MAPPING PROPERTIES FOR CONVOLUTIONS INVOLVING HYPERGEOMETRIC FUNCTIONS

J. A KIM and K. H. SHON

Received 4 March 2002

For $\mu \geq 0$, we consider a linear operator $L_\mu : A \to A$ defined by the convolution $f_\mu \ast f$, where $f_\mu = (1 - \mu)z_2F_1(a, b, c; z) + \mu z(z_2F_1(a, b, c; z))'$. Let $\psi^\ast (A, B)$ denote the class of normalized functions $f$ which are analytic in the open unit disk and satisfy the condition $zf'/f \prec (1 + Az)/1 + Bz$, $-1 \leq A < B \leq 1$, and let $R_\eta(\beta)$ denote the class of normalized analytic functions $f$ for which there exists a number $\eta \in (-\pi/2, \pi/2)$ such that $\text{Re}(e^{i\eta}(f'(z) - \beta)) > 0$, $\beta < 1$. The main object of this paper is to establish the connection between $R_\eta(\beta)$ and $\psi^\ast (A, B)$ involving the operator $L_\mu(f)$. Furthermore, we treat the convolution $I = \int_0^z (f_\mu(t)/t)dt \ast f(z)$ for $f \in R_\eta(\beta)$.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  (1.1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and $S$ denotes the subclass of functions in $A$ which are univalent in $U$. Moreover, let $S^\ast(\alpha)$ and $K(\alpha)$ be the subclasses of $S$ consisting, respectively, of functions which are starlike of order $\alpha$ and convex of order $\alpha$, where $0 \leq \alpha < 1$ in $U$. Clearly, we have $S^\ast(\alpha) \subseteq S^\ast(0) = S^\ast$, where $S^\ast$ denotes the class of functions in $A$ which are starlike in $U$ and $K(\alpha) \subseteq K(0) = K$, where $K$ denotes the class of functions in $A$ which are convex in $U$, and we mention the well-known inclusion chain $K \subset S^\ast(1/2) \subset S^\ast \subset S$. For the analytic functions $g$ and $h$ on $U$ with $g(0) = h(0)$, $g$ is said to be subordinate to $h$ if there exists an analytic function $w$ on $U$ such that $w(0) = 0$, $|w(z)| < 1$, and $g(z) = h(w(z))$ for $z \in U$. We denote this subordinated relation by

$$g < h \quad \text{or} \quad g(z) < h(z) \quad (z \in U).$$  (1.2)

For $-1 \leq A < B \leq 1$, a function $p$, which is analytic in $U$ with $p(0) = 1$, is said to belong to the class $P(A, B)$ if

$$p(z) < \frac{1 + Az}{1 + Bz} \quad (z \in U).$$  (1.3)
The above condition means that \( p \) takes the values in the disk with a center \( (1 - AB) / (1 - B^2) \) and a radius \( |A - B| / (1 - B^2) \). The boundary circle cuts the real axis at the points \((1 + A) / (1 + B)\) and \((1 - A) / (1 - B)\). A function \( f \in A \) is said to be in \( \phi^*(A,B) \) if \( z f' / f \in P(A,B) \), and in \( K(A,B) \) if \( z f' \in \phi^*(A,B) \). The class \( \phi^*(A,B) \) was introduced by N. Shukla and P. Shukla [4]. Also, Janowski [2] introduced the class \( P(A,B) \). For the fixed natural number \( n \), the subclass \( P_n(A,B) \) of \( P(A,B) \) containing functions \( p \) of the form

\[
p(z) = 1 + p_n z^n + \cdots,
\]

\( z \in U \), was defined by Stankiewicz and Waniurski [7]. In addition, Stankiewicz and Trojnar-Spelina [6] investigated a function \( p(z) = 1 - p_n z^n - \cdots \) belongs to the class \( R(n,A,B) \), where \( A \in R \) and \( B \in [0, 1] \) if \( p(z) \prec (1 + Az) / (1 - Bz) \).

