \quad \forall x \in I. \tag{2.4}$$

Since $g^m(1) \leq 1$, the set $S = \{x \in I : g^m(x) \leq x\}$ is not empty. Thus, since S is closed, S has a minimum element c . Clearly, $c = g^m(c)$. Hence $f^n(c) = f^n(g^m(c)) = g^m(f^n(c))$ so that $f^n(c) \in S$. Consequently, $f^n(c) \geq c = g^m(c)$. Since $f^n(c) \geq g^m(c)$ contradicts (2.4), the assertion that $A_{m,n}$ is empty is false. Now suppose that (i) does not hold, that is, f and g do not have a common periodic point, and let $x \in A_{m,n}$. Then $f^n(x) = g^m(x)$, and hence $f^n(f(x)) = f(g^m(x)) = g^m(f(x))$ and $f^n(g(x)) = g(f^n(x)) = g(g^m(x)) = g^m(g(x))$. Thus if $x \in A_{m,n}$, $f(x), g(x) \in A_{m,n}$, and $f^{pn}(x) = g^{pm}(x)$ for each positive integer p . From this, it follows that $O(g, x)$ is contained in $A_{m,n}$ whenever $x \in A_{m,n}$. From closeness of $A_{m,n}$, it follows that $\omega(g, x) \subseteq A_{m,n}$. If $\omega(g, x)$ has an isolated point, then the isolated point is a periodic point of g which is contained in $A_{m,n}$, thus also a periodic point of f , which is a contradiction. So for each $x \in A_{m,n}$, $\omega(g, x)$ is a perfect set. Suppose for some $x \in A_{m,n}$, $\omega(g, x)$ is somewhere dense, then it contains an interval I_0 . Let I_1 be the largest interval such that $I_0 \subseteq I_1 \subseteq A_{m,n}$ and $x_0 \in I_1^o$, then there are positive integers k_1, k_2 with $k_1 < k_2$ so that $g^{k_1}(x)$ and $g^{k_2}(x)$ are in I_1 . This implies that $g^{k_2-k_1}(I_1) \subseteq I_1$, and thus g should have a periodic point in I_1 that is also a periodic point of f , which is again a contradiction. It is also easy to see that the set $\omega(g, x)$ is an invariant set under g . Suppose A_1 is a minimal set contained in $\omega(g, x)$. Since f and g have no common periodic points, the set A_1 cannot be the orbit of a periodic point. Hence A_1 is also a perfect set contained in $A_{m,n} \cap \omega(g, x)$. Thus, for each $x \in A_{m,n}$, $\omega(g, x) = \omega(f, x)$ is a perfect nowhere dense set containing a perfect minimal set. □

REMARK 2.13. Bruckner and Ceder [5] proved that ω_f is Baire 1 if and only if any infinite ω -limit set for f is perfect. Motivated by this result, one may extend the above Baire-1 classification of ω_f to the closed subsets of I . If so, the above theorem tells us either the two commuting continuous functions f and g have a common periodic point or both ω_f and ω_g are Baire 1 on the closed set $A_{m,n}$ with $\omega(f, x)$ and $\omega(g, x)$ being nowhere dense for each $x \in A_{m,n}$.

REMARK 2.14. An argument similar to the proof of the above theorem could give the same result if the role of $A_{m,n}$ is replaced with any nonempty f -invariant closed subset of $P(g)$ or g -invariant closed subset of $P(f)$. In

particular, [Theorem 2.12](#) is also true if in its statement we replace $A_{m,n}$ with $F(f)$ or $F(g)$.

The family of all ω -limit sets of a continuous function with the Hausdorff metric forms a metric space. Bruckner and Ceder [5] introduced a kind of chaos in terms of the map $\omega_f : x \rightarrow \omega(f, x)$. In the same paper, they proved [Lemma 2.15](#) and used it to prove [Theorem 2.16](#) which characterizes the continuity of ω_f .

