NEW CHARACTERIZATIONS FOR HANKEL TRANSFORMABLE SPACES OF ZEMANIAN

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ABSTRACT. In this paper we obtain new characterizations of the Zemanian spaces $H_{\mu}$ and $H'_{\mu}$.

KEY WORDS AND PHRASES. Hankel transform, distribution, Zemanian spaces

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A H Zemanian [7, Ch 5] introduced the space $H_{\mu} \quad (\mu \in \mathbb{R})$ of functions as follows a complex valued smooth function $\phi(x) \quad x \in I = (0, \infty)$, is in $H_{\mu}$ if, and only if, the quantity

$$\gamma_{n,k}^{\mu}(\phi) = \sup_{x \in I} |x^n(x^{-1}D)^k(x^{-\mu-1/2}\phi(x))| < \infty$$

is finite, for every $n, k \in \mathbb{N}$. This space endowed the topology generated by $\{\gamma_{n,k}^{\mu}\}_{n,k \in \mathbb{N}}$ is a Fréchet space. In the sequel we will refer to the above topology as the usual topology of $H_{\mu}$. Zemanian introduced the space $H'_{\mu}$ to extend the Hankel integral transformation defined by

$$h_{\mu}\phi(x) = \int_{0}^{\infty} (xt)^{1/2} J_{\mu}(xt) \phi(t) \, dt$$

where $J_{\mu}$ denotes the Bessel function of the first kind and order $\mu$, to generalized functions. He proved that $h_{\mu}$ is an automorphism of $H_{\mu}$ provided that $\mu > -\frac{1}{2}$. The generalized Hankel transform of $f \in H'_{\mu}$, the dual space of $H_{\mu}$, is defined as the transposed of $H'_{\mu}$ through

$$\langle h_{\mu}f, \phi \rangle = \int_{0}^{\infty} (xt)^{1/2} J_{\mu}(xt) \phi(t) \, dt$$

Thus if $\mu \geq -\frac{1}{2}$, $h_{\mu}$ is an automorphism of $H'_{\mu}$ when this space is equipped with the weak* topology or with the strong topology.

In [2] J. J. Betancor and I. Marrero have studied the main topological properties of the spaces $H_{\mu}$ and $H'_{\mu}$. Amongst other results, it is established (Theorem 3.3) that the space $H_{\mu}, \mu \geq -\frac{1}{2}$, is constituted by all those complex valued smooth functions $\phi(x) \quad x \in I$, such that

$$\tau_{n,k}^{\mu}(\phi) = \sup_{x \in I} |x^n N_{\mu+k-1}N_{\mu+\ldots}N_{\mu}(x)| < \infty$$

for every $n, k \in \mathbb{N}$. Moreover, the system of seminorms $\{\tau_{n,k}^{\mu}\}_{n,k \in \mathbb{N}}$ generates of $H_{\mu}$ its usual topology. Moreover in [4] they gave new descriptions for the usual topology of $H_{\mu}$ through $L_2$-norms

A. H. Zemanian [7, p. 134] defined the space $O$ formed by all those complex valued smooth functions $\nu(x) \quad x \in I$, satisfying that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $(1 + x^2)^{n_k}(x^{-1}D)^k\nu(x)$ is a bounded function on $I$. He proved that $O$ is a space of multiplier of $H_{\mu}$. Recently J. J. Betancor and I. Marrero [2, Theorems 2.3 and 4.9] have characterized $O$ as the space of multipliers of $H_{\mu}$ and $H'_{\mu}$.

In this paper we characterize the smooth complex valued functions in $H_{\mu}, \mu \geq -\frac{1}{2}$, as the ones satisfying

$$Z_{n}(\phi) = \sup_{x \in I} |x^n \phi(x)| < \infty \quad (1)$$
and
\[ y_n^\mu(\phi) = \sup_{x \in I} |N_{\mu+n-1}^\mu \phi(x)| < \infty \]  \hspace{1cm} (2)

for every \( n \in \mathbb{N} \). Moreover we prove that the usual topology of \( H_\mu \) can be defined by the family of seminorms \( \{ Z_n, y_n^\mu \}_{n \in \mathbb{N}} \) and a new characterization for the elements of \( H'_\mu \) is obtained. In the sequel we will assume that \( \mu \geq -\frac{1}{2} \).

