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EXTENDED BLOCKER, DELETION, AND CONTRACTION MAPS ON ANTICHAINS

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Families of maps on the lattice of all antichains of a finite bounded poset that extend the blocker, deletion, and contraction maps on clutters are considered. Influence of the parameters of the maps is investigated. Order-theoretic extensions of some principal relations for the set-theoretic blocker, deletion, and contraction maps on clutters are presented.

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1. Introduction and preliminary. Let P be a finite bounded poset of cardinality greater than one. We can define some maps on the lattice of all antichains $\mathfrak{A}(P)$ of the poset P that naturally extend the set-theoretic blocker, deletion, and contraction maps on clutters; such maps were considered in [4, 5].

A set H is called a *blocking set* for a nonempty family $\mathcal{G} = \{G_1,\dots,G_m\}$ of nonempty subsets of a finite set if, for each $k \in \{1,\dots,m\}$, it holds $|H \cap G_k| > 0$. The family of all inclusionwise minimal blocking sets for $\mathcal{G}$ is called the *blocker* of $\mathcal{G}$. We denote the blocker of $\mathcal{G}$ by $\mathfrak{B}(\mathcal{G})$.

A family of subsets of a finite *ground set* S is called a *clutter* or a *Sperner family* if no set from that family contains another. The empty clutter $\emptyset$ containing no subsets of S and the clutter $\{\hat{0}\}$ whose unique set is the empty subset $\hat{0}$ of S are called the *trivial clutters* on S. The set-theoretic *blocker map* reflects a nontrivial clutter to its blocker, and that map reflects a trivial clutter to the other trivial clutter: $\mathfrak{B}(\emptyset) = \{\hat{0}\}$ and $\mathfrak{B}(\{\hat{0}\}) = \emptyset$.

Let $X \subseteq S$ and $|X| > 0$. The set-theoretic *deletion* $(\backslash X)$ and *contraction* $(/X)$ *maps* are defined in the following way: if $\mathcal{G}$ is a nontrivial clutter on S, then the *deletion* $\mathcal{G}\backslash X$ is the family $\{G \in \mathcal{G} : |G \cap X| = 0\}$ and the *contraction* $\mathcal{G}/X$ is the family of all inclusionwise minimal sets from the family $\{G - X : G \in \mathcal{G}\}$. The *deletion* and *contraction* for the trivial clutters coincide with the clutters $\emptyset\backslash X = \emptyset/X = \emptyset$ and $\{\hat{0}\}\backslash X = \{\hat{0}\}/X = \{\hat{0}\}$. The maps $(\backslash\hat{0})$ and $(/\hat{0})$ are the identity map on clutters; for any clutter $\mathcal{G}$, we by definition have $\mathcal{G}\backslash\hat{0} = \mathcal{G}/\hat{0} = \mathcal{G}$.

Let $\mathcal{G}$ be a clutter on the ground set S. Given a subset $X \subseteq S$, we have

$$\mathfrak{B}(\mathfrak{B}(\mathcal{G})) = \mathcal{G}, \tag{1.1}$$

$$\mathfrak{B}(\mathcal{G})\backslash X = \mathfrak{B}(\mathcal{G}/X), \qquad \mathfrak{B}(\mathcal{G})/X = \mathfrak{B}(\mathcal{G}\backslash X). \tag{1.2}$$

Recall that the atoms of the poset P are the elements covering its least element. Let X be a subset of the atom set P^{a} of P. (We denote the empty subset of P^{a} by $\varnothing^{\mathrm{a}}$.) We use the denotation $\mathfrak{b}:\mathfrak{A}(P)\to\mathfrak{A}(P)$ for the order-theoretic blocker map from [4], and we use the denotations $(\backslash X),(/X):\mathfrak{A}(P)\to\mathfrak{A}(P)$ for the order-theoretic operators of deletion and contraction from [5], respectively. We do not recall those concepts here because the map $\mathfrak{b}$ is the $(\varnothing^{\mathrm{a}},0)$-blocker map from Definition 2.1 of the present paper and the maps $(\backslash X)$ and $(/X)$ are the $(X,0)$-deletion and $(X,0)$-contraction maps from Definition 3.1 of the present paper, respectively.