LEMMA 2.15. *If ω_f is continuous, then $F_1(f)$ is connected.*

THEOREM 2.16. *The following conditions are equivalent:*

- (1) ω_f is continuous;
- (2) $\{f^n\}_{n=1}^\infty$ is equicontinuous;
- (3) ω_{f^2} is continuous;
- (4) $F_1(f^2) = \bigcap_{n=1}^\infty f^n(I)$;
- (5) $F_1(f^2)$ is connected and, for all x , $\{f^{2n}(x)\}_{n=1}^\infty$ converges to a point of $F_1(f^2)$;
- (6) $F_1(f^2)$ is connected;
- (7) ω_f is lower semicontinuous;
- (8) ω_f is upper semicontinuous.

Using the above theorem, we show that the recurrent set of a continuous function defined on a closed interval with continuous ω_f is a closed interval.

LEMMA 2.17. *For any natural number n , ω_{f^n} is continuous when ω_f is continuous.*

PROOF. From the lower semicontinuity of ω_f , it follows that $\{f^k\}_{k=1}^\infty$ is equicontinuous, implying that $\{f^{nk}\}_{k=1}^\infty$ is equicontinuous for each $n \in \mathbb{N}$. Hence ω_{f^n} is also lower semicontinuous. Thus the lemma follows from [Theorem 2.16](#). \square

THEOREM 2.18. *Suppose f is a continuous self-map of the unit interval with $\omega_f : x \rightarrow \omega(f, x)$ continuous. Then $R(f)$ is a closed interval.*

PROOF. For each $m \geq 1$, ω_{f^m} is continuous by [Lemma 2.17](#), thus $F_1(f^m) = F_m(f)$ is connected by [Lemma 2.15](#). We also have $F_1(f) \subseteq F_n(f)$ for all $n \geq 1$, implying that $\emptyset \neq F_1(f) \subseteq \bigcap_{n=1}^\infty F_n(f)$, thus $P(f) = \bigcup_{n=1}^\infty F_n(f)$ is connected. Since $P(f) \subseteq R(f) \subseteq \overline{P(f)}$, we have $R(f)$ connected. Now let $\{x_n\}_{n=1}^\infty$ be a sequence in $R(f)$ with $\lim_{n \rightarrow \infty} x_n = x_0$. We show that $x_0 \in R(f)$. Let ϵ be an arbitrary positive number. From the continuity of ω_f and the fact that $\lim_{n \rightarrow \infty} x_n = x_0$, there exists $N_1 \in \mathbb{N}$ so that $|x_n - x_0| < \epsilon$ and $d_H(\omega(f, x_n), \omega(f, x_0)) < \epsilon$ for all $n \geq N_1$. Since $\{x_n\} \subseteq R(f)$, we have $x_n \in \omega(f, x_n)$ for each n . Thus, for each $n \geq N_1$, there exists $z_n \in \omega(f, x_0)$ so that $|x_n - z_n| < \epsilon$, implying that $|z_n - x_0| < 2\epsilon$. Hence $d(x_0, \omega(f, x_0)) \leq 2\epsilon$, for every $\epsilon > 0$, implying that $x_0 \in \omega(f, x_0)$, hence $x_0 \in R(f)$. Thus we have shown that $R(f)$ is a closed and connected subset of the real line, hence it is a closed interval. \square

REMARK 2.19. Due to the fact that $R(f) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in I : |f^n(x) - x| \leq 1/m\}$, we see that the recurrent set of continuous self-maps of a closed interval is an $F_{\sigma\delta}$ set. A question of considerable interest is the classification of such maps with F_{σ} recurrent set. This class of functions contains nomadic functions. A continuous self-map of an interval is called nomadic if it has a dense orbit at some point. It is also known that if $R(f)$ is closed, then f has topological entropy zero. Thus [Theorem 2.18](#) provides another subclass of the class of continuous self-maps of the interval with an F_{σ} recurrent set as well as zero topological entropy.

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