**Proposition 1.** A complex valued smooth function \( \phi(x), \ x \in I \), is in \( H_\mu \) if, and only if, \( \phi \) satisfies (1) and (2) for every \( n \in \mathbb{N} \).

**Proof.** It is clear that if \( \phi \in H_\mu \) then \( \phi \) satisfies (1) and (2) for every \( n \in \mathbb{N} \).

Let now \( \phi \) be a complex valued smooth function defined on \( I \). To see that (1) and (2) \((n \in \mathbb{N})\) are sufficient conditions for \( \phi \) belongs to \( H_\mu \) we proceed by induction. Suppose, as induction hypothesis, that
\[ \sup_{x \in I} |z^n N_{\mu+n-1}^\mu \phi(x)| < \infty, \quad m \in \mathbb{N} \quad \text{and} \quad n \in \mathbb{N}, \quad 0 \leq n < \ell \]
for certain \( \ell \in \mathbb{N} \), \( \ell \geq 1 \).

By using partial integration we can obtain
\[ \|x^m N_{\mu+\ell-1}^\mu \phi(x)\|_2^2 = \int_0^{\infty} |x^m N_{\mu+\ell-1}^\mu \phi(x)|^2 \, dx \]
\[ = \int_0^{\infty} x^{2m} N_{\mu+\ell-1}^\mu (\phi(x)) N_{\mu+\ell-1}^\mu (\bar{\phi}(x)) \, dx \]
\[ = \int_0^{\infty} (Dx^{-1})^\ell (x^{2m+\ell+1/2} N_{\mu+\ell-1}^\mu (\phi(x))) x^{-m-1/2} \bar{\phi}(x) \, dx \]
for every \( m \in \mathbb{N} \), \( \ell < 2m + 2 \), because
\[ \left[ \left( D^{-1/2} (x^{2m+\ell+1/2} N_{\mu+\ell-1}^\mu (\phi(x))) (x^{-1} D)^{\ell-1} (x^{-m-1/2} \bar{\phi}(x)) \right)_0^{\infty} = 0 \]  \hspace{1cm} (3)
for each \( i, m \in \mathbb{N} \), \( 0 \leq i < \ell < 2m + 2 \). In effect, if \( m, i \in \mathbb{N} \), \( 0 \leq i < \ell < 2m + 2 \) then Leibniz's rule leads to
\[ (Dx^{-1})^\ell (x^{2m+\ell+1/2} N_{\mu+\ell-1}^\mu (\phi(x))) (x^{-1} D)^{\ell-i} (x^{-m-1/2} \bar{\phi}(x)) \]
\[ = \sum_{j=0}^{\ell} a_j x^{2m+\ell+1/2-j} (x^{-1} D)^{\ell-i} (x^{-m-1/2} \bar{\phi}(x)) (x^{2m+\ell+1/2-j} N_{\mu+\ell-i-1}^\mu \phi(x)) \]
where \( a_j, \ j \in \mathbb{N} \), \( 0 \leq j \leq i \), are suitable real numbers, and by virtue of induction hypothesis (3) follows.

Most straightforward manipulations allow us to write
\[ (Dx^{-1})^\ell (x^{2m+\ell+1/2} N_{\mu+\ell-1}^\mu (\phi(x))) x^{-m-1/2} \bar{\phi}(x) = \sum_{j=0}^{\ell} a_j x^{2m-j} \bar{\phi}(x) N_{\mu+2\ell-j-1}^\mu \phi(x) \]
with \( m \in \mathbb{N} \) and \( a_j \in \mathbb{R} \), \( j \in \mathbb{N} \), \( 0 \leq j \leq \ell \).

Hence we can establish
\[ \|x^m N_{\mu+\ell-1}^\mu \phi(x)\|_2^2 \leq C_1 \sum_{j=0}^{\ell} \int_0^{\infty} |x^{2m-j} \bar{\phi}(x)| |N_{\mu+2\ell-j-1}^\mu \phi(x)| \, dx \]
\[ \leq C_2 \sum_{j=0}^{\ell} \sup_{x \in I} |(1 + x^2)^{2m-j} \phi(x)| \sup_{x \in I} |N_{\mu+2\ell-j-1}^\mu \phi(x)| < \infty, \]  \hspace{1cm} (4)
provided that \( m \in \mathbb{N}, \ 2m \geq \ell \). Here \( C_i, \ i = 1, 2 \), denotes suitable positive constants.