For any antichain A of P, the following relations hold in $\mathfrak{A}(P)$:

$$\mathfrak{b}(\mathfrak{b}(\mathfrak{b}(A)))=\mathfrak{b}(A), \tag{1.3}$$

$$\mathfrak{b}(A)\backslash X\le\mathfrak{b}(A/X)\le\mathfrak{b}(A)\le\mathfrak{b}(A)/X\le\mathfrak{b}(A\backslash X). \tag{1.4}$$

Equality (1.3) from [4] goes back to (1.1) from [2, 3]. Comparison (1.4) from [5] goes back to (1.2) from [6].

In the present paper, we consider families of the so-called (X,k)-blocker, (X,k)-deletion, and (X,k)-contraction maps on $\mathfrak{A}(P)$ parametrized by subsets $X\subseteq P^{\mathrm{a}}$ and numbers $k\in\mathbb{N}$, $k<|P^{\mathrm{a}}|$. We show that for all pairs of the above-mentioned parameters X and k, the essential properties of the maps remain similar to those of the $(\varnothing^{\mathrm{a}},0)$-blocker, $(X,0)$-deletion, and $(X,0)$-contraction maps on $\mathfrak{A}(P)$ that were investigated in [4, 5]. In particular, we present analogues of relations (1.3) and (1.4) in Proposition 2.6(ii) and Theorem 3.7.

We refer the reader to [7, Chapter 3] for basic information and terminology in the theory of posets.

We use $\mathbf{min}\,Q$ to denote the set of all minimal elements of a poset Q. If Q has a least element, then it is denoted $\hat{0}_Q$; if Q has a greatest element, then it is denoted $\hat{1}_Q$.

Throughout the paper, P stands for a finite bounded poset of cardinality greater than one, that is, P by definition has the least and greatest elements that are distinct. We denote by $\mathfrak{I}(A)$ and $\mathfrak{F}(A)$ the order ideal and filter of P generated by an antichain A, respectively.

All antichains of P compose a distributive lattice denoted $\mathfrak{A}(P)$; in the present paper, antichains are by definition partially ordered in the following way; if $A',A''\in\mathfrak{A}(P)$, then we set

$$A'\le A''\quad\text{iff}\ \mathfrak{F}(A')\subseteq\mathfrak{F}(A''). \tag{1.5}$$

We call the least and greatest elements $\hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}$ of $\mathfrak{A}(P)$ the *trivial antichains* of P because, in the context of the present paper, they are counterparts of the trivial clutters. Here, $\hat{0}_{\mathfrak{A}(P)}$ is the empty antichain of P and $\hat{1}_{\mathfrak{A}(P)}$ the one-element antichain $\{\hat{0}_P\}$. We denote by $\vee$ and $\wedge$ the operations of join and meet in the lattice $\mathfrak{A}(P)$; if $A', A'' \in \mathfrak{A}(P)$, then

$$\begin{aligned} A' \vee A'' &= \mathbf{min}\,(A' \cup A''), \\ A' \wedge A'' &= \mathbf{min}\,(\mathfrak{F}(A') \cap \mathfrak{F}(A'')). \end{aligned} \tag{1.6}$$

2. (X,k)-blocker map. In this section, we consider a family of maps on antichains of a finite bounded poset that extend the set-theoretic blocker map on clutters. From now on, X is always a subset of P^{a} and k is a nonnegative integer less than $|P^{\mathrm{a}}|$.

Definition 2.1. The (X,k)*-blocker map* on $\mathfrak{A}(P)$ is the map $\mathfrak{b}_k^X : \mathfrak{A}(P) \to \mathfrak{A}(P)$,

$$A \longmapsto \mathbf{min}\,\{b \in P : |\mathfrak{I}(b) \cap \mathfrak{I}(a) \cap (P^{\mathrm{a}} - X)| > k \ \forall a \in A\} \tag{2.1}$$

if A is nontrivial, and

$$\hat{0}_{\mathfrak{A}(P)} \longmapsto \hat{1}_{\mathfrak{A}(P)}, \qquad \hat{1}_{\mathfrak{A}(P)} \longmapsto \hat{0}_{\mathfrak{A}(P)}. \tag{2.2}$$

Given an antichain $A \in \mathfrak{A}(P)$, the antichain $\mathfrak{b}_k^X(A)$ is the (X,k)*-blocker of* A in P.

We use the denotations $\mathfrak{b}_k$ and $\mathfrak{b}^X$ instead of the denotations $\mathfrak{b}_k^{\varnothing^{\mathrm{a}}}$ and $\mathfrak{b}_0^X$, respectively. The $(\varnothing^{\mathrm{a}},0)$-blocker map is the *blocker map* $\mathfrak{b}$ on $\mathfrak{A}(P)$ considered in [4]. Given $A \in \mathfrak{A}(P)$, the antichain $\mathfrak{b}(A)$ is called the *blocker of* A in P.

