

## ON LOCALLY $s$ -CLOSED SPACES

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**ABSTRACT.** In the present paper, the concepts of  $s$ -closed sub-spaces, locally  $s$ -closed spaces have been introduced and characterized. We have seen that local  $s$ -closedness is a semi-regular property; the concept of  $s$ - $\theta$ -closed mapping has been introduced here and the following important properties are established :-

Let  $f : X \rightarrow Y$  be an  $s$ - $\theta$ -closed surjection with  $s$ -set (Maio and Noiri [8]) point inverses. Then :

- (a) If  $f$  is completely continuous (Arya and Gupta [1]) and  $Y$  is a locally compact  $T_2$ -space, then,  $X$  is locally  $s$ -closed.
- (b) If  $f$  is  $\gamma$ -continuous (Ganguly and Basu [5]) and  $X$  is a locally compact  $T_2$ -space, then,  $Y$  is locally  $s$ -closed.

**KEY WORDS AND PHRASES.**  $s$ -closed subspace,  $s$ -set, locally  $s$ -closed,  $s$ - $\theta$ -closed mapping,  $\gamma$ -continuous and completely continuous mapping, regular open set,  $s$ - $\theta$ -open set, local compactness.

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1. INTRODUCTION.  $S$ -closed spaces (Thompson [14]) and  $s$ -closed (Maio and Noiri [8]) spaces originated from almost compact spaces by the use of semi-open sets as introduced by Levine [7]. Ganster and Reilly [6] had shown, towards the distinction between these concepts, that every infinite topological space can be embedded as a closed connected subspace of an  $S$ -closed space which is not an  $s$ -closed space. Noiri [13] further generalized  $S$ -closed spaces to locally  $S$ -closed spaces. In this paper we generalize  $s$ -closed spaces to locally  $s$ -closed spaces and study  $s$ -closed subspaces. Certain important characterizations and properties of locally  $s$ -closed spaces have also been established.  $s$ - $\theta$ -closed mapping is introduced and characterized and we have seen, under certain conditions on the domain and co-domain spaces, that an  $s$ - $\theta$ -closed mapping would be a continuous mapping. Completely continuous and  $\gamma$ -continuous mappings were introduced respectively by Arya and Gupta [1] and Ganguly and Basu [5]; by the help of these mappings we have been able to establish certain properties which correlate locally compact  $T_2$ -spaces with locally  $s$ -closed spaces.

