THE MIDPOINT SET OF A CANTOR SET

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ABSTRACT. A non-endpoint of the Cantor ternary set is any Cantor point which is not an endpoint of one of the remaining closed intervals obtained in the usual construction process of the Cantor ternary set in the unit interval. It is shown that the set of points in the unit interval which are not midway between two distinct Cantor ternary points is precisely the set of Cantor non-endpoints. It is also shown that the generalized Cantor set $C_\lambda$, for $1/3 < \lambda < 1$, has void intersection with its set of midpoints obtained from distinct members of $C_\lambda$.

KEY WORDS AND PHRASES. Cantor set, midpoint set, distance set.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. Primary 00 (General), Secondary 04 (Set Theory).

1. INTRODUCTION.

J. Randolph [6] and N. C. Bose Majumdar [1] have shown that every point in the unit interval is the mean value of a pair of (not necessarily distinct) Cantor ternary points. In 1936, V. Jarnik [5] noted that the set of all Cantor
points which represent an irrational number has void intersection with its set of distinct midpoints. Here we will characterize all points in the unit interval which are not midway between two distinct Cantor points. We shall also present a class of Cantor-type sets with the property that each member of the class has void intersection with its set of distinct midpoints.

For certain linear sets, it has been shown [1] that there exists a definite relationship between the distance set and the midpoint set of the given set. We will present examples to demonstrate that this relationship cannot be extended to n-dimensional sets as purported in [1].

2. BASIC CONCEPTS.

DEFINITIONS. Let $A \subseteq \mathbb{R}$. We define the following sets:

$$M(A) = \{m: x+y=2m, x, y \in A\}$$

$$M^*(A) = \{m: x+y=2m, x \neq y, x, y \in A\}$$

$$D(A) = \{d: d=|x-y|, x, y \in A\}$$

CONSTRUCTION OF THE SET $C_\lambda$. Let $0 < \lambda < 1$. From the closed unit interval delete the open middle segment of length $\lambda$, leaving two closed intervals $I_{11}$ and $I_{12}$ each of length $(1-\lambda)/2$. From each of the intervals $I_{11}$ and $I_{12}$ delete the middle open segment of length $\lambda(1-\lambda)/2$, leaving four closed, congruent intervals $I_{21}, I_{22}, I_{23}$, and $I_{24}$ of length $(1-\lambda)/2$. Continue this process inductively. If we let

$$A_1 = I_{11} \cup I_{12}$$

$$A_2 = \bigcup_{k=1}^{4} I_{2k}$$

$$A_n = \bigcup_{k=1}^{2^n} I_{nk}$$
then the set \( C_\lambda \) is defined to be \( \bigcap_{n=1}^{\infty} A_n \). Those points in \( C_\lambda \) which are not endpoints of any \( I_{nk} \) in the construction of \( C_\lambda \) shall be termed non-endpoints of \( C_\lambda \). We will denote the set of non-endpoints of \( C_{1/3} \) by \( N \).

3. \( M^*(C_{1/3}) = (0,1) - N \).

In this section we show that the set of points in the unit interval which are not midway between two distinct Cantor ternary points is precisely the set of non-endpoints of \( C_{1/3} \).

**THEOREM.** (B.M. [2]). \( M^*(C_{1/3}) \supseteq (0,1) - N \).

We complete the characterization with the following

**THEOREM.** \( N \subseteq (0,1) - M^*(C_{1/3}) \).

**PROOF.** We need only show that if \( z \in N \) and \( x = y = 2z \) for \( x, y \in C_{1/3} \), then \( x = y = z \). To accomplish this we shall show that the ternary expansion of \( 2z \) satisfies the following property:

\[
(*) \text{ Every } w \in [0,1] \text{ which can be expressed as } w = .\delta_1 1\delta_2 1\delta_3 1 \ldots \text{ (base 3),}
\]

where \( \delta_j \) is a complex of 0's if \( j \) is odd (or is empty) and \( \delta_j \) is a complex of 2's if \( j \) is even (or is empty), and the digit 1 appears an infinite number of times, is uniquely expressible as \( w = x+y \), where \( x, y \in C_{1/3} \) (see [2].).

Since \( z \in N \) and if \( 0 < z < 1/3 \), then
where $\alpha_i = 0$, $\alpha_i \in \{0,1\}$ for $i > 1$, and the values 0 and 1 are both assumed an infinite number of times, thus, $2z < 2/3$ and

$$2z = \sum_{i=1}^{\infty} \frac{4\alpha_i}{3^i} = \sum_{i=1}^{\infty} \left(\frac{\alpha_i}{3^{i-1}} + \frac{\alpha_i}{3^i}\right) = \sum_{j=1}^{\infty} \left(\frac{\alpha_j + \alpha_{j+1}}{3^j}\right),$$

where $(\alpha_1 + \alpha_2) \in \{0,1\}$ and $(\alpha_j + \alpha_{j+1}) \in \{0,1,2\}$ for $j > 1$. Expressed as a ternary decimal

$$2z = \left(\delta_1 + \delta_2\right) \left(\delta_2 + \delta_3\right) \left(\delta_3 + \delta_4\right) \ldots \text{(base 3)}.$$

Since $\alpha_1 = 0$ and $\alpha_j (j > 1)$ assumes only the values 0 or 1, clearly in the ternary expansion of $2z$, the digits 0 and 2 can never appear in succession and consequently $2z$ satisfies (*). Hence if $x+y=2z$, $x,y \in C_{1/3}$, then since $z \in C_{1/3}$, it follows that $x=y=z$.