If $\{a\}$ is a one-element antichain of P, then we write $\mathfrak{b}_k^X(a)$ instead of $\mathfrak{b}_k^X(\{a\})$. Let $a \neq \hat{0}_P$. Since the blocker map on $\mathfrak{A}(P)$ is antitone, for every $E \subseteq \mathfrak{b}(a) - X$, we have $\{a\} \le \mathfrak{b}(\mathfrak{b}(a)) \le \mathfrak{b}(\mathfrak{b}(a) - X) \le \mathfrak{b}(E) \le \mathfrak{b}(a)$.

The following statement immediately follows from Definition 2.1.

Lemma 2.2. *Let A be a nontrivial antichain of P. If $\mathfrak{b}_k^X(A) \neq \hat{0}_{\mathfrak{A}(P)}$, then, for each $a \in A$ and for all $b \in \mathfrak{b}_k^X(A)$, it holds that*

$$|\mathfrak{I}(a) \cap \mathfrak{I}(b) \cap (P^{\mathrm{a}} - X)| > k. \tag{2.3}$$

Let $a \in P$, $a \neq \hat{0}_P$. From now on, $\mathcal{T}_a$ denotes the family of subsets of the atom set P^{a} defined as follows:

$$\mathcal{T}_a = \{E \subseteq \mathfrak{b}(a) - X : |E| = k+1\}. \tag{2.4}$$

Let $\mathfrak{L}(P^{\mathrm{a}})$ denote the Boolean lattice of all subsets of the atom set P^{a}, and let $\mathfrak{L}(P^{\mathrm{a}})^{(k+1)}$ denote the subset of all elements of rank $k+1$ of $\mathfrak{L}(P^{\mathrm{a}})$. Given a $(k+1)$-subset $E \subseteq P^{\mathrm{a}}$, we denote by $\varepsilon(E)$ the least upper bound for E in $\mathfrak{L}(P^{\mathrm{a}})$; conversely, given an element $\mathfrak{e} \in \mathfrak{L}(P^{\mathrm{a}})^{(k+1)}$, we denote by $\varepsilon^{-1}(\mathfrak{e})$ the $(k+1)$-subset of all atoms of $\mathfrak{L}(P^{\mathrm{a}})$ that are comparable with $\mathfrak{e}$.

Let A be a nontrivial antichain of P. If $|\mathfrak{b}(a)-X| \le k$ for some $a \in A$, then Definition 2.1 implies $\mathfrak{b}_k^X(A) = \hat{0}_{\mathfrak{A}(P)}$. In the case $|\mathfrak{b}(a)-X| > k$ for all $a \in A$, Proposition 2.3 describes two alternative ways of elementwise finding the (X,k)-blocker of A; it involves the set-theoretic blocker $\mathscr{B}(\cdot)$ of a set family.

Proposition 2.3. *Let A be a nontrivial antichain of P. If $|\mathfrak{b}(a)-X| > k$, for all $a \in A$, then*

$$\mathfrak{b}_k^X(A) = \bigwedge_{a\in A}\bigvee_{E\in\mathcal{T}_a}\mathfrak{b}(E) = \bigvee_{\mathfrak{E}\in\mathscr{B}(\{\{\varepsilon(E):E\in\mathcal{T}_a\}:\, a\in A\})}\ \bigwedge_{\mathfrak{e}\in\mathfrak{E}}\mathfrak{b}(\varepsilon^{-1}(\mathfrak{e})). \tag{2.5}$$

Proof. We have

$$\mathfrak{b}_k^X(A) = \bigwedge_{a\in A}\mathfrak{b}_k^X(a), \tag{2.6}$$

and an order-theoretic argument shows that, for every $a \in A$, it holds that

$$\mathfrak{b}_k^X(a) = \bigvee_{E\in\mathcal{T}_a}\mathfrak{b}(E), \tag{2.7}$$

where $\mathfrak{b}(E) = \bigwedge_{e\in E}\{e\}$.