Throughout the present paper, by  $(X, T)$  or simply by  $X$  we shall mean a topological space. A subset  $A$  of a topological space is said to be regular open (resp. regular closed) if  $\text{int}(\text{cl}(A))=A$  (resp.  $\text{cl}(\text{int}(A))=A$ ), where  $\text{cl}(A)$  (resp.  $\text{int}(A)$ ) denotes the closure (resp. interior) of  $A$ . A subset  $A$  of a space  $X$  is said to be semi-open [7] if there exists an open set  $O$  such that  $O \subset A \subset \text{cl}(O)$ . The complement of a semi-open set is called semi-closed (Crossley and Hildebrand [3]). The semi-closure of a subset  $A$  of a space, denoted by  $\text{scl}A$ , is the intersection of all semi-closed sets containing  $A$  (Crossley and Hildebrand [3]). A set  $A$  which is both semi-open as well as semi-closed is called a semi-regular set (Maio and Noiri [8]). The collection of all semi-open (resp. semi-regular, regular open) sets containing a point  $x$  of  $X$  will be denoted by  $\text{SO}(x)$  (resp.  $\text{SR}(x)$ ,  $\text{RO}(x)$ ) and for the whole space  $X$  these will be denoted by  $\text{SO}(X)$  (resp.  $\text{SR}(X)$ ,  $\text{RO}(X)$ ). A point  $x$  of  $X$  is said to be s- $\theta$ -cluster [8] point of a subset  $A$  of  $X$  if for every  $U \in \text{SO}(x)$ ,  $\text{scl}U \cap A \neq \emptyset$ . Since, for a semi-open set  $U$ ,  $\text{scl}U$  is a semi-regular set [8], a point  $x$  of  $X$  is said to be an s- $\theta$ -cluster point of  $A$  iff  $R \cap A \neq \emptyset$ , for all  $R \in \text{SR}(x)$ . The collection of all s- $\theta$ -cluster points of  $A$  will be denoted by s- $\theta$ -cl $A$  ( $[A]_{\text{s-}\theta}$ , for short). A set  $A$  is s- $\theta$ -closed if  $A = [A]_{\text{s-}\theta}$ . A complement of an s- $\theta$ -closed set is called an s- $\theta$ -open set. For a space  $(X, T)$ ,  $\text{RO}(X, T)$  is a base for a topology  $T_S$  on  $X$  coarser than  $T$  and  $(X, T_S)$  is called the semi-regularization space of  $(X, T)$ . A topological property  $P$  is said to be semi-regular property if whenever a space  $(X, T)$  possesses that property  $P$  so does its semi-regularization space  $(X, T_S)$ . A subset  $A$  of  $X$  is s-closed [8] (resp. S-closed (Noiri [11])) relative to  $X$  or simply an s-set (resp. S-set) if every cover  $\mathcal{U}$  of  $A$  by sets of  $\text{SO}(X)$  admits a finite subfamily  $\mathcal{U}_0$  such that  $A \subset \bigcup_{U \in \mathcal{U}_0} \text{scl}U$  (resp.  $A \subset \bigcup_{U \in \mathcal{U}_0} \text{cl}U$ ). In case  $A = X$  and  $A$  is an s-set (resp. S-set), then  $X$  is called s-closed [8] (resp. S-closed [14]). A subset  $A$  is called Nearly compact (NC-set (Carnahan [2]), for short) if every cover  $\mathcal{U}$  of  $A$  by means of open sets of  $X$  has a finite subfamily  $U_1, \dots, U_n$  (say) such that  $A \subset \bigcup_{i=1}^n \text{intcl}U_i$ . Clearly every s-set (resp. compact) set, is an NC-set, but not conversely. A subset  $A$  of a space  $X$  is said to be an  $\alpha$ -set (Noiri [10]) if  $A \subset \text{int}(\text{cl}(\text{int}(A)))$ .

2. s-CLOSED SUBSPACES. At the very outset, an example is given to assert that, every set, s-closed relative to  $X$ , is not necessarily an s-closed subspace of  $X$ .

EXAMPLE 1. Every countable set in an uncountable set  $X$  with co-countable topology  $T$  is s-closed relative to  $(X, T)$ , but is not even an S-closed subspace.

DEFINITION 1. A subset  $A$  of  $X$  is said to be pre-open (Mashour et al. [9]) if  $A \subset \text{intcl}A$ . This collection includes all open sets and, even more, all  $\alpha$ -open sets.

LEMMA 1. (See Dorsett [4]) Let  $(X, T)$  be a topological space and let  $A$  be pre-open set in  $(X, T)$ , then  $\text{SR}(A, T_A) = \text{SR}(X, T) \cap A$ , where  $T_A$  is the subspace topology on  $A$ .

THEOREM 1. A pre-open set  $A$  of  $X$  is s-closed as a subspace iff it is s-closed relative to  $X$ .

PROOF. Let  $A$  be s-closed relative to  $X$  and also let  $\{V_\alpha : \alpha \in I\}$  be a cover of  $A$  by semi-regular sets of the subspace  $A$ . Then by Lemma 1, there exists a semi-regular set  $U_\alpha$  in  $X$ , for each  $\alpha \in I$ , such that  $V_\alpha = U_\alpha \cap A$ . Therefore,  $A \subset \bigcup_{\alpha \in I} U_\alpha$ . Since  $A$  is s-closed relative to  $X$ , there exists a finite subset  $I_0$  of  $I$  such that  $A \subset \bigcup_{\alpha \in I_0} U_\alpha$ , which shows that  $A \subset \bigcup_{\alpha \in I_0} (U_\alpha \cap A)$  i.e.,  $A \subset \bigcup_{\alpha \in I_0} V_\alpha$ . Therefore,  $A$  is s-closed as a subspace.

