

PROPER SHAPE INVARIANTS: TAMENESS AND MOVABILITY

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ABSTRACT. We study geometric properties of topological spaces called proper $N_C^{\mathcal{B}}$ -tameness, proper $P_C^{\mathcal{B}}$ -tameness, and proper $N^{\mathcal{B}}$ -movability, where \mathcal{B} and \mathcal{C} denote classes of spaces. They are related to proper $M_C^{\mathcal{B}}$ -tameness and proper $M^{\mathcal{B}}$ -movability from [5] and could be regarded as their dual forms. All three are invariants of a recently invented author's proper shape theory and are described by the use of proper multi-valued functions. We explore their basic properties and prove several results on their preservation under proper maps.

KEY WORDS AND PHRASES. Proper multi-valued function, σ -small, proper σ -homotopy, proper multi-net, properly $N_C^{\mathcal{B}}$ -tame, properly $N^{\mathcal{B}}$ -movable.

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1. INTRODUCTION.

In this paper we shall continue exploration of invariants in the proper shape theory of arbitrary topological spaces from [3]. As in the article [5], we study proper tameness and proper movability. However, this time instead of looking at maps from some objects into a space we take a dual approach where we utilize maps from a space into those objects.

The objects are spaces from classes \mathcal{B} and \mathcal{C} of topological spaces and for maps we take multi-valued functions. The dualization procedure when applied to invariants of proper $M_C^{\mathcal{B}}$ -tameness and proper $M^{\mathcal{B}}$ -movability from [5] gives new invariants which we call proper $N_C^{\mathcal{B}}$ -tameness, proper $P_C^{\mathcal{B}}$ -tameness and proper $N^{\mathcal{B}}$ -movability. The change from the letter M to the letters N and P should reflect duality between these notions. As the reader will see this duality is striking.

Let us describe the content of the paper in greater detail. In §2 we recall notions and results from Čerin [3] that are necessary in further developments. The next §3 studies properly $N_C^{\mathcal{B}}$ -tame and properly $P_C^{\mathcal{B}}$ -tame spaces. The origin for these notions could be traced back to L. Siebenmann's tame at infinity spaces [7]. The idea is that we require that small enough proper multi-valued functions from a given space X into members of a class of spaces \mathcal{B} factor through some member of another class \mathcal{C} also through sufficiently small proper multi-valued functions. This concept is related to the notion of shape dimension Borsuk [2] and could be regarded as a substitute for it in the proper shape theory. We prove that these are invariants in the proper shape category $\mathcal{S}h_p$, explore the role of classes \mathcal{B} and \mathcal{C} , and study what kind of maps will preserve and inversely preserve properly $N_C^{\mathcal{B}}$ -tame and properly $P_C^{\mathcal{B}}$ -tame spaces. The classes of proper $M^{\mathcal{B}}$ -surjections, proper $N^{\mathcal{B}}$ -surjections, proper $M^{\mathcal{B}}$ -injections, and proper $N^{\mathcal{B}}$ -injections from [4] are of key importance here.

In the following §4 we consider properly $N^{\mathcal{B}}$ -movable spaces which are the analogue of movable compacta Borsuk [2] in proper shape theory.

Finally, in §5 we consider dependence of these notions on classes \mathcal{B} and \mathcal{C} under the assumption that they are connected with each other by morphisms from [4].

2. PRELIMINARIES ON PROPER SHAPE THEORY.

In this section we shall introduce notions and results from Čerin [3] that are required for our theory.

Let X and Y be topological spaces. By a *multi-valued function* $F : X \rightarrow Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of X . A multi-valued function $F : X \rightarrow Y$ is *S-proper* provided for every compact subset C of Y its small counterimage $F'(C) = \{x \in X \mid F(x) \subset C\}$ is a compact subset of X . On the other hand, F is *B-proper* provided for every compact subset C of Y its big counterimage $F''(C) = \{x \in X \mid F(x) \cap C \neq \emptyset\}$ is a compact subset of X . We shall use the term *proper* to name either *S-proper* or *B-proper*. However, in a given situation, once we make a selection between two different kinds of properness it is understood that it will be retained throughout.

Observe that for single-valued functions the two notions of properness coincide. Classes of *S-proper* and *B-proper* multi-valued functions are completely unrelated [3]. It follows that each of our notions and results involving proper multi-valued functions actually has two versions.

Let $Cov(Y)$ denote the collection of all numerable covers of a topological space Y . Let $Inc(Y)$ denote the collection of all finite subsets c of $Cov(Y)$ which have a unique (with respect to the refinement relation) maximal element $[c] \in Cov(Y)$. We consider $Inc(Y)$ ordered by the inclusion relation and regard $Cov(Y)$ as a subset of single-element subsets of $Inc(Y)$.

Let $\sigma \in Cov(Y)$. Let σ^+ denote the set of all numerable covers of Y refining σ while σ^* denotes the set of all numerable covers τ of Y such that the star $st(\tau)$ of τ refines σ . Similarly, for a natural number n , σ^{*n} denotes the set of all numerable covers τ of Y such that the n -th star $st^n(\tau)$ of τ refines σ .

For our approach to proper shape theory the following notion of size for multi-valued functions will play the most important role.

Let $F : X \rightarrow Y$ be a multi-valued function and let $\alpha \in Cov(X)$ and $\gamma \in Cov(Y)$. We shall say that F is an (α, γ) -function provided for every $A \in \alpha$ there is a $C_A \in \gamma$ with $F(A) \subset C_A$. On the other hand, F is γ -small provided there is an $\alpha \in Cov(X)$ such that F is an (α, γ) -function. For a σ -small multi-valued function $F : X \rightarrow Y$ we use $S(F, \sigma)$ to denote the family of all numerable covers α of X such that F is an (α, σ) -function.