Let \( R_\eta(\beta) \) denote the class of functions \( f \in A \) for which there exists a number \( \eta \in (-\pi/2, \pi/2) \) such that

\[
\text{Re}[e^{i\eta} (f'(z) - \beta)] > 0 \quad (z \in U, \ \beta < 1).
\]

(1.4)

Clearly, we have \( R_\eta(\beta) \subset S \ (0 \leq \beta < 1) \). Furthermore, if a function \( f \) of the form (1.1) belongs to the class \( R_\eta(\beta) \), then

\[
|a_n| \leq \frac{2(1 - \beta) \cos \eta}{n} \quad (n \in N \setminus \{1\}).
\]

(1.5)

The class \( R_\eta(\beta) \) was studied by Kanas and Srivastava [3].

The hypergeometric function \( _2F_1(a,b,c;z) \) is given as a power series, converging in \( U \), in the following way

\[
_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n,
\]

(1.6)

where \( a, b, \) and \( c \) are complex numbers with \( c \neq 0, -1, -2, \ldots \), and \( (\lambda)_n \) denotes the Pochhammer symbol (or the generalized factorial since \( (1)_n = n! \)) defined, in terms of the Gamma function \( \Gamma \), by

\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1 & \text{if } n = 0, \\
\lambda(\lambda+1) \cdots (\lambda+n-1) & \text{if } n \in N = \{1, 2, \ldots \}.
\end{cases}
\]

(1.7)

Note that \( _2F_1(a,b,c;z) \), for \( a = c \) and \( b = 1 \) (or, alternatively, for \( a = 1 \) and \( b = c \)), reduces to the relatively more familiar geometric function. We also
note that \( _2F_1(a, b, c; 1) \) converges for \( \text{Re}(c - a - b) > 0 \) and is related to the Gamma functions by

\[
_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \quad (1.8)
\]

The Hadamard product (or convolution) of two power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is defined as the power series

\[
(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \quad (1.9)
\]

N. Shukla and P. Shukla [4] studied the mapping properties of a function \( f_\mu \) to be as given in

\[
f_\mu(z) = (1 - \mu)z_2F_1(a, b, c; z) + \mu z(z_2F_1(a, b, c; z))' \quad (\mu \geq 0), \quad (1.10)
\]

and investigated the geometric properties of an integral operator of the form

\[
I(z) = \int_0^z f_\mu(t) \frac{t}{t} dt. \quad (1.11)
\]

We now consider a linear operator \( L_\mu : A \to A \) defined by

\[
L_\mu(f) = f_\mu(z) \ast f(z). \quad (1.12)
\]

For \( \mu = 0 \) in (1.12), \( L_\mu(f) = [I_{a, b, c}(f)](z) \), which was introduced by Hohlov [1]. Also, Kanas and Srivastava [3], and Srivastava and Owa [5] showed that the operator \( I_{a, b, c}(f) \) is the natural extensions of the Alexander, Libera, Bernardi, and Carlson-Shaffer operators. In this paper, we find a relation between \( R_\eta(\beta) \) and \( \varphi^*(A, B) \) involving the operator \( L_\mu(f) \). Furthermore, we study to obtain some conditions for the starlikeness and convexity of the convolution of \( I \) and \( f \), which are given by (1.11) and (1.1), respectively, for \( f \in R_\eta(\beta) \).

2. Main results. We make use of the following lemma.

**Lemma 2.1** [4]. Sufficient conditions for \( f \) of the form (1.1) to be in \( \varphi^*(A, B) \) and \( K(A, B) \) are

\[
\sum_{n=2}^{\infty} \left| a_n \right| \leq B - A, \quad (2.1)
\]

respectively.
Theorem 2.2. Let \( a > 1, \ b > 1, \) and \( c > a + b + 1. \) If \( f \in R_\eta(\beta) \) and the inequality
\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1+B) \left( 1 + \frac{\mu ab}{c-a-b-1} \right) - (A+1) \left( \mu - \frac{(\mu-1)(c-a-b)}{(a-1)(b-1)} \right) \right] \\
\leq (B-A) \left( \frac{1}{2(1-\beta)\cos \eta} + 1 \right) + \frac{(A+1)(\mu-1)(c-1)}{(a-1)(b-1)} \\
\tag{2.2}
\]
is satisfied, then \( L_\mu(f) \in \varphi^*(A,B). \)