Conversely, let  $A$  be  $s$ -closed as a subspace. Let  $\{V_\alpha : \alpha \in I\}$  be a cover of  $A$  by semi-regular sets of  $X$ . Then  $A = \bigcup_{\alpha \in I} (V_\alpha \cap A)$ . Since  $A$  is  $s$ -closed as a subspace, there exists a finite subset  $I_0$  of  $I$  such that  $A = \bigcup_{\alpha \in I_0} (V_\alpha \cap A)$ , which shows that  $A \subset \bigcup_{\alpha \in I_0} V_\alpha$ . Therefore  $A$  is  $s$ -closed relative to  $X$ .

**THEOREM 2.** Let  $B$  be a pre-open set in  $(X, T)$ . Then a subset  $A$  of  $B$  is  $s$ -closed relative to the subspace  $B$  iff  $A$  is  $s$ -closed relative to  $X$ .

**PROOF.** The proof follows by Lemma 1.

**COROLLARY 1.** Let  $A$  and  $B$  be open sets of a space  $X$  such that  $A \subset B$ . Then  $A$  is an  $s$ -closed subspace of  $B$  iff  $A$  is an  $s$ -closed subspace of  $X$ .

**PROOF.** Applying Theorem 1 and Theorem 2, we get the result.

**DEFINITION 2.** Let  $(X, T)$  be a topological space, then  $SR(X, T)$  forms a sub-base for a topology called  $T_{SR}$ -topology on  $X$ .

**LEMMA 2.** A subset  $A$  of a space  $(X, T)$  is  $s$ -closed relative to  $(X, T)$  iff  $A$  is compact in  $(X, T_{SR})$ .

**PROOF.** Let  $A$  be  $s$ -closed relative to  $(X, T)$ . Then every cover of  $A$  by sets of  $SR(X, T)$  has a finite subcover. But  $SR(X, T)$  forms a sub-base for  $(X, T_{SR})$ . So every sub-basic open cover of  $(X, T_{SR})$  has a finite subcover. Therefore by Alexander sub-base theorem  $A$  is compact in  $(X, T_{SR})$ .

Conversely, if  $A$  is compact in  $(X, T_{SR})$  then every sub-basic open cover has a finite subcover. So every cover by sets of  $SR(X, T)$  has a finite subcover. Therefore  $A$  is  $s$ -closed relative to  $(X, T)$ .

**THEOREM 3.** Let  $B$  be a  $T_{SR}$ -closed set in  $X$  and let  $A$  be any subset of  $X$  which is  $s$ -closed relative to  $(X, T)$ . Then  $A \cap B$  is  $s$ -closed relative to  $(X, T)$ .

**PROOF.** Let  $\{U_\alpha : \alpha \in I\}$  be a  $T_{SR}$ -open cover of  $A \cap B$ . Then clearly  $\{U_\alpha : \alpha \in I\} \cup (X - B)$  is a  $T_{SR}$ -open cover of  $A$ . By Lemma 2,  $A$  is compact relative to  $(X, T_{SR})$ ; and so, there exists a finite subset  $I_0$  of  $I$  such that  $A \subset \{ \bigcup_{\alpha \in I_0} U_\alpha \} \cup (X - B)$ , which implies that  $A \cap B \subset \bigcup_{\alpha \in I_0} U_\alpha$ . Therefore  $A \cap B$  is compact in  $(X, T_{SR})$ . Then by Lemma 2,  $A \cap B$  is  $s$ -closed relative to  $(X, T)$ .

**COROLLARY 2.** If  $B$  is regular open or regular closed and  $A$  is any subset of  $X$  which is  $s$ -closed relative to  $X$ , then  $A \cap B$  is  $s$ -closed relative to  $X$ .

**PROOF.** Since every regular closed or regular open set is semi-regular, the corollary follows from Theorem 2.

**COROLLARY 3.** If  $X$  is an  $s$ -closed space and  $A$  is a regular open set of  $X$ , then  $A$  is an  $s$ -closed subspace of  $X$ .

**PROOF.** The proof follows from Theorem 1 and Theorem 3.

**COROLLARY 4.** If  $A$  is  $s$ -closed open subspace of  $X$  and  $B$  is a regular open set of  $X$ , then  $A \cap B$  is an  $s$ -closed subspace of  $X$  and (hence of  $A$  and  $B$ ).