Next we introduce the notions which correspond to the equivalence relation of proper homotopy for proper maps.

Let F and G be proper multi-valued functions from a space X into a space Y and let γ be a numerable cover of Y . We shall say that F and G are *properly γ -homotopic* and write $F \stackrel{\gamma}{\simeq} G$ provided there is a γ -small proper multi-valued function H from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $F(x) \subset H(x, 0)$ and $G(x) \subset H(x, 1)$ for every $x \in X$. We shall say that H is a *proper γ -homotopy* that *joins* F and G or that it *realizes* the relation (or proper homotopy) $F \stackrel{\gamma}{\simeq} G$.

The following two definitions correspond to Ball and Sher's definitions of proper fundamental net and proper homotopy for proper fundamental nets from [1].

Let X and Y be topological spaces. By a *proper multi-net* from X into Y we shall mean a collection $\varphi = \{F_c \mid c \in Inc(Y)\}$ of proper multi-valued functions $F_c : X \rightarrow Y$ such that for every $\gamma \in Cov(Y)$ there is a $c \in Inc(Y)$ with $F_d \stackrel{\gamma}{\simeq} F_c$ for every $d > c$. We use functional notation $\varphi : X \rightarrow Y$ to indicate that φ is a proper multi-net from X into Y . Let $M_p(X, Y)$ denote all proper multi-nets $\varphi : X \rightarrow Y$.

Two proper multi-nets $\varphi = \{F_c\}$ and $\psi = \{G_c\}$ between topological spaces X and Y are *properly homotopic* and we write $\varphi \simeq \psi$ provided for every $\gamma \in Cov(Y)$ there is a $c \in Inc(Y)$ such that $F_d \stackrel{\gamma}{\simeq} G_d$ for every $d > c$. On the other hand, we write $\varphi \stackrel{\gamma}{\simeq} \psi$ and call φ and ψ *properly γ -homotopic* provided there is a $c \in Inc(Y)$ such that $F_d \stackrel{\gamma}{\simeq} G_d$ for every $d > c$.

The relation of proper homotopy is an equivalence relation on the set $M_p(X, Y)$. The proper homotopy class of a proper multi-net φ is denoted by $[\varphi]$ and the set of all proper homotopy classes by $Sh_p(X, Y)$.

The topological spaces as objects together with the proper homotopy classes of proper multi-nets as morphisms form the proper shape category Sh_p (see [3]).

3. PROPERLY N_C^B -TAME AND PROPERLY P_C^B -TAME CLASSES.

The notion of a properly M_C^B -tame class of spaces from [5] allow us to obtain two new properties that are preserved under proper M -domination. They could be considered as dual

to the notion of a properly M_C^B -tame space. While in [5] we investigated a space X by looking at small proper multi-valued functions from members of a given class of spaces \mathcal{B} into X , we now change our point of view by concentrating on small proper multi-valued functions from X into members of \mathcal{B} .

Let \mathcal{B} and \mathcal{C} be classes of spaces. A class of spaces \mathcal{X} is (1) *properly N_C^B -tame* and (2) *properly P_C^B -tame* provided the class \mathcal{B} is (1) properly M_C^X -tame and (2) properly M_X^C -tame, respectively. In other words, provided that

- (1) for every $B \in \mathcal{B}$ and every $\sigma \in Cov(B)$ there is a $\tau \in Cov(B)$ such that for every $X \in \mathcal{X}$ and every proper τ -small multi-valued function $F : X \rightarrow B$ there is a $C \in \mathcal{C}$ and a proper σ -small multi-valued function $H : C \rightarrow B$ so that for every $\gamma \in Cov(C)$ there is a proper γ -small multi-valued function $G : X \rightarrow C$ with $F \stackrel{\sigma}{\simeq} H \circ G$.
- (2) for every $B \in \mathcal{B}$ and every $\sigma \in Cov(B)$ there is a $\tau \in Cov(B)$ such that for every $C \in \mathcal{C}$ and every proper τ -small multi-valued function $F : C \rightarrow B$ there is an $X \in \mathcal{X}$ and a proper σ -small multi-valued function $H : X \rightarrow B$ so that for every $\xi \in Cov(X)$ there is a proper ξ -small multi-valued function $G : C \rightarrow X$ with $F \stackrel{\sigma}{\simeq} H \circ G$.

We shall say that a space X has one of the above properties provided the class $\{X\}$ consisting just of a space X has this property.

The three versions of proper tameness share many properties. We shall now state and prove the N and the P versions of most results from §3 in [5].

THEOREM 3.1. A class \mathcal{X} of spaces is properly N_C^B -tame if and only if a properly N_C^B -tame class of spaces properly M -dominates it.

PROOF. Suppose that \mathcal{X} is properly M -dominated by a properly N_C^B -tame class \mathcal{Y} . Then \mathcal{B} is properly M_C^Y -tame so that \mathcal{B} is properly M_C^X -tame by theorem 3.5 in [5]. Hence, \mathcal{X} is properly N_C^B -tame. \square

COROLLARY 3.2. A space is properly N_C^B -tame if and only if it is quasi Sh_p -dominated or Sh_p -dominated by a properly N_C^B -tame class of spaces.

THEOREM 3.3. A class \mathcal{X} of spaces is properly P_C^B -tame if and only if a properly P_C^B -tame class of spaces properly M -dominates it.

PROOF. Similar to the proof of theorem 3.1. \square

COROLLARY 3.4. A space is properly P_C^B -tame if and only if it Sh_p -dominates or quasi Sh_p -dominates a properly P_C^B -tame class of spaces.