Assume now that \( m \in \mathbb{N}, \ 2m < \ell \). We have

\[
\|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2^2 = \left( \int_0^1 + \int_1^{\infty} \right) |x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)|^2 \, dx
\]

\[
\leq \int_0^1 |N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)|^2 \, dx + \int_{\ell}^{\infty} |x^\ell N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)|^2 \, dx .
\]

Therefore, by invoking (4) and the induction hypothesis we infer that

\[
\|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2 < \infty , \quad \text{when} \quad m \in \mathbb{N}, \ 2m < \ell .
\]

Thus it is concluded that \( \|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2 < \infty , \ m \in \mathbb{N} \).

Also, for every \( m \in \mathbb{N}, \ m \geq 1, \) and \( x \in I, \)

\[
(x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x))^2 = \int_0^x D_t (t^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(t))^2 \, dt
\]

\[
= \int_0^x 2t^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(t) \left( |m + \mu + \frac{1}{2} + \ell t^{m-1} N_{\mu+\ell-1} \cdots N_{\mu} \phi(t) + t^m N_{\mu+\ell} \cdots N_{\mu} \phi(t)) \right) \, dt .
\]

Hence if \( m \in \mathbb{N}, \ m \geq 1, \) and \( x \in I \) by using Holder’s inequality we can find \( C \geq 0 \) such that

\[
|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)|^2 \leq C \left( \|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2 \|x^{m-1} N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2 \right)
\]

\[
+ \sup_{x \in I} |N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)| \left( \|x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2 + \|x^{m+1} N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)\|_2 \right)
\]

and then \( \sup_{x \in I} |x^m N_{\mu+\ell-1} \cdots N_{\mu} \phi(x)| < \infty, \ m \in \mathbb{N} . \)

Thus the proof is finished.

The last proposition allows us to define the usual topology of \( H_\mu \) through a family of seminorms simpler than \( \{ \gamma_{m,k}^\mu \}_{m,k \in \mathbb{N}} \).

**PROPOSITION 2.** The usual topology of \( H_\mu \) is defined by the system of seminorms \( \{ Z_n, y_n^\mu \}_{n \in \mathbb{N}} \).

**PROOF.** It is clear that the topology generated by \( \{ \gamma_{m,k}^\mu \}_{m,k \in \mathbb{N}} \) is finer than the one defined by \( \{ Z_n, y_n^\mu \}_{n \in \mathbb{N}} \) on \( H_\mu \). Moreover by proceeding in a way similar to A. H. Zemanian [7, Lemma 5.2-2] we can prove that \( H_\mu \) endowed with the topology generated by \( \{ Z_n, y_n^\mu \}_{n \in \mathbb{N}} \) is a Fréchet space. Hence the desired result is an immediate consequence of the Open Mapping Theorem [6, Corollary 2.12].

We now prove a new characterization for the elements of \( H_\mu' \), the dual space of \( H_\mu \). The procedure employed is analogous to the one used by the author [1] and by J. J. Betancor and I. Marrero [2].

**PROPOSITION 3.** Let \( f \) be a linear functional defined on \( H_\mu \). Then \( f \) is in \( H_\mu' \) if, and only if, there exist \( r \in \mathbb{N} \) and \( f_k, g_k \in L_\infty(0,\infty) \) (the space of essentially bounded functions on \((0,\infty)\)), \( k \in \mathbb{N}, \ 0 \leq k \leq r \), such that

\[
f = \sum_{k=0}^r h_k^\mu (x^k f_k + x^{-\mu+1/2} (x^{-1} D)^k x^{k+\mu-1/2} g_k) . \quad (5)
\]