**PROOF.** The proof follows from Corollary 2 and Theorem 1 and second part follows from Corollary 1.

**THEOREM 4.** If  $A_i$ ,  $i = 1, 2, \dots, n$  are  $s$ -sets i.e.,  $s$ -closed relative to  $X$ . then  $\bigcup_{i=1}^n A_i$  is  $s$ -closed relative to  $X$ .

**PROOF.** Straightforward.

**THEOREM 5.** Let  $X$  be an  $s$ -closed space and let  $A$  be a closed set of  $X$  and let frontier of  $A$ , denoted by  $Fr(A)$ , be  $s$ -closed relative to  $X$ . Then  $A$  is  $s$ -closed relative to  $X$ .

PROOF. Since  $X$  is  $s$ -closed, by Corollary 3 and Theorem 1,  $\text{int}A$  is  $s$ -closed relative to  $X$  whenever  $A$  is a closed set. Since  $A = \text{int}A \cup \text{Fr}(A)$ , by Theorem 4,  $A$  is  $s$ -closed relative to  $X$ .

### 3. LOCALLY $s$ -CLOSED SPACES

DEFINITION 3. A space  $X$  is said to be locally  $s$ -closed iff each point belongs to a regular open neighbourhood (nbd. for short) which is an  $s$ -closed subspace of  $X$ .

REMARK 1. Clearly every  $s$ -closed space is a locally  $s$ -closed space. However, the converse is not true, in general, because any uncountable set with discrete topology is locally  $s$ -closed but not  $s$ -closed.

THEOREM 6. A topological space  $(S, T)$  is locally  $s$ -closed iff for each point  $x \in X$ , there exists a regular open set  $U$  containing  $x$  such that  $U$  is locally  $s$ -closed.

PROOF. Sufficiency : At first we prove that if  $A$  is a regular-open set in  $(X, T)$  then every regular-open set in the subspace  $(A, T_A)$  is also regular-open in  $(X, T)$ . Let  $V \subset A$  be regular-open in the subspace  $(A, T_A)$ . Then  $V = \text{int}_A \text{cl}_A V = \text{int}_A (A \cap \text{cl}_X V) = \text{int}_X (A \cap \text{cl}_X V) = \text{int}_X A \cap \text{int}_X \text{cl}_X V = A \cap \text{int}_X \text{cl}_X V = \text{int}_X \text{cl}_X V$  (as  $V \subset A$  implies  $\text{int}_X \text{cl}_X V \subset \text{int}_X \text{cl}_X A = A$ ). Therefore  $V$  is regular open in  $(X, T)$ . Now let  $x$  be any point of  $X$ . Then, by hypothesis, there exists a regular-open set  $U$  of  $(X, T)$  containing  $x$  such that  $U$  is locally  $s$ -closed. Then there exists a regular open set  $V$  in  $U$  such that  $x \in V$  and  $V$  is an  $s$ -closed subspace of  $U$ . Therefore  $V$  is a regular-open set in  $(X, T)$  and by Corollary 1,  $V$  is  $s$ -closed subspace of  $X$ . Therefore  $(X, T)$  is locally  $s$ -closed.

Necessity : The proof is straightforward.

THEOREM 7. Let  $(X, T)$  be a topological space. The following are equivalent :

- (i)  $X$  is locally  $s$ -closed;
- (ii) every point has a regular-open set which is  $s$ -closed relative to  $X$ ;
- (iii) every point  $x$  of  $X$  has an open nbd  $U$  such that  $\text{int}_X \text{cl}_X U$  is  $s$ -closed relative to  $X$ ;
- (iv) every point  $x$  of  $X$  has an open nbd  $U$  such that  $\text{scl}U$  is  $s$ -closed relative to  $X$ ;
- (v) for every point  $x$  of  $X$ , there exists an  $\alpha$ -open set  $V$  containing  $x$  such that  $\text{scl}V$  is  $s$ -closed relative to  $X$ ;
- (vi) for every point  $x$  of  $X$ , there exists an  $\alpha$ -open set  $V$  containing  $x$  such that  $\text{int}_X \text{cl}_X V$  is  $s$ -closed relative to  $X$ ;
- (vii) for each  $x \in X$ , there exists a pre-open set  $V$  containing  $x$  such that  $\text{scl}V$  is  $s$ -closed relative to  $X$ ;
- (viii) for every  $x$  of  $X$ , there exists a pre-open set  $V$  containing  $x$  such that  $\text{int}_X \text{cl}_X V$  is  $s$ -closed relative to  $X$ ;
- (ix) for every  $x \in X$ , there exists a pre-open set  $V$  containing  $x$  such that  $\text{int}_X \text{cl}_X V$  is an  $s$ -closed subspace of  $X$ .