THEOREM 3.5. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be classes of spaces. If a class \mathcal{X} of spaces is both properly N_A^B -tame and properly N_C^A -tame, then \mathcal{X} is also properly N_C^B -tame.

PROOF. Let a member B of \mathcal{B} and a numerable cover σ of B be given. Let $\mu \in \sigma^*$. Since X is properly N_A^B -tame, there is a $\tau \in Cov(B)$ such that for every $X \in \mathcal{X}$ and every proper τ -small multi-valued function $F : X \rightarrow B$ there is a $A \in \mathcal{A}$ and a proper μ -small multi-valued function $K : A \rightarrow B$ so that for every $\xi \in Cov(A)$ there is a proper γ -small multi-valued function $L : X \rightarrow A$ with $F \stackrel{\mu}{\simeq} L \circ K$. Then τ is the cover we have been looking for.

Indeed, let $X \in \mathcal{X}$ and let $F : X \rightarrow B$ be a proper τ -small multi-valued function. Pick an $A \in \mathcal{A}$ and a K as above. Let $\theta \in S(K, \mu)$. Now we utilize the fact that \mathcal{X} is also properly N_C^A -tame to select a numerable cover ξ of A such that for every $X \in \mathcal{X}$ and every proper ξ -small multi-valued function $L : X \rightarrow A$ there is a $C \in \mathcal{C}$ and a proper θ -small multi-valued function $M : C \rightarrow A$ so that for every $\gamma \in Cov(C)$ there is a proper γ -small multi-valued function $G : X \rightarrow C$ with $L \stackrel{\theta}{\simeq} M \circ G$. Choose an L and then a C and an M as above. Let H denotes the composition $K \circ M$. Then C and H are the required space and proper function.

In fact, let a $\gamma \in \mathcal{C}$ be given. We first choose a G as above. Our choices give us the following chain of relations $F \stackrel{\mu}{\simeq} L \circ K \stackrel{\theta}{\simeq} K \circ M \circ G = H \circ G$. Hence, $F \stackrel{\sigma}{\simeq} H \circ G$. \square

THEOREM 3.6. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be classes of spaces. If a class \mathcal{X} of spaces is properly P_A^B -tame and the class \mathcal{B} is properly M_A^C -tame, then \mathcal{X} is also properly P_C^B -tame.

PROOF. The assumption on \mathcal{X} can be expressed by saying that the class \mathcal{B} is properly M_X^A -tame. It follows from theorem 3.4 in [5] that \mathcal{B} is properly M_X^C -tame or that \mathcal{X} is properly P_C^B -tame. \square

THEOREM 3.7. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be classes of spaces such that \mathcal{B} and \mathcal{D} are properly M -dominated by \mathcal{A} and \mathcal{C} , respectively. If a class \mathcal{X} of spaces is properly $N_{\mathcal{D}}^{\mathcal{A}}$ -tame, then it is also properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame.

PROOF. The assumption that \mathcal{X} is properly $M_{\mathcal{D}}^{\mathcal{A}}$ -tame means that \mathcal{A} is properly $M_{\mathcal{D}}^{\mathcal{X}}$ -tame. Since \mathcal{A} properly M -dominates \mathcal{B} , it follows from theorem 3.1 in [5] that \mathcal{B} is properly $M_{\mathcal{D}}^{\mathcal{X}}$ -tame. But, since \mathcal{C} properly M -dominates \mathcal{D} , we get that \mathcal{B} is properly $M_{\mathcal{C}}^{\mathcal{X}}$ -tame and therefore that \mathcal{X} is properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame. \square

COROLLARY 3.8. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be classes of spaces such that \mathcal{B} and \mathcal{D} are quasi Sh_p -dominated or Ssh_p -dominated by \mathcal{A} and \mathcal{C} , respectively. If a space X is properly $N_{\mathcal{D}}^{\mathcal{A}}$ -tame, then it is also properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame.

THEOREM 3.9. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be classes of spaces such that \mathcal{B} and \mathcal{C} are properly M -dominated by \mathcal{A} and \mathcal{D} , respectively. If a class \mathcal{X} of spaces is properly $P_{\mathcal{D}}^{\mathcal{A}}$ -tame, then it is also properly $P_{\mathcal{C}}^{\mathcal{B}}$ -tame.

PROOF. See the proof of theorem 3.7. \square

COROLLARY 3.10. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and \mathcal{D} be classes of spaces such that \mathcal{B} and \mathcal{C} are quasi Sh_p -dominated or Ssh_p -dominated by \mathcal{A} and \mathcal{D} , respectively. If a space X is properly $P_{\mathcal{D}}^{\mathcal{A}}$ -tame, then it is also properly $P_{\mathcal{C}}^{\mathcal{B}}$ -tame.

There seems to be no analogue of theorem 3.7 in [5] for properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame and properly $P_{\mathcal{C}}^{\mathcal{B}}$ -tame spaces. In order to state versions of theorem 3.9 in [5] we need the following dual form of the notion of a proper $M^{\mathcal{B}}$ -surjection from [4].

Let \mathcal{B} be a class of topological spaces. A class \mathcal{F} of proper maps is *properly $N^{\mathcal{B}}$ -surjective* provided for every $B \in \mathcal{B}$ and every $\sigma \in Cov(B)$ there is a $\tau \in Cov(B)$ such that for every $f : X \rightarrow Y$ from \mathcal{F} and every proper τ -small multi-valued function $F : X \rightarrow B$ there is a proper σ -small multi-valued function $G : Y \rightarrow B$ with $F \stackrel{\sigma}{\simeq} G \circ f$. A proper map $f : X \rightarrow Y$ is a *proper $N^{\mathcal{B}}$ -surjection* provided the class $\{f\}$ is properly $N^{\mathcal{B}}$ -surjective.