The inclusion $\mathfrak{b}_k^X(A) \supseteq \bigvee_{\mathfrak{E}\in\mathscr{B}(\{\{\varepsilon(E):E\in\mathcal{T}_a\}:a\in A\})}\bigwedge_{\mathfrak{e}\in\mathfrak{E}}\mathfrak{b}(\varepsilon^{-1}(\mathfrak{e}))$ follows from Definition 2.1. To prove the inclusion

$$\mathfrak{b}_k^X(A) \subseteq \bigvee_{\mathfrak{E}\in\mathscr{B}(\{\{\varepsilon(E):E\in\mathcal{T}_a\}:a\in A\})}\ \bigwedge_{\mathfrak{e}\in\mathfrak{E}}\mathfrak{b}(\varepsilon^{-1}(\mathfrak{e})), \tag{2.8}$$

assume that it does not hold. Consider an element $b \in \mathfrak{b}_k^X(A)$ such that it does not belong to the right-hand side of (2.8). In this case, there is an element $a \in A$ such that $|\mathfrak{I}(b)\cap\mathfrak{I}(a)\cap(P^{\mathrm{a}}-X)| \le k$. It means that the left-hand side of (2.8) is not an (X,k)-blocker of A, a contradiction. □

The following lemma clarifies how the parameters of the (X,k)-blocker map influence the image of $\mathfrak{A}(P)$; additionally, the lemma states that $\mathfrak{b}_k^X$ is antitone.

Lemma 2.4. (i) *Let $Y \subseteq P^{\mathrm{a}}$, $Y \supseteq X$, and let j be a nonnegative integer, $j \le k$. If $A \in \mathfrak{A}(P)$, then*

$$\mathfrak{b}_j^X(A) \ge \mathfrak{b}_k^X(A) \ge \mathfrak{b}_k^Y(A). \tag{2.9}$$

(ii) *For all $A', A'' \in \mathfrak{A}(P)$ such that $A' \le A''$, it holds that*

$$\mathfrak{b}_k^X(A') \ge \mathfrak{b}_k^X(A''). \tag{2.10}$$

Proof. (i) There is nothing to prove if A is trivial. Suppose that A is a nontrivial antichain of P. For each element $a \in A$, we by (2.7) have

$$\mathfrak{b}_k^X(a) = \bigvee_{E \in \mathcal{T}_a} \mathfrak{b}(E) \geq \bigvee_{\substack{E \subseteq \mathfrak{b}(a) - Y: \\ |E| = k+1}} \mathfrak{b}(E) = \mathfrak{b}_k^Y(a). \tag{2.11}$$

With respect to (2.6), this yields

$$\mathfrak{b}_k^X(A) = \bigwedge_{a \in A} \mathfrak{b}_k^X(a) \geq \bigwedge_{a \in A} \mathfrak{b}_k^Y(a) = \mathfrak{b}_k^Y(A). \tag{2.12}$$

The relation $\mathfrak{b}_j^X(A) \geq \mathfrak{b}_k^X(A)$ is proved in a similar way.

(ii) If A' is a trivial antichain, then the assertion immediately follows from Definition 2.1. Suppose that A' is nontrivial. For every $a' \in A'$, there is $a'' \in A''$ such that $\{a'\} \leq \{a''\}$ and, as a consequence, it holds the inclusion $\mathfrak{b}(a') \supseteq \mathfrak{b}(a'')$, (2.7) implies $\mathfrak{b}_k^X(a') \geq \mathfrak{b}_k^X(a'')$, and the proof is completed by applying (2.6). □

In addition to Lemma 2.4(ii), we need the following statement to describe the structure of the image of $\mathfrak{A}(P)$ under the (X,k)-blocker map.

Lemma 2.5. *For any $A \in \mathfrak{A}(P)$, it holds that*

$$\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) \geq A. \tag{2.13}$$

Proof. If A is a trivial antichain of P, then the lemma follows from Definition 2.1 because, in this case, we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) = A$. Suppose that A is nontrivial. If $\mathfrak{b}_k^X(A) = \hat{0}_{\mathfrak{A}(P)}$, then we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) = \hat{1}_{\mathfrak{A}(P)} \geq A$ and we are done. Finally, suppose that $\mathfrak{b}_k^X(A)$ is a nontrivial antichain. On the one hand, according to Lemma 2.2, for each $a \in A$ and for all $b \in \mathfrak{b}_k^X(A)$, it holds that

$$|\mathfrak{I}(a) \cap \mathfrak{I}(b) \cap (P^{\mathrm{a}} - X)| > k. \tag{2.14}$$

On the other hand, we, by Definition 2.1, have

$$\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) = \mathbf{min}\{g \in P : |\mathfrak{I}(g) \cap \mathfrak{I}(b) \cap (P^{\mathrm{a}} - X)| > k \ \forall b \in \mathfrak{b}_k^X(A)\}. \tag{2.15}$$

Hence, we have $\mathfrak{b}_k^X(\mathfrak{b}_k^X(A)) \geq A$. □

We complete this section by applying a standard technique of the theory of posets to the lattice $\mathfrak{A}(P)$ and the (X,k)-blocker map on it. See, for instance, [1, Chapter IV] on (co)closure operators.