PROOF. (i)  $\rightarrow$  (ii) : Follows from Theorem 1 and from the fact that every regular-open set is pre-open set. (ii)  $\rightarrow$  (iii) is obvious.

(iii)  $\rightarrow$  (iv) : Follows from the fact that for an open set  $U$ ,  $\text{scl}U = \text{intcl}U$  (Maio and Noiri [8]). (iv)  $\rightarrow$  (v) is evident, since every open set is  $\alpha$ -open.

(v)  $\rightarrow$  (vi), (vi)  $\rightarrow$  (vii), (vii)  $\rightarrow$  (viii) and (viii)  $\rightarrow$  (ix) are straightforward because of the facts that every  $\alpha$ -open set is pre-open and a set  $V$  is pre-open iff  $\text{scl}V = \text{intcl}V$  (Dorsett [4]). (ix)  $\rightarrow$  (i) follows from Theorem 1.

**THEOREM 8.** A topological space  $(X, T)$  is locally s-closed iff, its semi-regularization space  $(X, T_S)$  is locally s-closed.

**PROOF.** Let  $(X, T)$  be locally s-closed. Dorsett [4] proved that  $SR(X, T) = SR(X, T_S)$  and hence a subset  $A$  of  $X$  is s-closed relative to  $(X, T)$  iff  $A$  is s-closed relative to  $(X, T_S)$ . We know that if  $U$  is an open and  $V$  a closed subset of  $(X, T)$ , then  $cl_T U = cl_{T_S} U$  and  $int_T V = int_{T_S} V$ . Using these results we can easily prove that for a regular-open set  $U$  of  $(X, T)$ ,  $int_T cl_T U = int_{T_S} cl_{T_S} U$ . Therefore every regular-open set in  $(X, T)$  is regular open in  $(X, T_S)$  and vice-versa. So  $(X, T)$  and  $(X, T_S)$  have the same collection of regular-open sets. Therefore, by definition,  $(X, T)$  is locally s-closed iff  $(X, T_S)$  is locally s-closed.

**REMARK 2.** Local s-closedness is a semi-regular property.

**DEFINITION 4.** A function  $f : X \rightarrow Y$  is said to be s- $\theta$ -closed if image of each s- $\theta$ -closed set in  $X$  is closed in  $Y$ .

**THEOREM 9.** A function  $f : X \rightarrow Y$  is s- $\theta$ -closed iff  $cl(f(A)) \subset f([A]_{s-\theta})$  for any subset  $A$  of  $X$ .

**PROOF.** Let  $f$  be s- $\theta$ -closed and  $A$  be any subset of  $X$ . Then  $f([A]_{s-\theta})$  is closed in  $Y$  (since  $[A]_{s-\theta}$  is s- $\theta$ -closed set). Clearly  $f(A) \subset f([A]_{s-\theta})$  and hence  $cl(f(A)) \subset f([A]_{s-\theta})$ .

Conversely, let  $A$  be an arbitrary s- $\theta$ -closed set in  $X$ . By hypothesis  $f(A) \subset cl(f(A)) \subset f([A]_{s-\theta}) = f(A)$ . Therefore  $f(A) = cl(f(A))$  and hence  $f(A)$  is closed in  $Y$ .

**THEOREM 10.** A surjective function  $f : X \rightarrow Y$  is s- $\theta$ -closed iff for each subset  $A$  in  $Y$  and each s- $\theta$ -open set  $U$  in  $X$  containing  $f^{-1}(A)$ , there exists an open set  $V$  in  $Y$  containing  $A$  such that  $f^{-1}(V) \subset U$ .