THEOREM 3.11. If \mathcal{F} is a properly $N^{\mathcal{B}}$ -surjective class of proper maps and the class \mathcal{F}'' is properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame, then the class \mathcal{F}' is also a properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame.

PROOF. Let a member B of \mathcal{B} and a numerable cover σ of B be given. Let $\mu \in \sigma^*$. Since \mathcal{F}'' is properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame, there is a $\xi \in \mu^+$ such that for every $Y \in \mathcal{F}''$ and every proper ξ -small multi-valued function $K : Y \rightarrow B$ there is a $C \in \mathcal{C}$ and a proper μ -small multi-valued function $H : C \rightarrow B$ so that for every $\gamma \in Cov(C)$ there is a proper γ -small multi-valued function $L : Y \rightarrow C$ so that $K \stackrel{\mu}{\simeq} H \circ L$. Finally, we utilize the fact that \mathcal{F} is a properly $N^{\mathcal{B}}$ -surjective to select the required numerable cover τ of B so that for every member $f : X \rightarrow Y$ of \mathcal{F} and every proper τ -small multi-valued function $F : X \rightarrow B$ there is a proper ξ -small multi-valued function $K : Y \rightarrow B$ with $F \stackrel{\xi}{\simeq} K \circ f$.

Consider a proper τ -small multi-valued function $F : X \rightarrow B$ from a member X of \mathcal{F}' into B . Choose a proper map $f : X \rightarrow Y$ from \mathcal{F} , a K , and then a C and an H as above. Let $\gamma \in Cov(C)$. Pick an L and let G denote the composition of f and L . Notice that G is a proper γ -small multi-valued function. Then we obtain the following chain of relations. $F \stackrel{\xi}{\simeq} K \circ f \stackrel{\mu}{\simeq} H \circ L \circ f = H \circ G$. Hence, $F \stackrel{\sigma}{\simeq} H \circ G$ and \mathcal{F}' is properly \mathcal{BCN} -tame. \square

In a similar fashion one can prove the following result for properly $P_{\mathcal{C}}^{\mathcal{B}}$ -tame spaces.

THEOREM 3.12. If \mathcal{F} is a properly $N^{\mathcal{B}}$ -surjective class of proper maps and the class \mathcal{F}'' is properly $P_{\mathcal{C}}^{\mathcal{B}}$ -tame, then the class \mathcal{F}' is also a properly $P_{\mathcal{C}}^{\mathcal{B}}$ -tame.

THEOREM. 3.13. If \mathcal{F} is a class of properly right M -placid proper maps and the class \mathcal{F}'' is properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame, then the class \mathcal{F}' is also properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame.

PROOF. Let a member B of \mathcal{B} and a numerable cover σ of B be given. Let $\mu \in \sigma^*$. Since \mathcal{F}' is properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame, there is a $\tau \in \mu^+$ such that for every $X \in \mathcal{F}'$ and every proper τ -small multi-valued function $P : X \rightarrow B$ there is a $C \in \mathcal{C}$ and a proper μ -small multi-valued function $H : C \rightarrow B$ so that for every $\gamma \in Cov(C)$ there is a proper γ -small multi-valued function

$Q : X \rightarrow C$ and a proper μ -homotopy W joining P and $H \circ Q$. Then τ is the required cover.

Indeed, consider a member Y of \mathcal{F}'' and a proper τ -small multi-valued function $F : Y \rightarrow B$. Let $f : X \rightarrow Y$ be from the class \mathcal{F} . Let P denote the composition of f and F . Then P is a proper τ -small multi-valued function from X into B so that a $C \in \mathcal{C}$ and an H as above exist.

Let a $\gamma \in \text{Cov}(C)$ be given. Now, we pick the function Q and a proper μ -homotopy W . Let $\lambda \in S(F, \tau)$ and let θ be from the intersection of sets $D(W, \mu)$ and $S(Q, \gamma)$. Since f is properly right M -placid, there is a proper θ -small multi-valued function $J : Y \rightarrow X$ such that $f \circ J \stackrel{\lambda}{\simeq} id_Y$. Let G denote the composition of J and Q . Then we have the following relations. $F \stackrel{\tau}{\simeq} F \circ f \circ J \stackrel{\mu}{\simeq} H \circ Q \circ J = G \circ H$. Hence, $F \stackrel{\sigma}{\simeq} H \circ G$. \square

COROLLARY 3.14. A proper M -retract of a properly $N_C^{\mathcal{B}}$ -tame space is properly $N_C^{\mathcal{B}}$ -tame.

THEOREM 3.15. If \mathcal{F} is a class of properly right M -placid proper maps and the class \mathcal{F}'' is properly $N_C^{\mathcal{B}}$ -tame, then the class \mathcal{F}' is also properly $N_C^{\mathcal{B}}$ -tame.

PROOF. The proof is similar to the proof of theorem 3.13. \square

In the next result that corresponds to theorem 3.13 in [5] we shall use a notion of properly $N^{\mathcal{B}}$ -injective class of proper maps from [4] whose definition we now recall.