Proposition 2.6. (i) *The composite map* $\mathfrak{b}_k^X \circ \mathfrak{b}_k^X$ *is a closure operator on* $\mathfrak{A}(P)$.

(ii) *The poset* $\mathfrak{B}_k^X(P) = \{\mathfrak{b}_k^X(A) : A \in \mathfrak{A}(P)\}$ *is a self-dual lattice; the restriction map* $\mathfrak{b}_k^X|_{\mathfrak{B}_k^X(P)}$ *is an anti-automorphism of* $\mathfrak{B}_k^X(P)$. *The lattice* $\mathfrak{B}_k^X(P)$ *is a meet-subsemilattice of the lattice* $\mathfrak{A}(P)$.

(iii) *For every* $B \in \mathfrak{B}_k^X(P)$, *its preimage* $(\mathfrak{b}_k^X)^{-1}(B)$ *under the* (X,k)*-blocker map is a convex join-subsemilattice of the lattice* $\mathfrak{A}(P)$. *The greatest element of* $(\mathfrak{b}_k^X)^{-1}(B)$ *is* $\mathfrak{b}_k^X(B)$.

Proof. In view of Lemmas 2.4(ii) and 2.5, assertions (i) and (ii) are a corollary of [1, Propositions 4.36 and 4.26]. To prove (iii), choose arbitrary elements $A', A'' \in (\mathfrak{b}_k^X)^{-1}(B)$, where $B = \mathfrak{b}_k^X(A)$ for some $A \in \mathfrak{A}(P)$, and note that $\mathfrak{b}_k^X(A' \vee A'') = \mathfrak{b}_k^X(A') \wedge \mathfrak{b}_k^X(A'') = B$. If $B = \hat{0}_{\mathfrak{A}(P)}$, then $\mathfrak{b}_k^X(B) = \hat{1}_{\mathfrak{A}(P)}$ is the greatest element of $(\mathfrak{b}_k^X)^{-1}(B)$. If $B = \hat{1}_{\mathfrak{A}(P)}$, then $(\mathfrak{b}_k^X)^{-1}(B)$ is the one-element subposet $\{\hat{0}_{\mathfrak{A}(P)}\}$ of $\mathfrak{A}(P)$. Finally, if B is a nontrivial antichain of P, then the element $\mathfrak{b}_k^X(B) = \mathfrak{b}_k^X(\mathfrak{b}_k^X(A))$ is by (2.15) the greatest element of $(\mathfrak{b}_k^X)^{-1}(B)$. Since the (X,k)-blocker map is antitone, we can see that the subposet $(\mathfrak{b}_k^X)^{-1}(B)$ of $\mathfrak{A}(P)$ is convex. □

We call the poset $\mathfrak{B}_k^X(P)$ from Proposition 2.6(ii) the *lattice of* (X,k)*-blockers* in P. The poset $\mathfrak{B}(P) = \mathfrak{B}_0^{\varnothing^a}(P)$ is called in [4] the *lattice of blockers* in P.

3. (X,k)-deletion and (X,k)-contraction maps.

In this section, we consider order-theoretic extensions of the set-theoretic deletion and contraction maps on clutters.

Definition 3.1. (i) If $\{a\}$ is a nontrivial one-element antichain of P, then the (X,k)*-deletion* $\{a\}\backslash_k X$ and (X,k)*-contraction* $\{a\}/_k X$ of $\{a\}$ in P are the antichains

$$\{a\}\backslash_k X = \begin{cases} \{a\}, & \text{if } |\mathfrak{b}(a) \cap X| \le k, \\ \hat{0}_{\mathfrak{A}(P)}, & \text{if } |\mathfrak{b}(a) \cap X| > k, \end{cases} \tag{3.1}$$

$$\{a\}/_k X = \begin{cases} \{a\}, & \text{if } |\mathfrak{b}(a) \cap X| \le k, \\ \mathfrak{b}_k^X(\mathfrak{b}_k^X(a)), & \text{if } |\mathfrak{b}(a) \cap X| > k,\ \mathfrak{b}(a) \nsubseteq X, \\ \hat{1}_{\mathfrak{A}(P)}, & \text{if } |\mathfrak{b}(a) \cap X| > k,\ \mathfrak{b}(a) \subseteq X. \end{cases} \tag{3.2}$$