**PROOF.** Let \( f \in H_\mu' \). By virtue of a well-known result ([7, Theorem 1.8-1]) there exist \( r \in \mathbb{N} \) and \( C > 0 \) such that

\[
|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq r} \{ Z_k(\phi), y_k^\mu(\phi) \} , \quad \phi \in H_\mu . \quad (6)
\]

According to [7, Lemma 5.4-1(2), (3) and Theorem 5.4-1] and since \( z^{1/2} J_\mu(z) \) is a bounded function on \( I \) for every \( k \in \mathbb{N} \) one has
\[
\sup_{x \in I} |x^k \phi(x)| = \sup_{x \in I} |x^k h_{\mu}(h_{\mu}\phi)(x)| \leq C \int_0^\infty |N_{\mu+k-1} \ldots N_{\mu}(h_{\mu}\phi)(t)| \, dt
\] (7)
and
\[
\sup_{x \in I} |N_{\mu+k-1} \ldots N_{\mu}\phi(x)| = \sup_{x \in I} |N_{\mu+k-1} \ldots N_{\mu} h_{\mu}(h_{\mu}\phi)(x)| \leq C \int_0^\infty |t^k (h_{\mu}\phi)(t)| \, dt
\] (8)
for a suitable \( C > 0 \).

The linear mapping
\[ j : H_{\mu} \to JH_{\mu} \subset L_1(0, \infty)^{2r+2} \]
\[ \phi \to (x^k h_{\mu}\phi, N_{\mu+k-1} \ldots N_{\mu} h_{\mu}\phi)_{k=0} \]

is one to one because \( h_{\mu} \) is an automorphism of \( H_{\mu} \) ([7, Theorem 5.4-1]). Here \( L_1(0, \infty) \) denotes the usual Lebesgue space of order 1.

On the other hand, the inequalities (6), (7) and (8) imply that the linear mapping
\[ L : JH_{\mu} \subset L_1(0, \infty)^{2r+2} \to \mathbb{C} \]
\[ (x^k h_{\mu}\phi, N_{\mu+k-1} \ldots N_{\mu} h_{\mu}\phi)_{k=0} \to \langle f, \phi \rangle \]
is continuous when \( JH_{\mu} \) is endowed with the topology induced by \( L_1(0, \infty)^{2r+2} \). Hence, by invoking the Hahn-Banach Theorem \( L \) can be extended to \( L_1(0, \infty)^{2r+2} \) as a member of \( (L_1(0, \infty)^{2r+2})' \), the dual space of \( L_1(0, \infty)^{2r+2} \). Since, as it is well known, \( L_1(0, \infty)' = L_\infty(0, \infty) \) there exist \( f_k, g_k \in L_\infty(0, \infty), \ k \in \mathbb{N}, \ 0 \leq k \leq r, \) such that
\[ \langle f, \phi \rangle = \sum_{k=0}^r \langle f_k, x^k h_{\mu}\phi \rangle + \langle g_k, x^{k+1/2} (x^{-1} D)^k (x^{-1/2} \phi) \rangle, \quad \phi \in H_{\mu}. \]
Therefore
\[ f = \sum_{k=0}^r h_{\mu}' (x^k f_k + (-1)^k x^{-1/2} (x^{-1} D)^k x^{k+1/2} g_k). \]

Thus the proof of necessity is finished.

Conversely, if \( f \) is a linear functional defined on \( H_{\mu} \) by (5) for certain \( r \in \mathbb{N} \) and \( f_k, g_k \in L_\infty(0, \infty), \ k \in \mathbb{N}, \ 0 \leq k \leq r, \) then
\[ |\langle f, \phi \rangle| \leq C \sum_{k=0}^r \left( \|f_k\|_{\infty} \sup_{x \in I} |(1+x^2) x^k h_{\mu}(h_{\mu}\phi)(x)| + \|g_k\|_{\infty} \sup_{x \in I} |(1+x^2) N_{\mu+k-1} \ldots N_{\mu}(h_{\mu}\phi)(x)| \right) \]
for \( \phi \in H_{\mu} \), where \( \|\cdot\|_{\infty} \) denotes the usual norm in \( L_\infty(0, \infty) \). Hence, according to [7, Theorem 5.4-1] and [2, Theorem 3.3], \( f \) is in \( H_{\mu}' \).

REFERENCES

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