Let \mathcal{B} be a class of spaces. A class \mathcal{F} of proper maps is *properly $N^{\mathcal{B}}$ -injective* provided for every $B \in \mathcal{B}$ and every $\sigma \in \text{Cov}(B)$ there is a $\tau \in \text{Cov}(B)$ such that for every $f : X \rightarrow Y$ from \mathcal{F} and for every proper τ -small multi-valued functions $F, G : Y \rightarrow B$ the relation $F \circ f \stackrel{\tau}{\simeq} G \circ f$ implies the relation $F \stackrel{\sigma}{\simeq} G$. A proper map $f : X \rightarrow Y$ is a *proper $N^{\mathcal{B}}$ -injection* provided the class $\{f\}$ consisting of f alone is properly $N^{\mathcal{B}}$ -injective.

Also, a class of proper maps which is both properly $N^{\mathcal{B}}$ -injective and properly $N^{\mathcal{C}}$ -surjective is called *properly $N_C^{\mathcal{B}}$ -bijective*. We shall use a shorter name *properly $N^{\mathcal{B}}$ -bijective* for a properly $N_B^{\mathcal{B}}$ -bijective class of proper maps. A proper map f is a *proper $N_C^{\mathcal{B}}$ -bijection* provided the class $\{f\}$ is properly $N_C^{\mathcal{B}}$ -bijective. A *proper $N^{\mathcal{B}}$ -bijection* is defined analogously.

THEOREM 3.16. If \mathcal{F} is a properly $N^{\mathcal{B}}$ -injective class of proper $N^{\mathcal{C}}$ -surjections and the class \mathcal{F}' is properly $N_C^{\mathcal{B}}$ -tame, then the class \mathcal{F}'' is also properly $N_C^{\mathcal{B}}$ -tame.

PROOF. Let a member B of \mathcal{B} and a numerable cover σ of B be given. Since \mathcal{F} is properly $N^{\mathcal{B}}$ -injective, there is a $\varrho \in \text{Cov}(B)$ such that for every map $f : X \rightarrow Y$ from the class \mathcal{F} and all proper ϱ -small multi-valued functions $P, Q : Y \rightarrow B$ the relation $P \circ f \stackrel{\varrho}{\simeq} Q \circ f$ implies the relation $P \stackrel{\sigma}{\simeq} Q$. Let $\mu \in \varrho^*$. We now utilize the assumption that \mathcal{F}' is properly $N_C^{\mathcal{B}}$ -tame to select a $\tau \in \text{Cov}(B)$ such that for every member X of the class \mathcal{F}' and every proper τ -small multi-valued function $K : X \rightarrow B$ there is a $C \in \mathcal{C}$ and a proper μ -small multi-valued function $H : C \rightarrow B$ so that for every $\delta \in \text{Cov}(C)$ there is a proper δ -small multi-valued function $L : X \rightarrow C$ so that $K \stackrel{\mu}{\simeq} H \circ L$.

In order to check that τ is a cover we were looking for, let Y be from the class \mathcal{F}'' and let $F : Y \rightarrow B$ be a proper τ -small multi-valued function. Select a proper map $f : X \rightarrow Y$ from the class \mathcal{F} . Let K denote the composition of f and F . Notice that K is a proper τ -small multi-valued function from X into B . Pick a space C from the class \mathcal{C} and a proper μ -small multi-valued function H as above. Let a $\gamma \in \text{Cov}(C)$ be given. Let $\theta \in S(H, \mu)$ refines γ . Since f is a proper \mathcal{CN} -surjection, there is a $\delta \in \theta^+$ such that for every proper δ -small multi-valued function $L : X \rightarrow C$ there is a proper θ -small multi-valued function $G : Y \rightarrow C$ with $L \stackrel{\theta}{\simeq} G \circ f$. Pick an L and a G as above. Our choices imply $F \circ f = K \stackrel{\mu}{\simeq} H \circ L \stackrel{\mu}{\simeq} H \circ G \circ f$. It follows that $F \circ f \stackrel{\varrho}{\simeq} H \circ G \circ f$ and therefore $F \stackrel{\sigma}{\simeq} H \circ G$. \square

The situation with properly $P_C^{\mathcal{B}}$ -tame classes of spaces is much simpler as the following theorem shows. The proof of it is left to the reader.

THEOREM 3.17. If \mathcal{F} is a class of proper $M^{\mathcal{C}}$ -surjections and the class \mathcal{F}'' is properly $P_C^{\mathcal{B}}$ -tame, then the class \mathcal{F}' is also properly $P_C^{\mathcal{B}}$ -tame.

4. PROPERLY $N^{\mathcal{B}}$ -MOVABLE CLASSES.

In this section we shall do for properly $M^{\mathcal{C}}$ -movable spaces what we have done in §3 for properly $M_C^{\mathcal{B}}$ -tame spaces. In other words, we shall introduce a dual notion called proper

N^B -movability. It applies to classes of spaces and it satisfies six theorems which are analogues of results in [5, §5].

Let \mathcal{X} and \mathcal{B} be a classes of spaces. The class \mathcal{X} is called *properly N^B -movable* provided the class \mathcal{B} is properly $M^{\mathcal{X}}$ -movable, i. e., provided for every $B \in \mathcal{B}$ and every $\sigma \in Cov(B)$ there is a $\tau \in Cov(B)$ such that for every $X \in \mathcal{X}$, every $\varrho \in Cov(B)$, and every proper τ -small multi-valued function $F : X \rightarrow B$ there is a proper ϱ -small multi-valued function $G : X \rightarrow B$ with $F \stackrel{\sigma}{\simeq} G$. A space X is *properly N^B -movable* provided the class $\{X\}$ is properly N^B -movable.

The following two theorems are immediate consequences of 5.2 and 5.1 in [5], respectively.

THEOREM 4.1. If a class of spaces \mathcal{C} quasi Sh_p -dominates another such class \mathcal{B} and a class of spaces \mathcal{X} is properly N^C -movable, then the class \mathcal{X} is also properly N^B -movable.

