ON FINITELY EQUIVALENT CONTINUA

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For positive integers \(m\) and \(n\), relations between (hereditary) \(m\)- and \(n\)-equivalence are studied, mostly for arc-like continua. Several structural and mapping problems concerning (hereditarily) finitely equivalent continua are formulated.

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A continuum means a compact connected metric space. For a positive integer \(n\), a continuum \(X\) is said to be \(n\)-equivalent provided that \(X\) contains exactly \(n\) topologically distinct subcontinua. A continuum \(X\) is said to be hereditarily \(n\)-equivalent provided that each nondegenerate subcontinuum of \(X\) is \(n\)-equivalent. If there exists a positive integer \(n\) such that \(X\) is \(n\)-equivalent, then \(X\) is said to be finitely equivalent. Thus, for \(n = 1\), the concepts of “1-equivalent” and “hereditarily 1-equivalent” coincide, and they mean the same as “hereditarily equivalent” in the sense considered, for example, by Cook in [2].

Observe the following statement.

**Statement 1.** Each subcontinuum of an \(n\)-equivalent continuum is \(m\)-equivalent for some \(m \leq n\). Thus, each finitely equivalent continuum is hereditarily finitely equivalent.

Some structural results concerning finitely equivalent continua are obtained by Nadler Jr. and Pierce in [9]. They have shown that if a continuum \(X\) is (a) semi-locally connected at each of its noncut points, then it is finitely equivalent if and only if it is a graph; (b) aposyndetic at each of its noncut points and finitely equivalent, then it is a graph. Furthermore, in both cases (a) and (b), if \(X\) is \(n\)-equivalent, then each subcontinuum of \(X\) is a \(\Theta_{n+1}\)-continuum. Recall that Nadler Jr. and Pierce in [9, page 209] posed the following problem.

**Problem 2.** Determine which graphs, or at least how many, are \(n\)-equivalent for each \(n\).

The arc and the pseudo-arc are the only known 1-equivalent continua. In [10] Whyburn has shown that each planar 1-equivalent continuum is tree-like, and planarity assumption has been deleted after 40 years by Cook [2] who proved tree-likeness of any 1-equivalent continuum. But it is still not known whether or not the arc and the pseudo-arc are the only ones among 1-equivalent continua.

In contrast to 1-equivalent case, 2-equivalent continua need not be hereditarily 2-equivalent, a simple closed curve is 2-equivalent while not hereditarily
2-equivalent. The 2-equivalent continua were studied by Mahavier in [5] who proved that if a 2-equivalent continuum contains an arc, then it is a simple triod, a simple closed curve or irreducible, and that the only locally connected 2-equivalent continua are a simple triod and a simple closed curve. It is also shown that if \( X \) is a decomposable, not locally connected, 2-equivalent continuum containing an arc, then \( X \) is arc-like and it is the closure of a topological ray \( R \) such that the remainder \( \text{cl}(R) \setminus R \) is an end continuum of \( X \). Furthermore, two examples of 2-equivalent continua are presented in [5]: the first, [5, Example 1, page 246], is a decomposable continuum \( X \) which is the closure of a ray \( R \) such that the remainder \( \text{cl}(R) \setminus R \) is homeomorphic to \( X \); the second, [5, Example 2, page 247], is an arc-like hereditarily decomposable continuum containing no arc.

Looking for an example of a hereditarily 2-equivalent continuum note that the former example surely is not hereditarily 2-equivalent because it contains an arc. We analyze the latter one.

The continuum \( M \) constructed in [5, Example 2, page 247] does not contain any arc, and it contains a continuum \( N \) such that each subcontinuum of \( M \) is homeomorphic to \( M \) or to \( N \), see [5, the paragraph following Lemma 3, page 249]. Further, by its construction, \( N \) does contain continua homeomorphic to \( M \) (see [5, the final part of the proof, page 251]). Therefore, the following statement is established.

**Theorem 3.** The continuum \( M \) constructed in [5, Example 2, page 247] has the following properties:
(a) \( M \) is an arc-like;
(b) \( M \) is hereditarily decomposable;
(c) \( M \) does not contain any arc;
(d) \( M \) is hereditarily 2-equivalent.

In connection with the above theorem, the following problem can be posed.

**Problem 4.** Determine for what integers \( n \geq 3 \), there exists a continuum \( M \) satisfying conditions (a), (b), and (c) of **Theorem 3** and being hereditarily \( n \)-equivalent.

The following results are consequences of [1, Theorem, page 35].

**Theorem 5.** For each hereditarily \( n \)-equivalent continuum \( X \), that does not contain any arc, there exists an \((n + 2)\)-equivalent continuum \( Y \) such that each of its subcontinua is homomorphic either to a subcontinuum of \( X \) or to \( Y \), or to an arc.

**Proof.** Indeed, a compactification \( Y \) of a ray \( R \) having the continuum \( X \) as the remainder, that is, such that \( X = \text{cl}(R) \setminus R \) is such a continuum. \( \square \)

Since if \( M \) is arc-like and hereditarily decomposable, then so is any of compactifications \( Y \) of a ray having the continuum \( X \) as the remainder, we get the next result as a consequence of **Theorem 5**.
**Corollary 6.** If a continuum $M$ satisfies conditions (a), (b), and (c) of Theorem 3 and is hereditarily $n$-equivalent, then any of compactifications of a ray having the continuum $M$ as the remainder satisfies conditions (a) and (b) of Theorem 3 and is $(n + 2)$-equivalent.