**PROOF.** Sufficiency : Suppose that the given hypothesis holds. Let  $A$  be any s- $\theta$ -closed set in  $X$ . Let  $y$  be an arbitrary point in  $Y - f(A)$ ; then  $X - A$  is an s- $\theta$ -open set containing  $f^{-1}(y)$ ; by hypothesis, there exists an open set  $V_y$  containing  $y$  such that  $f^{-1}(V_y) \subset X - A$ . This shows that  $y \in V_y \subset Y - f(A)$ . Therefore  $Y - f(A) = \bigcup \{ V_y : y \in Y - f(A) \}$ . Hence  $Y - f(A)$  is an open set i.e.,  $f(A)$  is closed in  $Y$ .

Necessity : Let  $V = Y - f(X - U)$ . Since  $f^{-1}(A) \subset U$ , it shows that  $A \subset V$ . As  $f$  is s- $\theta$ -closed,  $f(X - U)$  is closed and hence  $V$  is open in  $Y$ . Therefore,  $f^{-1}(V) \subset X - f^{-1}\{f(X - U)\} \subset U$ .

**LEMMA 3.** A subset  $A$  of a space  $X$  is an s-set iff every cover of  $A$  by s- $\theta$ -open sets has a finite subfamily which covers  $A$ .

**PROOF.** Sufficiency part is straightforward.

Necessity : Let  $A$  be an s-set. Let  $\mathcal{U} = \{ U_\alpha : \alpha \in I \}$  be an s- $\theta$ -open cover of  $A$  and also let  $x \in A$ . Then there exists  $U_\alpha \in \mathcal{U}$  such that  $x \in U_\alpha$ . But  $U_\alpha$  being an s- $\theta$ -open set, there exists a semi-open set  $V_x$  such that  $x \in V_x \subset scl V_x \subset U_\alpha$ . Therefore the family  $\{ V_x : x \in A \}$  is a cover of  $A$  by semi-open sets of  $X$ . Hence there exist points say  $x_1, \dots, x_n$  such that  $A \subset \bigcup_{i=1}^n scl V_{x_i}$ . Hence  $A \subset \bigcup_{i=1}^n U_{\alpha_i}$ . Therefore  $\mathcal{U}$  has a finite subfamily which covers  $A$ .

**THEOREM 11.** Let  $f : X \rightarrow Y$  be an s- $\theta$ -closed surjection with s-set point inverses; if  $A$  is any compact set in  $Y$  then  $f^{-1}(A)$  is an s-set in  $X$ .

**PROOF.** Let  $\mathcal{U} = \{ U_\alpha : \alpha \in I \}$  be any cover of  $f^{-1}(A)$  by s- $\theta$ -open sets of  $X$ . For each point  $y \in A$ ,  $f^{-1}(y) \subset \bigcup_{\alpha \in I} U_\alpha$ . By hypothesis  $f^{-1}(y)$  is an s-set, by Lemma 3,

there exists a finite subfamily  $I_0$  of  $I$  such that  $f^{-1}(y) \subset \bigcup_{\alpha \in I_0} U_\alpha$ . Since we know that Union of any collection  $s$ - $\theta$ -open sets is  $s$ - $\theta$ -open and since  $f$  is an  $s$ - $\theta$ -closed function, by Theorem 10, there exists an open set  $V_y$  of  $Y$  containing  $y$  such that  $f^{-1}(V_y) \subset \bigcup_{\alpha \in I_0} U_\alpha$ .  $\{V_y : y \in A\}$  is a cover of a compact set  $A$  and hence there exist points  $y_1, \dots, y_n$  of  $A$  such that  $A \subset \bigcup_{i=1}^n V_{y_i}$  which shows that  $f^{-1}(A)$  is covered

by a finite number of  $s$ - $\theta$ -open sets from  $\mathcal{U}$  and hence  $f^{-1}(A)$  is an  $s$ -set.