THEOREM 4.2. A class of spaces \mathcal{X} will be properly N^B -movable if and only if it is properly M -dominated by a properly N^B -movable class of spaces \mathcal{Y} .

THEOREM 4.3. Let \mathcal{B} and \mathcal{C} be classes of topological spaces. If a class of spaces \mathcal{X} is at the same time properly N^C -tame and properly N^C -movable, then it is also properly N^B -movable.

PROOF. Let a member B of the class \mathcal{B} and a numerable cover σ of B be given. Let $\pi \in \sigma^*$. Since \mathcal{X} is properly N^C -movable, there is a numerable cover $\xi \in \pi^+$ such that for every proper ξ -small multi-valued function K from a member C of \mathcal{C} into B and every $\varrho \in Cov(X)$ there is a proper ϱ -small multi-valued function $L : C \rightarrow X$ with $K \stackrel{\pi}{\simeq} L$. Since \mathcal{X} is properly N^B -tame, there is a numerable cover τ of B such that for every proper τ -small multi-valued function F of a member X of \mathcal{X} into B we can find a $C \in \mathcal{C}$ and a proper ξ -small multi-valued function $R : C \rightarrow B$ with the property that for every $\alpha \in Cov(C)$ there is a proper α -small multi-valued function $P : X \rightarrow C$ so that $F \stackrel{\xi}{\simeq} R \circ P$. Then τ is the required numerable cover of B .

Indeed, let F be a proper τ -small multi-valued function from a member X of \mathcal{X} into B and let ϱ be a numerable cover of B . Choose a C and an R as above. Then we pick a proper ϱ -small multi-valued function K and a proper π -homotopy $L : C \times I \rightarrow B$ joining R and K . Let α be a numerable cover of C that satisfies $\alpha \in D(L, \pi)$ and $\alpha \in S(K, \varrho)$. With respect to α we now select a P as above and put $G = K \circ P$. Then G is a proper ϱ -small multi-valued function and we have $F \stackrel{\xi}{\simeq} R \circ P \stackrel{\pi}{\simeq} K \circ P = G$. Hence, $F \stackrel{\sigma}{\simeq} G$. \square

THEOREM 4.4. If a class of maps \mathcal{F} is properly N^B -surjective and the class \mathcal{F}'' is properly N^B -movable, then the class \mathcal{F}' is also properly N^B -movable.

PROOF. Let a member B of \mathcal{B} and a numerable cover σ of B be given. Let $\mu \in \sigma^*$. Since the class \mathcal{F}'' is properly N^B -movable, there is a numerable cover $\xi \in \mu^+$ of B such that for every proper ξ -small multi-valued function $K : Y \rightarrow B$ from a member Y of \mathcal{F}'' into B and every $\varrho \in Cov(B)$ there is a proper ϱ -small multi-valued function $L : Y \rightarrow B$ with $K \stackrel{\mu}{\simeq} L$. On the other hand, since \mathcal{F} is properly N^B -surjective, there is a $\tau \in Cov(B)$ such that for every proper map $f : X \rightarrow Y$ from the class \mathcal{F} and every proper τ -small multi-valued function $F : X \rightarrow B$ there is a proper ξ -small multi-valued function $K : Y \rightarrow B$ with $F \stackrel{\xi}{\simeq} K \circ f$. Then τ is the numerable cover of B we were looking for.

In fact, consider a proper τ -small multi-valued function $F : X \rightarrow B$ from a member X of \mathcal{F}' into B and a $\varrho \in Cov(B)$. Pick a proper map $f : X \rightarrow Y$ in \mathcal{F} and proper functions K and L as above. Let G denote the composition of f and L . Then G is a proper ϱ -small multi-valued function and we have $G = L \circ f \stackrel{\mu}{\simeq} K \circ f \stackrel{\xi}{\simeq} F$. Hence, $F \stackrel{\sigma}{\simeq} G$. \square

COROLLARY 4.5. The image of a properly N^B -movable space by a refinable proper map is properly N^B -movable space.

THEOREM 4.6. If \mathcal{F} is a class of properly right M -placid proper maps and the class \mathcal{F}' is properly N^B -movable, then the class \mathcal{F}'' is also properly N^B -movable.

PROOF. Let a member B of \mathcal{B} and a $\sigma \in Cov(B)$ be given. Let $\mu \in \sigma^*$. Since the class \mathcal{F}' is properly N^B -movable, there is a numerable cover $\tau \in \mu^+$ with the property that for every proper τ -small multi-valued function K from a member X of \mathcal{F}' into B and every $\varrho \in Cov(B)$ there is a proper ϱ -small multi-valued function $L : X \rightarrow B$ with $K \stackrel{\mu}{\simeq} L$. Then τ is the required numerable cover.

In fact, let a member Y of \mathcal{F}'' , a proper τ -small multi-valued function $F : Y \rightarrow B$, and a numerable cover ϱ of B be given. Let $f : X \rightarrow Y$ be from the class \mathcal{F} . We let K denote the composition $F \circ f$ and pick an L as above. Let W be a proper μ -homotopy joining K and L . Let $\theta \in \text{Cov}(X)$ be from the intersection of sets $S(L, \varrho)$ and $D(W, \mu)$. Let $\alpha \in S(F, \tau)$. Since the map f is properly right M -placid, there is a proper θ -small multi-valued function $J : Y \rightarrow X$ such that $\text{id}_Y \stackrel{\alpha}{\simeq} f \circ J$. Let G be the composition of J and L . The function G is proper and ϱ -small and we have relations $G = L \circ J \stackrel{\mu}{\simeq} K \circ J = F \circ f \circ J \stackrel{\tau}{\simeq} F$. Hence, $F \stackrel{\sigma}{\simeq} G$. \square

THEOREM 4.7. If \mathcal{F} is a properly $N^{\mathcal{B}}$ -injective class of proper $N^{\mathcal{B}}$ -surjections and \mathcal{F}' is properly $N^{\mathcal{B}}$ -movable, then \mathcal{F}'' is also properly $N^{\mathcal{B}}$ -movable.