In [7], an uncountable family $\mathcal{F}$ is constructed of compactifications of the ray with the remainder being the pseudo-arc.

**Statement 7.** Each member $X$ of the (uncountable) family $\mathcal{F}$ constructed in [7] is an arc-like 3-equivalent continuum. Any subcontinuum of $X$ is homeomorphic to an arc, to a pseudo-arc, or to the whole $X$.

A continuum $X$ has the **RNT-property** (retractable onto near trees) provided that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if a tree $T$ is $\delta$-near to $X$ with respect to the Hausdorff distance, then there is an $\varepsilon$-retraction of $X$ onto $T$, see [6, Definition 0]. It is shown in [6, Theorem 5] that if a continuum $X$ is a compactification of the ray $R$ and $X$ has the RNT-property, then the remainder $\text{cl}(R) \setminus R \subset X = \text{cl}(R)$ is the pseudo-arc. Therefore, Theorem 5 implies the following proposition.

**Proposition 8.** Each compactification $X$ of the ray having the RNT-property is a 3-equivalent continuum. Each subcontinuum of $X$ is homeomorphic to an arc, a pseudo-arc, or to the whole $X$.

Observe that $M$ of Theorem 3 being an arc-like is hereditarily unicoherent, and being hereditarily decomposable, it is a $\lambda$-dendroid (containing no arc). Another (perhaps the first) example of a $\lambda$-dendroid, in fact, an arc-like, containing no arc, has been constructed by Janiszewski in 1912, [3] but his description was rather intuitive than precise. It would be interesting to investigate if that old example of Janiszewski is or is not $n$-equivalent (hereditarily $n$-equivalent) for some $n$.

The following problems can be considered as a program of a study in the area rather than particular questions.

**Problems 9.** For each positive integer $n$, characterize continua which are (a) $n$-equivalent; (b) hereditarily $n$-equivalent.

**Problem 10.** Characterize continua which are finitely equivalent.

Sometimes a characterization of a class of spaces (or of spaces having a certain property) can be expressed in terms of containing some particular spaces. A classical illustration of this is a well-known characterization of nonplanar graphs by containing the two Kuratowski’s graphs: $K_5$ and $K_{3,3}$, see, for example, [8, Theorem 9.36, page 159]. To be more precise, recall the following concept. Let $\mathcal{A}$ be a class of spaces and let $\mathcal{P}$ be a property. Then $\mathcal{P}$ is said to be **finite (or countable) in the class $\mathcal{A}$** provided that there is a finite (or countable,
respectively) set \( \mathcal{F} \) of members of \( \mathcal{A} \) such that a member \( X \) has the property \( \mathcal{F} \) if and only if \( X \) contains a homeomorphic copy of some member of \( \mathcal{F} \). The result of [7] mentioned above in Statement 7 shows that this is not the way of characterizing 3-equivalent continua. Namely, the existence of the family \( \mathcal{F} \) shows the following theorem.

**Theorem 11.** The property of being 3-equivalent is neither finite nor countable in the class of (a) all continua; (b) arc-like continua.

A mapping \( f : X \to Y \) between continua \( X \) and \( Y \) is said to be
(i) **atomic** provided that for each subcontinuum \( K \) of \( X \), either \( f(K) \) is degenerate or \( f^{-1}(f(K)) = K \);
(ii) **monotone** provided that the inverse image of each subcontinuum of \( Y \) is connected;
(iii) **hereditarily monotone** provided that for each subcontinuum \( K \) of \( X \), the partial mapping \( f|K : K \to f(K) \) is monotone.

It is known that each atomic mapping is hereditarily monotone, see, for example, [4, (4.14), page 17]. Since each arcwise connected 2-equivalent continuum is either a simple closed curve or a simple triod, see [5, Theorem 2, page 244], each semilocally connected 3-equivalent continuum is either a simple 4-od [8, Definition 9.8, page 143] (i.e., a letter \( X \)) or a letter \( H \), see [9, page 209]. And since these continua are preserved under atomic mappings (as it is easy to see), we conclude that atomic mappings preserve the property of being 2-equivalent and being 3-equivalent for locally connected continua. However, this is not an interesting result, because each atomic mapping of an arcwise connected continuum onto a nondegenerate continuum is a homeomorphism, see [4, (6.3), page 51]. But the result cannot be extended to hereditarily monotone mappings, because a mapping that shrinks one arm of a simple triod to a point is hereditarily monotone and not atomic, and it maps a 2-equivalent continuum onto an arc that is 1-equivalent. On the other hand, if \( X \) is the 2-equivalent continuum which is the closure of a ray \( R \) as described in [5, Example 1, page 246], then the mapping \( f : X \to [0,1] \), that shrinks the remainder \( \text{cl}(R) \setminus R \) to a point (and is a homeomorphism on \( R \)), is atomic and it maps 2-equivalent continuum \( X \) onto the 1-equivalent continuum \([0,1]\). Therefore, atomic mappings do not preserve the property of being a 2-equivalent continuum. In connection with these examples, the following question can be asked.

**Question 12.** Let a continuum \( X \) be \( n \)-equivalent and let a mapping \( f : X \to Y \) be an atomic surjection. Must then \( Y \) be \( m \)-equivalent for some \( m \leq n \)?

In general, we can pose the following problems.

**Problems 13.** What kinds of mappings between continua preserve the property of being: (a) \( n \)-equivalent? (b) hereditarily \( n \)-equivalent? (c) finitely equivalent?
REFERENCES


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