Proof. By Lemma 2.1, it suffices to show that
\[
T_1 := \sum_{n=2}^{\infty} \left[ (1+B)n - (A+1) \right] \frac{(1+(n-1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \leq B-A. \tag{2.3}
\]
Since \( f \in R_\eta(\beta) \) and \( |a_n| \leq 2(1-\beta)\cos \eta/n. \) Hence,
\[
T_1 \leq \sum_{n=2}^{\infty} \left[ (1+B)n - (A+1) \right] \frac{(1+(n-1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \frac{2(1-\beta)\cos \eta}{n} \\
= 2(1-\beta)\cos \eta \left\{ (1+B) \left( \sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}} - 1 \right) \\
- \frac{(A+1)(c-1)}{(a-1)(b-1)} \left( \sum_{n=0}^{\infty} \frac{(a-1)_{n}(b-1)_{n}}{(c-1)_{n}(1)_{n}} - 1 - \frac{(a-1)(b-1)}{c-1} \right) \\
+ \frac{(1+B)\mu ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n}} \\
- (A+1)\mu \left[ \sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}} - 1 \right. \\
\left. \quad \quad - \frac{c-1}{(a-1)(b-1)} \left( \sum_{n=0}^{\infty} \frac{(a-1)_{n}(b-1)_{n}}{(c)_{n}(1)_{n}} - 1 - \frac{(a-1)(b-1)}{c-1} \right) \right] \right\} \\
= 2(1-\beta)\cos \eta \left\{ \Gamma(c)\Gamma(c-a-b) \left[ (1+B) \left( 1 + \frac{\mu ab}{c-a-b-1} \right) \\
+ (A+1) \left( \mu - \frac{(\mu-1)(c-a-b)}{(a-1)(b-1)} \right) \\
- \left[ 1+B-(A+1) \left( 1 - \frac{(\mu-1)(c-1)}{(a-1)(b-1)} \right) \right] \right\}. \tag{2.4}
\]
Now, this last expression is bounded above by \( B-A \) if (2.2) holds. \( \square \)
If we take $\mu = 0, A = 2\alpha - 1,$ and $B = 1$ in Theorem 2.2, we have the following corollary.

**Corollary 2.3.** Let $a > 1, b > 1,$ and $c > a + b + 1.$ If $f \in R(\beta)$ and the inequality

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)} \left[ 1 - \frac{\alpha(c-a-b)}{(a-1)(b-1)} \right] \leq (1-\alpha) \left( \frac{1}{2(1-\beta) \cos \eta + 1} \right) - \frac{\alpha(c-1)}{(a-1)(b-1)}$$

(2.5)

is satisfied, then $z_2 F_1 (a,b,c;z) \ast f \in S(\alpha).$

If we take $\alpha = 0, \beta = 0,$ and $\eta = 0$ in Corollary 2.3, we get the following corollary.

**Corollary 2.4.** Let $a > 1, b > 1,$ and $c > a + b + 1.$ If $f \in S,$ and the inequality

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)} \leq \frac{3}{2}$$

(2.6)

is satisfied, then $z_2 F_1 (a,b,c;z) \ast f \in S.$

**Theorem 2.5.** Let $a > 0, b > 0,$ and $c > a + b + 2.$ If $f \in R(\beta),$ and the inequality

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)} \left[ B - A + ((1+B)(1+2\mu) - (A+1)\mu) \frac{ab}{c-a-b-1} + \frac{(1+B)\mu(a)_2(b)_2}{(c-a-b-2)_2} \right] \leq (B-A) \left( \frac{1}{2(1-\beta) \cos \eta + 1} \right)$$