PROOF. Let a member B of \mathcal{B} and a numerable cover σ of B be given. Since the class \mathcal{F} is properly $N^{\mathcal{B}}$ -injective, there is a $\mu \in \text{Cov}(B)$ such that for every proper map $f : X \rightarrow Y$ from \mathcal{F} and all proper μ -small multi-valued functions $F, G : Y \rightarrow B$ the relation $F \circ f \stackrel{\xi}{\simeq} G \circ f$ implies the relation $F \stackrel{\sigma}{\simeq} G$. Let $\lambda \in \mu^*$. Since the class \mathcal{F}' is properly $N^{\mathcal{B}}$ -movable, there is a $\tau \in \text{Cov}(B)$ such that for every member X of \mathcal{F}' , every proper τ -small multi-valued function $P : X \rightarrow B$ and every $\pi \in \text{Cov}(B)$ there is a proper π -small multi-valued function $Q : X \rightarrow B$ with $P \stackrel{\lambda}{\simeq} Q$. Then τ is the required numerable cover of B .

In order to verify this, consider a member Y of \mathcal{F}'' , a proper τ -small multi-valued function $F : Y \rightarrow B$, and a numerable cover $\varrho \in \text{Cov}(B)$. We can assume that ϱ refines λ . Let $f : X \rightarrow Y$ be from \mathcal{F} . Since f is a proper $N^{\mathcal{B}}$ -surjection, there is a $\pi \in \text{Cov}(B)$ such that for every proper π -small multi-valued function $Q : X \rightarrow B$ there is a proper ϱ -small multi-valued function $G : Y \rightarrow B$ with $Q \stackrel{\delta}{\simeq} G \circ f$. Let P denote the composition of f and F . Observe that P is a proper τ -small multi-valued function from X into B . Pick a Q and a G as above. Our selections imply the following relations : $F \circ f = P \stackrel{\lambda}{\simeq} Q \stackrel{\delta}{\simeq} G \circ f$. It follows that $F \circ f \stackrel{\xi}{\simeq} G \circ f$ and therefore $F \stackrel{\sigma}{\simeq} G$. \square

5. COVERED AND EXTENDED CLASSES.

In this section we shall explore dependence of all proper shape invariants which were defined on classes of spaces involved under the assumption that these classes are connected by either surjections or injections. The connection can be through one of the following two notions.

Let \mathcal{F} be a class of proper maps and let \mathcal{B} and \mathcal{C} be classes of spaces. We shall say that the class \mathcal{C} is \mathcal{F} -covered by \mathcal{B} provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and an $h : B \rightarrow C$ from \mathcal{F} . Similarly, the class \mathcal{C} is \mathcal{F} -extended by \mathcal{B} provided for every $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ and a $k : C \rightarrow B$ from \mathcal{F} .

For a class of spaces \mathcal{B} we shall use $\mathcal{B}_i, \mathcal{B}_s,$ and \mathcal{B}_b to denote the classes of all proper $M^{\mathcal{B}}$ -injections, proper $M^{\mathcal{B}}$ -surjections, and proper $M^{\mathcal{B}}$ -bijections. Also, $\mathcal{B}^i, \mathcal{B}^s,$ and \mathcal{B}^b denote the classes of all proper $N^{\mathcal{B}}$ -injections, proper $N^{\mathcal{B}}$ -surjections, and proper $N^{\mathcal{B}}$ -bijections. Moreover, if \mathcal{F} and \mathcal{G} are classes of maps we let $\mathcal{F}\mathcal{G}$ denote the intersection $\mathcal{F} \cap \mathcal{G}$.

We begin with the result on properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame spaces and continue to cover all our proper shape invariants. The proofs are mostly omitted.

THEOREM 5.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D},$ and \mathcal{X} be classes of spaces such that \mathcal{X} is properly $N_{\mathcal{D}}^{\mathcal{A}}$ -tame and either

- (cc) \mathcal{B} is \mathcal{X}_s -covered by \mathcal{A} and \mathcal{D} is \mathcal{X}_s -covered by \mathcal{C} ,
- (ce) \mathcal{B} is \mathcal{X}_s -covered by \mathcal{A} and \mathcal{D} is \mathcal{A}^s -extended by \mathcal{C} ,
- (ec) \mathcal{B} is $\mathcal{C}_s\mathcal{X}_i$ -extended by \mathcal{A} and \mathcal{D} is \mathcal{X}_s -covered by \mathcal{C} , or
- (ee) \mathcal{B} is $\mathcal{C}_s\mathcal{X}_i$ -extended by \mathcal{A} and \mathcal{D} is \mathcal{A}^s -extended by \mathcal{C} ,

then \mathcal{X} is also properly $N_{\mathcal{C}}^{\mathcal{B}}$ -tame.