**COROLLARY 5.** Let  $f : X \rightarrow Y$  be an  $s$ - $\theta$ -closed surjection with  $s$ -set point inverses; if  $X$  is  $T_2$  and  $Y$  is compact then  $f$  is continuous.

**PROOF.** Let  $A$  be a closed set in  $Y$ . Therefore  $A$  is also compact; by Theorem 11,  $f^{-1}(A)$  is an  $s$ -set in  $X$ . Since every  $s$ -set is an NC-set and  $X$  is  $T_2$ , by Theorem 2.1 of T. Noiri [12],  $f^{-1}(A)$  is closed and hence  $f$  is continuous.

**DEFINITION 5.** A function  $f : X \rightarrow Y$  is said to be completely continuous (Arya and Gupta [1]) if inverse image of each open set in  $Y$  is regular-open in  $X$ .

**THEOREM 12.** Let  $f : X \rightarrow Y$  be a completely-continuous  $s$ - $\theta$ -closed surjection with  $s$ -set point inverses. If  $Y$  is locally compact  $T_2$ ,  $X$  is locally  $s$ -closed.

**PROOF.** Since  $Y$  is locally compact  $T_2$ , for each point  $x \in X$ , there exists a closed compact nbd.  $U$  of  $f(x)$ . Since  $f$  is completely continuous,  $f^{-1}(\text{int } U)$  is a regular open set containing  $x$ . But it is easy to see that every regular-open set is semi-regular and hence an  $s$ - $\theta$ -closed set (see Maio and Noiri [8]). Since  $U$  is compact and  $f$  is an  $s$ - $\theta$ -closed function, by Theorem 11,  $f^{-1}(U)$  is an  $s$ -set in  $X$  and  $x \in f^{-1}(\text{int } U) \subset f^{-1}(U)$ . Hence, by Corollary 2,  $f^{-1}(\text{int } U)$  is an  $s$ -set in  $X$ . Therefore  $X$  is locally  $s$ -closed.

**DEFINITION 6.** A function  $f : X \rightarrow Y$  is said to be  $\mathcal{V}$ -continuous (Ganguly and Basu [5]) if for each  $x \in X$  and each  $W \in \text{SO}(f(x))$ , there is an open set  $V$  containing  $x$  such that  $f(V) \subset \text{scl } W$ . Equivalently  $f$  is  $\mathcal{V}$ -continuous iff the inverse image of each semi-regular set is clopen.

**LEMMA 4.** If  $f : X \rightarrow Y$  is  $\mathcal{V}$ -continuous and  $K \subset X$  is compact; then  $f(K)$  is an  $s$ -set in  $Y$ .

**PROOF.** Let  $\{U_\alpha : \alpha \in I\}$  be a cover of  $f(K)$  by semi-regular sets of  $Y$ . Then  $\{f^{-1}(U_\alpha) : \alpha \in I\}$  is a cover of  $K$  by open sets of  $X$ . Since  $K$  is compact, there exists a finite subset  $I_0$  of  $I$  such that  $K \subset \bigcup_{\alpha \in I_0} f^{-1}(U_\alpha)$  i.e.,  $f(K) \subset \bigcup_{\alpha \in I_0} U_\alpha$ . So  $f(K)$  is an  $s$ -set in  $Y$ .

**LEMMA 5.** (See [12]) Let  $X$  be a  $T_2$ -space. Then for any disjoint NC-sets  $A$  and  $B$ , there exist disjoint regular open sets  $U$  and  $V$  such that  $A \subset U$  and  $B \subset V$ .

**THEOREM 13.** If  $f : X \rightarrow Y$  is an  $s$ - $\theta$ -closed,  $\mathcal{V}$ -continuous surjection with  $s$ -set point inverses and if  $X$  is locally compact  $T_2$ , then  $Y$  is locally  $s$ -closed.