(2.7)

is satisfied, then $L_\mu (f) \in K(A,B).$

**Proof.** The proof follows from Lemma 2.1. Using the method of the proof of Theorem 2.2, we omit the details involved. \qed

For $\mu = 0, A = 2\alpha - 1,$ and $B = 1,$ Theorem 2.5 yields the following corollary.
**Corollary 2.6.** Let \( a > 0, \ b > 0, \) and \( c > a + b + 2. \) If \( f \in R_\eta(\beta) \) and the inequality

\[
\Gamma(c) \Gamma(c-a-b) \frac{1 - \alpha + \frac{ab}{c-a-b-1}}{\Gamma(c-a) \Gamma(c-b)} \leq (1 - \alpha) \left( \frac{1}{2(1 - \beta) \cos \eta} + 1 \right)
\]

(2.8)

is satisfied, then \( z_2 F_1(a, b, c; z) * f \in K(\alpha). \)

For \( \alpha = 0, \beta = 0, \) and \( \eta = 0, \) **Corollary 2.6** yields the following corollary.

**Corollary 2.7.** Let \( a > 0, \ b > 0, \) and \( c > a + b + 1. \) If \( f \in S \) and the inequality

\[
\Gamma(c) \Gamma(c-a-b) \frac{1 + \frac{ab}{c-a-b-1}}{\Gamma(c-a) \Gamma(c-b)} \leq \frac{3}{2}
\]

(2.9)

is satisfied, then \( z_2 F_1(a, b, c; z) * f \in K. \)

In our next theorems, we find the sufficient conditions for \( I * f \) to be in \( \varphi^*(A,B) \) and \( K(A,B). \) From the definition of \( I \) given by (1.11), we obtain

\[
I(z) = z + \sum_{n=2}^{\infty} \left( (1-\mu + n\mu)(a)_{n-1} b_{n-1} \frac{1}{(c)_{n-1}} \right) \frac{z^n}{n} \quad (\mu \geq 0, \ z \in U).
\]

(2.10)

**Theorem 2.8.** Let \( a > 1, \ b > 1, \) and \( c > a + b. \) If \( f \in R_\eta(\beta) \) and the inequality

\[
(1 + B - (A + 1) \mu) F_1(a, b, c; 1) - (A + 1)(1-\mu) F_3(a, b, 1, 1, c, 2, 2; 1) \leq (B - A) \left( \frac{1}{2(1 - \beta) \cos \eta} + 1 \right)
\]

(2.11)

is satisfied, then \( I * f \in \varphi^*(A,B). \)

**Proof.** By Lemma 2.1, it satisfies to show that

\[
T_2 := \sum_{n=2}^{\infty} \left| \frac{(1-\mu + n\mu)(a)_{n-1} b_{n-1}}{(c)_{n-1}} a_n \right| \leq B - A.
\]

(2.12)
Suppose that \( f \in R_\eta(\beta) \). Then by (1.5) we observe that

\[
T_2 \leq \sum_{n=2}^\infty \left( (1+B)n - (A+1) \right) \frac{(1-\mu+n\mu)(a)_{n-1}(b)_{n-1} 2(1-\beta) \cos \eta}{(c)_{n-1}(1)_n n}
\]

\[
= 2(1-\beta) \cos \eta \left\{ ((1+B)(1-\mu) - (A+1)\mu) \sum_{n=2}^\infty \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n+1}} 
\right.
\]

\[
- (A+1)(1-\mu) \sum_{n=2}^\infty \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n+1}}
\]

\[
+ (1+B)\mu \sum_{n=2}^\infty \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n+1}}
\right\}
\]

\[
= 2(1-\beta) \cos \eta \left\{ ((1+B)(1-\mu) - (A+1)\mu) \left( \frac{c-1}{(a-1)(b-1)} + 2F_1(a, b, c; 1) \right) 
\right.
\]

\[
- (A+1)(1-\mu)4F_3(a, b, 1, 1, c, 2, 2; 1)
\]

\[
+ (1+B)\mu 2F_1(a, b, c; 1)
\]

\[
- \left[ ((1+B)(1-\mu) - (A+1)\mu) \frac{c-1}{(a-1)(b-1)} + B-A \right]
\right\}
\]

\[
\leq B - A
\]

(2.13)

by (2.11). This completes the proof. \( \square \)

Taking \( \mu = 0 \), \( A = 2\alpha - 1 \), and \( B = 1 \) in Theorem 2.8, we see the following corollary.