PROOF. (cc). Let a member B of \mathcal{B} and a numerable cover σ of B be given. Let $\mu \in \sigma^*$. Since the class \mathcal{B} is \mathcal{X}_s -covered by the class \mathcal{A} , there is an $A \in \mathcal{A}$ and a proper $M^{\mathcal{X}}$ -surjection $h : A \rightarrow B$. Let $\delta = h^{-1}(\mu)$. We now utilize the assumption that the class \mathcal{X} is properly $N_{\mathcal{D}}^{\mathcal{A}}$ -tame to select a numerable cover ε with the property that for every proper ε -small multi-valued function P from a member X of \mathcal{X} into A there is a $D \in \mathcal{D}$ and a proper δ -small

multi-valued function $K : D \rightarrow A$ so that for every $\eta \in \text{Cov}(D)$ there is a proper η -small multi-valued function $L : X \rightarrow D$ with $P \stackrel{\delta}{\simeq} K \circ L$. Since h is a proper $M^{\mathcal{X}}$ -surjection, there is a $\tau \in \text{Cov}(B)$ such that for every $X \in \mathcal{X}$ and every proper τ -small multi-valued function $F : X \rightarrow B$ there is a proper ε -small multi-valued function $P : X \rightarrow A$ with $F \stackrel{\mu}{\simeq} h \circ P$. Then τ is a numerable cover we were looking for.

Indeed, let $X \in \mathcal{X}$ and let $F : X \rightarrow B$ be a proper τ -small multi-valued function. Pick a P and then a D and a K as above. Since the class \mathcal{D} is \mathcal{X}_s -covered by the class \mathcal{C} , there is a $C \in \mathcal{C}$ and a proper $M^{\mathcal{X}}$ -surjection $k : C \rightarrow D$. Let $H = h \circ K \circ k$.

Let γ be a numerable cover of C . Let $\theta \in S(K, \delta)$. We use now the fact that k is a proper $M^{\mathcal{X}}$ -surjection to choose an $\eta \in \text{Cov}(D)$ with the property that for every $X \in \mathcal{X}$ and every proper η -small multi-valued function $K : X \rightarrow D$ there is a proper γ -small multi-valued function $G : X \rightarrow C$ with $L \stackrel{\theta}{\simeq} k \circ G$. Pick an L and then a G as above. Our selections imply the following chain of relations: $F \stackrel{\mu}{\simeq} h \circ P \stackrel{\mu}{\simeq} h \circ K \circ L \stackrel{\mu}{\simeq} h \circ K \circ k \circ G = H \circ G$. Hence, $F \stackrel{\sigma}{\simeq} H \circ G$. \square

THEOREM 5.2. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, and \mathcal{X} be classes of spaces such that \mathcal{X} is properly $P_{\mathcal{D}}^{\mathcal{A}}$ -tame and either

- (cc) \mathcal{B} is \mathcal{D}_s -covered by \mathcal{A} and \mathcal{C} is $\mathcal{B}_s \mathcal{X}^s$ -covered by \mathcal{D} ,
- (ce) \mathcal{B} is \mathcal{D}_s -covered by \mathcal{A} and \mathcal{C} is \mathcal{B}^s -extended by \mathcal{D} ,
- (ec) \mathcal{B} is $\mathcal{D}_s \mathcal{X}_s$ -extended by \mathcal{A} and \mathcal{D} is $\mathcal{B}^s \mathcal{X}^s$ -covered by \mathcal{D} , or
- (ee) \mathcal{B} is $\mathcal{C}_s \mathcal{D}_s$ -extended by \mathcal{A} and \mathcal{C} is \mathcal{B}^s -extended by \mathcal{D} ,

then \mathcal{X} is also properly $P_{\mathcal{C}}^{\mathcal{B}}$ -tame.

THEOREM 5.3. Let \mathcal{X}, \mathcal{B} , and \mathcal{C} be classes of spaces. If the class \mathcal{X} is properly $N^{\mathcal{B}}$ -movable and the class \mathcal{C} is either \mathcal{X}_s -covered or \mathcal{X}_b -extended by \mathcal{B} , then \mathcal{X} is also properly $N^{\mathcal{C}}$ -movable.

PROOF. (\mathcal{C} is \mathcal{X}_s -covered by \mathcal{B}). Let a member C of \mathcal{C} and a numerable cover γ of C be given. Let $\theta \in \gamma^*$. Since the class \mathcal{C} is \mathcal{X}_s -covered by the class \mathcal{B} , there is a $B \in \mathcal{B}$ and a proper $M^{\mathcal{X}}$ -surjection $h : B \rightarrow C$. Let $\sigma = h^{-1}(\theta)$. Since \mathcal{X} is properly $N^{\mathcal{B}}$ -movable, there is a numerable cover $\tau \in \text{Cov}(B)$ such that for every proper τ -small multi-valued function K from a member of X into B and every $\rho \in \text{Cov}(B)$ there is a proper ρ -small multi-valued function L with $K \stackrel{\sigma}{\simeq} L$. Since h is a proper $M^{\mathcal{X}}$ -surjection, there is a $\delta \in \text{Cov}(C)$ such that for every proper δ -small multi-valued function F from a member of \mathcal{X} into C there is a proper τ -small multi-valued function K with $F \stackrel{\theta}{\simeq} h \circ K$. Then δ is the required numerable cover of C .

In fact, let ε be an arbitrary numerable cover of C , let $X \in \mathcal{X}$, and let $F : X \rightarrow C$ be a proper δ -small multi-valued function. Let $\rho = h^{-1}(\varepsilon)$. Pick a K and an L as above and let G denote the composition $h \circ L$.

Then G is a proper ε -small multi-valued function and the relation $K \stackrel{\sigma}{\simeq} L$ implies $F \stackrel{\theta}{\simeq} h \circ K \stackrel{\theta}{\simeq} h \circ L = G$. Hence, $F \stackrel{\gamma}{\simeq} G$. \square

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