**PROOF.** We shall first prove that  $Y$  is  $T_2$ . Let  $y_1$  and  $y_2$  be two distinct points of  $Y$ . Then  $f^{-1}(y_1)$  and  $f^{-1}(y_2)$  are disjoint  $s$ -sets and hence disjoint NC-sets. By Lemma 5, there exist disjoint regular-open sets  $U_1$  and  $U_2$  such that  $f^{-1}(y_1) \subset U_1$  and  $f^{-1}(y_2) \subset U_2$ . But every regular-open set is an  $s$ - $\theta$ -open set and so, by Theorem 10, there exist open sets  $V_j$ ,  $j = 1, 2$  containing  $y_j$  in  $Y$  such that  $f^{-1}(V_j) \subset U_j$  where  $j=1, 2$ . Thus  $Y$  is  $T_2$ . Let  $X$  be locally compact  $T_2$ ; for each point  $x$  of  $f^{-1}(y)$ , there exists a compact closed nbd.  $U_x$  of  $x$  in  $X$ . Since interior of a closed nbd. is a regular-open set, it is semi-regular as well. Therefore the family  $\{\text{int } U_x : x \in f^{-1}(y)\}$  is a cover of an  $s$ -set  $f^{-1}(y)$  by semi-regular sets. By Proposition 4.1

of Maio and Noiri [8], there exist points  $x_1, \dots, x_n$  in  $t^{-1}(y)$  such that  $f^{-1}(y) \subset \bigcup_{i=1}^n \text{int}U_{x_i}$ . Let  $U = \bigcup_{i=1}^n U_{x_i}$ . Then  $f^{-1}(y) \subset \bigcup_{i=1}^n \text{int}U_{x_i} \subset \text{int}U$ . Since  $\text{int}U$  is clearly an s-open set containing  $t^{-1}(y)$  and since,  $t$  is an s- $\theta$ -closed function by Theorem 10, there exists an open set  $V_y$  containing  $y$  such that  $f^{-1}(V_y) \subset \text{int}U$  i.e.,  $y \in V_y \subset t(\text{int}U) \subset f(U)$ . But  $f$  being  $\mathcal{N}$ -continuous,  $t(U)$  is an s-set by Lemma 4. Since  $Y$  is  $T_2$ ,  $f(U)$  is closed by Theorem 2.1 of Noiri [12]. Therefore  $y \in V_y \subset \text{intcl}V_y \subset f(U)$ . Clearly  $\text{intcl}V_y$  is a regular-open set and hence by Corollary 2,  $\text{intcl}V_y$  is an s-set. Hence  $Y$  is locally s-closed.

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REFERENCES

1. ARYA, S.P. AND GUPTA, R. On strongly continuous mappings, Kyungpook Math. J. 14(1974), 131-143.
2. CARNAHAN, D., Locally nearly-compact spaces, Boll. Un. Mat. Ital., 4(6)1972, 146-153.
3. CROSSLEY, S.G. AND HILDEBRAND, S.K., Semi-closure, Texas J. Sci., 22(1971), 99-119.
4. DORSETT, C. Semi-regularization spaces and the semi-closure operator, s-closed spaces and Quasi-irresolute functions, Indian J. Pure Appl. Math. 21(5), 416-422, 1990.
5. GANGULY, S. AND BASU, C.K., More on s-closed spaces, Soochow J. Math., 18(4) 1992, 409-418.
6. GANSTER, M. AND REILLY, I.L., A note on S-closed spaces, Indian J. Pure Appl. Math., 19(10)(1988), 1031-1033.
7. LEVINE, N., Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963), 36-41.
8. MAIO, G.D. AND NOIRI, T., On s-closed spaces, Indian J. Pure Appl. Math., 18(3) 226-233 (1987).
9. MASHOUR, A., ABD.EL-MONSEF, M. AND EL-DEEB, S., Proc. Math. Phys. Soc. Egypt., 53(1982), 47-53.
10. NJASTAD, O. On some classes of nearly-open sets, Pacific J. Math., (1965) 15, 961-970.
11. NOIRI, T., On S-closed spaces, Ann. Soc. Sci. Bruxelles. T.91 IV(1977), 189-194.
12. NOIRI, T., N-closed sets and some separation axioms, Ann. Soc. Sci. Bruxelles Ser I. 88(1974), 195-199.
13. NOIRI, T., On S-closed subspaces, Acad. Naz. Del. Linei, VIII, LXIV, 1978, 157-162.
14. THOMPSON, T., S-closed spaces, Proc. Amer. Math. Soc., 60(1976), 335-338.