**Corollary 2.9.** Let \( a > 1 \), \( b > 1 \), and \( c > a + b \). If \( f \in R_\eta(\beta) \) and the inequality

\[
2F_1(a, b, c; 1) - \alpha 4F_3(a, b, 1, 1, c, 2, 2; 1) \leq (1 - \alpha) \left( \frac{1}{2(1-\beta) \cos \eta} + 1 \right)
\]

(2.14)

is satisfied, then \( \int_0^\infty 2F_1(a, b, c; t) dt * f \in S^*(\alpha) \).

Taking \( \alpha = 0 \), \( \beta = 0 \), and \( \eta = 0 \) in Corollary 2.9, we get the following corollary.

**Corollary 2.10.** Let \( a > 1 \), \( b > 1 \), and \( c > a + b \). If \( f \in S \) and the inequality

\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2}
\]

(2.15)

is satisfied, then \( \int_0^\infty 2F_1(a, b, c; t) dt * f \in S^* \).
Theorem 2.11. Let \( a > 1, \ b > 1, \) and \( c > a + b + 1. \) If \( f \in R_{\eta}(\beta) \) and the inequality

\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ (1 + B) \left( 1 - \frac{\mu ab}{c-a-b-1} \right) + (A+1) \left( \mu \frac{c-a-b}{(a-1)(b-1)} - \frac{c-a-b}{(a-1)(b-1)} \right) \right] \leq (B-A) \left( \frac{1}{2(1-\beta)\cos \eta} + 1 \right) - \frac{(1-\mu)(A+1)(c-1)}{(a-1)(b-1)} \]  

(2.16)
is satisfied, then \( I \ast f \in K(A,B). \)

Proof. The proof follows from Lemma 2.1 and by applying similar method as in the proof of Theorem 2.8; we omit the details involved. \( \square \)

If we let \( \mu = 0, \ A = 2\alpha - 1, \) and \( B = 1 \) in Theorem 2.11, we get the following corollary.

Corollary 2.12. Let \( a > 1, \ b > 1, \) and \( c > a + b + 1. \) If \( f \in R_{\eta}(\beta) \) and the inequality

\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[ 1 - \frac{\alpha(c-a-b)}{(a-1)(b-1)} \right] \leq (1-\alpha) \left( \frac{1}{2(1-\beta)\cos \eta} + 1 \right) - \frac{\alpha(c-1)}{(a-1)(b-1)} \]  

(2.17)
is satisfied, then \( \int_{0}^{\pi} 2F_1(a,b,c;t)dt \ast f \in K(\alpha). \)

If we let \( \alpha = 0, \ \beta = 0, \) and \( \eta = 0 \) in Corollary 2.12, we have the following corollary.

Corollary 2.13. Let \( a > 1, \ b > 1, \) and \( c > a + b + 1. \) If \( f \in S \) and the inequality

\[
\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2} \]  

(2.18)
is satisfied, then \( \int_{0}^{\pi} 2F_1(a,b,c;t)dt \ast f \in K. \)

Acknowledgment. This work was supported by the Korea Science and Engineering Foundation (KOSEF), project no. 2000-6-101-01-2.

References


J. A Kim: Department of Mathematics, Pohang University of Science Technology, Pohang, Kyungbuk 790-784, Korea

E-mail address: jiakim@postech.ac.kr

K. H. Shon: Department of Mathematics, College of Natural Sciences, Pusan National University, Pusan 609-735, Korea

E-mail address: khshon@hyowon.pusan.ac.kr
Special Issue on
Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>May 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie

Hindawi Publishing Corporation
http://www.hindawi.com