Now if $2/3 < z < 1$, then $0 < 1-z < 1/3$. By symmetry $1-z \in N$. Thus, if $x$ and $y$ are Cantor ternary points such that $x+y=2z$, then $(1-x) + (1-y) = 2(1-z)$ where $1-x, 1-y \in C_{1/3}$. It follows from the above argument that $x=y=z$.

4. A PROPERTY OF $M^*(C_\lambda)$, $\lambda > 1/3$.

In this section we demonstrate a class of sets with the property that each member of the class has void intersection with its set of distinct midpoints.

**Lemma.** Let $\lambda > 1/3$ and let $x, m, y \in C_\lambda$ be such that $x < m < y$ and $x+y = 2m$. Then $x, m, y$ are always contained in the same closed interval $I_{nk}$ for each construction stage $n$ of $C_{\lambda}$.

**Proof.** We induct on $n$. For $n=1$, it is easily seen that $x, m, y \in I_{11}$ or $x, m, y \in I_{12}$ since $\lambda > 1/3$. 

Assume that for \( n=t \), \( x \), \( m \), \( y \in I_{tp} \) for some \( p \). Let \( W \) denote the open segment deleted from \( I_{tp} \) and let \( I_{t+1,j} \) and \( I_{t+1,j+1} \) denote the remaining closed intervals obtained from \( I_{tp} \) during the \( n+1 \)-st construction stage of \( C_\lambda \). Since \( \lambda > 1/3 \), it follows that \( |W| > |I_{t+1,j}| = |I_{t+1,j+1}| \). It immediately follows that \( x \), \( m \), \( y \in I_{t+1,j} \) or \( x \), \( m \), \( y \in I_{t+1,j+1} \).

**THEOREM.** For \( \lambda > 1/3 \), \( M^*(C_\lambda) \cap C_\lambda = \emptyset \).

**PROOF.** Let \( x \in C_\lambda \) and let \( I_n(x) \) (\( n=1,2, \ldots \)) be the \( n \)-th stage closed interval in the construction of \( C_\lambda \) containing \( x \). If \( x \), \( m \), \( y \in C_\lambda \) are such that \( x+y = 2m \) and \( x=y \), then it follows that \( x=y=m \) by the preceding lemma, since

\[
\bigcap_{n=1}^{\infty} I_n(x) = \{x\}. \]

Consequently \( M^*(C_\lambda) \cap C_\lambda = \emptyset \).

5. **A NOTE ON MIDPOINT SETS AND DISTANCE SETS**

If \( A \) is a symmetric subset of the closed unit interval with \( 0 \), \( l \in A \), then it is known [1] that \( D(A) = [0,1] \) if, and only if, \( M(A) = [0,1] \). Also in [1], the author attempted to generalize this result to higher dimensions with the following statement.

If \( A_1, A_2, \ldots, A_n \) are symmetric subsets of the closed unit interval with \( 0 \), \( l \in A_k \) (\( k=1, \ldots, n \)), and if \( A = A_1 \times A_2 \times \ldots \times A_n \), then the distance set of \( A \), \( D(A) = \{d: d = |P-Q|, P, Q \in A\} \) where \( |P-Q| \) denotes the Euclidean distance from \( P \) to \( Q \), is the interval \( [0,\sqrt{n}] \) if, and only if, \( M(A_k) = [0,1] \) (\( k=1,2, \ldots, n \)).

While it is true that if \( M(A_k) = [0,1] \) for each \( k \), then \( D(A) = [0,\sqrt{n}] \), the converse does not hold.

**EXAMPLE 1.** If \( A_1 = [0,1] \) and \( A_2 = [0,1/4] \cup [3/4,1] \), then clearly \( D(A_1 \times A_2) = [0,\sqrt{2}] \), but \( M(A_2) = [0,1/4] \cup [3/8, 5/8] \cup [3/4,1] \).

**EXAMPLE 2.** It is known (see [3]) that \( D(C_\lambda) = [0,1] \) if, and only if, \( \lambda \leq 1/3 \); consequently \( M(C_\lambda) = [0,1] \) if, and only if, \( \lambda \leq 1/3 \). For \( \lambda > 1/3 \),
M(C_\lambda) \neq [0,1], but it has been shown [4] that \( D(C_\lambda \times C_\lambda) = [0, \sqrt{2}] \) for \( \lambda \leq \sqrt{2} - 1 \).

REFERENCES


Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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