(ii) If A is a nontrivial antichain of P, then the (X,k)*-deletion* $A\backslash_k X$ and (X,k)*-contraction* $A/_k X$ of A in P are the antichains

$$A\backslash_k X = \bigvee_{a \in A} (\{a\}\backslash_k X), \qquad A/_k X = \bigvee_{a \in A} (\{a\}/_k X). \tag{3.3}$$

(iii) The (X,k)-*deletion* and (X,k)-*contraction* of the trivial antichains of P are

$$\hat{0}_{\mathfrak{A}(P)}\setminus_k X = \hat{0}_{\mathfrak{A}(P)}/_k X = \hat{0}_{\mathfrak{A}(P)}, \quad \hat{1}_{\mathfrak{A}(P)}\setminus_k X = \hat{1}_{\mathfrak{A}(P)}/_k X = \hat{1}_{\mathfrak{A}(P)}. \tag{3.4}$$

(iv) The map

$$(\setminus_k X) : \mathfrak{A}(P) \longrightarrow \mathfrak{A}(P), \qquad A \longmapsto A\setminus_k X, \tag{3.5}$$

is the *operator of* (X,k)-*deletion* on $\mathfrak{A}(P)$.

The map

$$(/_k X) : \mathfrak{A}(P) \longrightarrow \mathfrak{A}(P), \qquad A \longmapsto A/_k X, \tag{3.6}$$

is the *operator of* (X,k)-*contraction* on $\mathfrak{A}(P)$.

Given an antichain $A \in \mathfrak{A}(P)$, we use the denotations $A\setminus X$ and A/X instead of the denotations $A\setminus_0 X$ and $A/_0 X$, respectively. The $(X,0)$-deletion map $(\setminus X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ and the $(X,0)$-contraction map $(/X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ are the *operators of deletion* and *contraction* on $\mathfrak{A}(P)$, respectively, considered in [5].

The following observation is an immediate consequence of Definition 3.1. If $a', a'' \in P$ and $\{a'\} \le \{a''\}$ in $\mathfrak{A}(P)$, then

$$\{a'\}\setminus_k X \le \{a''\}\setminus_k X, \qquad \{a'\}/_k X \le \{a''\}/_k X; \tag{3.7}$$

hence, in view of (3.3) and (3.4), we can formulate the following lemma.

Lemma 3.2. *If* $A', A'' \in \mathfrak{A}(P)$ *and* $A' \le A''$, *then*

$$A'\setminus_k X \le A''\setminus_k X, \qquad A'/_k X \le A''/_k X. \tag{3.8}$$

Moreover, if $\{a\}$ is a one-element antichain of P, then we have

$$\{a\}\setminus_k X \le \{a\} \le \{a\}/_k X, \tag{3.9}$$

and a more general statement is true.

Lemma 3.3. *If* $A \in \mathfrak{A}(P)$, *then*

$$A\setminus_k X \le A \le A/_k X. \tag{3.10}$$

Another consequence of Definition 3.1 is that, for a one-element antichain $\{a\}$ of P, it holds that

$$\mathfrak{b}_k^X(a)\setminus_k X \le \mathfrak{b}_k^X(\{a\}/_k X) \le \mathfrak{b}_k^X(a) \le \mathfrak{b}_k^X(a)/_k X \le \mathfrak{b}_k^X(\{a\}\setminus_k X). \tag{3.11}$$

Let $\{a\}$ be a nontrivial one-element antichain of P. We obviously have $(\{a\}\backslash_k X)\backslash_k X = \{a\}\backslash_k X$. We show that $(\{a\}/_k X)/_k X = \{a\}/_k X$. If $|\mathfrak{b}(a) \cap X| \le k$, then Definition 3.1 implies $(\{a\}/_k X)/_k X = \{a\}/_k X = \{a\}$; further, if $|\mathfrak{b}(a) \cap X| > k$ and $\mathfrak{b}(a) \subseteq X$, then Definition 3.1 implies $(\{a\}/_k X)/_k X = \{a\}/_k X = \hat{1}_{\mathfrak{A}(P)}$. Suppose that $|\mathfrak{b}(a) \cap X| > k$ and $\mathfrak{b}(a) \nsubseteq X$. In this case, on the one hand, we have $(\{a\}/_k X)/_k X \ge \{a\}/_k X$ by Lemma 3.3, on the other hand, for every element $b \in \{a\}/_k X = \mathfrak{b}_k^X(\mathfrak{b}_k^X(a))$, we have $\mathfrak{b}_k^X(b) \ge \mathfrak{b}_k^X(a)$, and, as a consequence, we have $(\{a\}/_k X)/_k X = \bigvee_{b \in \{a\}/_k X}(\{b\}/_k X) \le \mathfrak{b}_k^X(\mathfrak{b}_k^X(a)) = \{a\}/_k X$. We arrive at the conclusion that $(\{a\}/_k X)/_k X = \{a\}/_k X$. With respect to (3.3), we can formulate the following lemma.

Lemma 3.4. *If $A \in \mathfrak{A}(P)$, then*

$$(A\backslash_k X)\backslash_k X = A\backslash_k X, \qquad (A/_k X)/_k X = A/_k X. \tag{3.12}$$

Lemmas 3.2, 3.3, and 3.4 lead to a characterization of the (X,k)-deletion and (X,k)-contraction maps in terms of (co)closure operators.

Proposition 3.5. *The map $(\backslash_k X)$ is a coclosure operator on $\mathfrak{A}(P)$. The map $(/_k X)$ is a closure operator on $\mathfrak{A}(P)$.*

The following proposition is a counterpart of Lemma 2.4(i).

Proposition 3.6. *Let $Y \subseteq P^{\mathrm{a}}$, $Y \supseteq X$, and let m be an integer, $k \le m < |P^{\mathrm{a}}|$. If $A \in \mathfrak{A}(P)$, then*

$$\begin{gathered} A\backslash_m X \ge A\backslash_k X \ge A\backslash_k Y, \\ A/_k X \le A/_k Y \le A/_m Y. \end{gathered} \tag{3.13}$$

Proof. If A is a trivial antichain, then the proposition follows from (3.4). Suppose that A is nontrivial. For each $a \in A$, (3.1) implies $\{a\}\backslash_k X \ge \{a\}\backslash_k Y$, (3.2) implies $\{a\}/_k X \le \{a\}/_k Y$, and (3.3) yields

$$\begin{gathered} A\backslash_k X = \bigvee_{a \in A}(\{a\}\backslash_k X) \ge \bigvee_{a \in A}(\{a\}\backslash_k Y) = A\backslash_k Y, \\ A/_k X = \bigvee_{a \in A}(\{a\}/_k X) \le \bigvee_{a \in A}(\{a\}/_k Y) = A/_k Y. \end{gathered} \tag{3.14}$$

Other relations are proved in a similar way. □

We denote the images $(\backslash_k X)(\mathfrak{A}(P)) = \{A\backslash_k X : A \in \mathfrak{A}(P)\}$ and $(/_k X)(\mathfrak{A}(P)) = \{A/_k X : A \in \mathfrak{A}(P)\}$ by $\mathfrak{A}(P)\backslash_k X$ and $\mathfrak{A}(P)/_k X$, respectively. We can interpret well-known properties of (semi)lattice maps and (co)closure operators on lattices in the case of the (X,k)-deletion and (X,k)-contraction maps.

Definition 3.1 implies that the maps $(\backslash_k X), (/_k X) : \mathfrak{A}(P) \to \mathfrak{A}(P)$ are upper $\{\hat{0}_{\mathfrak{A}(P)}, \hat{1}_{\mathfrak{A}(P)}\}$-homomorphisms, that is, for all $A', A'' \in \mathfrak{A}(P)$, we have $(A' \vee A'')\backslash_k X = (A'\backslash_k X) \vee (A''\backslash_k X)$ and $(A' \vee A'')/_k X = (A'/_k X) \vee (A''/_k X)$, and, moreover, we have $\hat{0}_{\mathfrak{A}(P)}\backslash_k X = \hat{0}_{\mathfrak{A}(P)}/_k X = \hat{0}_{\mathfrak{A}(P)}$ and $\hat{1}_{\mathfrak{A}(P)}\backslash_k X = \hat{1}_{\mathfrak{A}(P)}/_k X = \hat{1}_{\mathfrak{A}(P)}$.

The posets $\mathfrak{A}(P)\backslash_k X$ and $\mathfrak{A}(P)/_k X$, with the partial orders induced by the partial order on $\mathfrak{A}(P)$, are lattices.

We call the poset $\mathfrak{A}(P)\backslash_k X$ the *lattice of* (X,k)*-deletions* in P, and we call the poset $\mathfrak{A}(P)/_k X$ the *lattice of* (X,k)*-contractions* in P.

The lattice $\mathfrak{A}(P)\backslash_k X$ is a join-subsemilattice of $\mathfrak{A}(P)$. Denote by $\wedge_{\mathfrak{A}(P)\backslash_k X}$ the operation of meet in $\mathfrak{A}(P)\backslash_k X$. If $D', D'' \in \mathfrak{A}(P)\backslash_k X$, then we have $D' \wedge_{\mathfrak{A}(P)\backslash_k X} D'' = (D' \wedge D'')\backslash_k X$.

The lattice $\mathfrak{A}(P)/_k X$ is a sublattice of $\mathfrak{A}(P)$.

If $D \in \mathfrak{A}(P)\backslash_k X$, then the preimage $(\backslash_k X)^{-1}(D)$ of D under the (X,k)-deletion map is the closed interval $[D, D \vee \bigvee_{E\subseteq X:|E|=k+1} \mathfrak{b}(E)]$ of $\mathfrak{A}(P)$.

If $D \in \mathfrak{A}(P)/_k X$, then the preimage $(/_k X)^{-1}(D)$ of D under the (X,k)-contraction map is a convex join-subsemilattice of the lattice $\mathfrak{A}(P)$, with the greatest element D.

Relations (1.2) and (1.4) have the following analogue.

Theorem 3.7. *If $A \in \mathfrak{A}(P)$, then*

$$\mathfrak{b}_k^X(A)\backslash_k X \le \mathfrak{b}_k^X(A/_k X) \le \mathfrak{b}_k^X(A) \le \mathfrak{b}_k^X(A)/_k X \le \mathfrak{b}_k^X(A\backslash_k X). \tag{3.15}$$

Proof. There is nothing to prove if A is a trivial antichain. Suppose that A is nontrivial. The relations

$$\mathfrak{b}_k^X(A)\backslash_k X \le \mathfrak{b}_k^X(A) \le \mathfrak{b}_k^X(A)/_k X, \qquad \mathfrak{b}_k^X(A/_k X) \le \mathfrak{b}_k^X(A) \le \mathfrak{b}_k^X(A\backslash_k X) \tag{3.16}$$

follow from Lemmas 3.3 and 2.4(ii).

We need the following auxiliary relations. If A' and A'' are arbitrary antichains of P, then

$$(A' \wedge A'')\backslash_k X \le (A'\backslash_k X) \wedge (A''\backslash_k X), \tag{3.17}$$

$$(A' \wedge A'')/_k X \le (A'/_k X) \wedge (A''/_k X). \tag{3.18}$$

To prove $\mathfrak{b}_k^X(A)\backslash_k X \le \mathfrak{b}_k^X(A/_k X)$, we use (3.17) and (3.11), and we see that

$$\begin{aligned}\mathfrak{b}_k^X(A)\backslash_k X &= \Big(\bigwedge_{a\in A} \mathfrak{b}_k^X(a)\Big)\backslash_k X \le \bigwedge_{a\in A} (\mathfrak{b}_k^X(a)\backslash_k X) \le \bigwedge_{a\in A} \mathfrak{b}_k^X(\{a\}/_k X)\\ &= \mathfrak{b}_k^X\Big(\bigvee_{a\in A} (\{a\}/_k X)\Big) = \mathfrak{b}_k^X(A/_k X).\end{aligned} \tag{3.19}$$

To prove $\mathfrak{b}_k^X(A)/_kX \le \mathfrak{b}_k^X(A\backslash_k X)$, we use (3.18) and (3.11), and we see that

$$\begin{aligned}\mathfrak{b}_k^X(A)/_kX &= \Big(\bigwedge_{a\in A}\mathfrak{b}_k^X(a)\Big)/_kX \le \bigwedge_{a\in A}\big(\mathfrak{b}_k^X(a)/_kX\big) \le \bigwedge_{a\in A}\mathfrak{b}_k^X(\{a\}\backslash_k X)\\ &= \mathfrak{b}_k^X\Big(\bigvee_{a\in A}(\{a\}\backslash_k X)\Big) = \mathfrak{b}_k^X(A\backslash_k X).\end{aligned} \tag{3.20}$$

□

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