Internat. J. Math. & Math. Sci.
VOL. 18 NO. 3 (1995) 475-484

COHOMOLOGY WITH L^p-BOUNDS ON POLYCYLINDERS

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(Received October 8, 1993 and in revised form March 15, 1994)

ABSTRACT. Let $\Omega = \Omega_1 \times \ldots \times \Omega_n$ be a polycylinder in $\mathbb{C}^n$, that is each Ω_j is bounded, non–empty and open in $\mathbb{C}$. The main result proved here is that, if B_p is the sheaf of germs of L^p–holomorphic functions on $\overline{\Omega}$ then $H^q(\overline{\Omega},B_p) = 0$ for $q \geq 1$. The proof of this is then used to establish a Leray's Isomorphism with L^p–bounds theorem.

KEY WORDS AND PHRASES. Sheaf Cohomology – L^p bounds, $\overline{\partial}$–equation.
AMS CLASSIFICATION. 32F20, 32C35, 35N15.

A polycylinder in $\mathbb{C}^n$ is a product set $\Omega = \Omega_1 \times \ldots \times \Omega_n$ such that each Ω_j is open and bounded in $\mathbb{C}$, $1 \leq j \leq n$. If B is the sheaf of germs of bounded holomorphic functions on the closure of a polycylinder Ω, it is proved, among other things, in [1] that the cohomology group $H^1(\overline{\Omega},B) = 0$. As part of the vanishing theorems in [7], this is generalized to $H^q(\overline{\Omega},B) = 0$, $q \geq 1$, where Ω is a member of a set of product domains not including all polycylinders and in [3] this is generalized to all polycylinders.

For the product domains Ω considered in [7], it is also shown that if B_p is the sheaf of germs of L^p – holomorphic functions on $\overline{\Omega}$, then $H^q(\overline{\Omega},B_p) = 0$ for $q \geq 1$ and $1 \leq p \leq \infty$. In this paper we extend this result to all polycylinders as an application of a result which we call Dolbeault–Grothendieck lemma with L^p – Bounds. As applications of the vanishing theorem $H^q(\overline{\Omega},B_\infty) = 0$, $q \geq 1$, we state a theorem on the characterization of the generators of the maximal ideals of the Banach Algebra $H^\infty(\Omega)$ of bounded holomorphic functions on Ω, which we have obtained elsewhere by different methods. Furthermore we state the fact that the Weak–Corona–Problem is solvable on all polycylinders and not merely on the product domains considered in [7]. We also state and prove a Leray's Isomorphism Theorem with L^p – bounds. This is influenced by the work in [4] and [8] in which the acyclic covers for which the theorem is proved, are more general, noting at this point that the acyclic covers made up of polycylinders here cannot as yet be replaced by acyclic covers made up of strongly pseudoconvex domains, even though L^p – estimates on strongly pseudoconvex domains are more advanced than L^p – estimates on polycylinders.

§1. Definitions and Statements of The Theorems

1. If $U \subset \mathbb{C}^n$ is an open set and $f \in C^\infty(U)$ and $1 \leq p \leq \infty$ we define

$$\|f\|^{(0)}_{L^p(U)} = \|f\|^{(0,0)}_{L^p(U)} = \|f\|_{L^p(U)}$$

$\|f\|_{L^p(U)}$ being the L^p – norm of f on U,

$$\|f\|^{(0,r)}_{L^p(U)} = \max_{i_1<\dots<i_r} \left\| \frac{\partial^r f}{\partial \bar{z}_{i_1} \dots \partial \bar{z}_{i_r}} \right\|^{(0)}_{L^p(U)} \quad \text{for } 1 \leq r \leq n$$

and

$$\|f\|^{(n)}_{L^p(U)} = \max_{0 \leq r \leq n} \|f\|^{(0,r)}_{L^p(U)}$$

If $f = \sum_{(i_1,\dots,i_q)} f_{i_1 \dots i_q} d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_q}$ is a $C^\infty(0,q)$ – form on U where Σ' means the summation is over increasing multi–indices, we write f as $\Sigma'_I f_I d\bar{z}^I$ for short $I = (i_1,\dots, i_q)$, and set

$$\|f\|^{(n)}_{L^p_{(0,q)}(U)} = \max_I \|f_I\|^{(n)}_{L^p(U)}$$

Then corresponding to theorem 1 in [3], there is the following:

THEOREM 1: Let Ω be a polycylinder in $\mathbb{C}^n$ and $1 \leq p \leq \infty$. There is a $K_* > 0$ such that if f is a smooth $\bar{\partial}$–closed $(0,q+1)$–form on Ω with $\|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} < \infty$, then there is a smooth $(0,q)$–form u on Ω with $\bar{\partial}u = f$ and

$$\|u\|^{(n)}_{L^p_{(0,q)}(\Omega)} \leq K_* \|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)}$$

2. Let Ω be a polycylinder and $U \neq \phi$ a set open in $\bar{\Omega}$, then $B^p_\Omega(U)$ is the Banach space of holomorphic functions f on $\Omega \cap U$ such that $\|f\|_{L^p(U \cap \Omega)} < \infty$, $1 \leq p \leq \infty$. If V is open in $\bar{\Omega}$ with $\phi \neq V \subset U$, the restriction map r^U_V: $B^p_\Omega(U) \to B^p_\Omega(V)$ is defined. Then $B^p_0 := \{B^p_\Omega(\Omega); r^U_V\}$ is then the canonical presheaf of L^p – holomorphic functions on $\bar{\Omega}$. The associated sheaf B_p is the sheaf of germs of L^p – holomorphic functions on $\bar{\Omega}$. From Theorem 1 there is the following:

THEOREM 2: Let $\Omega \subset \mathbb{C}^n$ be a polycylinder and B_p, the sheaf of germs of L^p – holomorphic functions on $\bar{\Omega}$. Then $H^q(\bar{\Omega},B_p) = 0$ for $q \geq 1$ and $1 \leq p \leq \infty$.

3. Let Ω be a polycylinder and $w \in \Omega$. M_w denotes the maximal ideal of the ring $\mathcal{O}_w$ of germs of holomorphic functions at w and $M_w(\Omega)$ is the maximal ideal of functions in $H^\infty(\Omega)$ vanishing at w. If f is holomorphic in Ω, f_w denotes the germ at w.

Using Theorem 2 and the Koszul's complex constructed in [9] and used in [6] to solve the Gleason Problem on strongly pseudo convex domains, we get

THEOREM 3: Let $w \in \Omega \subset \mathbb{C}^n$ and $f_1,\dots, f_n \in H^\infty(\Omega)$. Then $f_1,\dots, f_n$ generate $M_w(\Omega)$ if and only if (i) $f_{1w},\dots, f_{nw}$ generate M_w and (ii) w is the only common zero of $f_1,\dots, f_n$ in Ω.

In particular $z_1 - w_1,\dots, z_n - w_n$ generate $M_w(\Omega)$.

4. The Weak Corona Problem is formulated in [2]: Let X be a relatively compact domain of a topological space Y. Let $f_0,\dots, f_N$ be complex–valued continuous functions on X; $f_1,\dots, f_N$ verify the weak corona assumption (on X) if the following two conditions hold:

a) $f_0,\dots, f_N$ have no common zeros on X;

b) a positive number $\delta > 0$ exists so that for each $z \in \partial X$(= boundary of X in Y), an index $i \in \{0,\dots, N\}$ $i = i(z)$ and an open neighborhood V_z of z in Y are given such that $|f_i(w)| \geq \delta$ on $V_z \cap X$.

Let A be a function algebra on X. The weak corona problem is solvable in A (on X) when for $f_0,\dots, f_N \in A$ which verify the weak corona assumption, $f_0,\dots, f_N$ represent 1 in A. From the work in [2] and Theorem 2 we have

THEOREM 4: Let $\Omega \subset \mathbb{C}^n$ be any polycylinder. Then the weak corona problem is solvable in $H^\infty(\Omega)$.

5. Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions in $\mathbb{C}^n$. If $U \subset \mathbb{C}^n$ is open and $r > 0$ is an integer let $\Gamma(U,\mathcal{O}^r)$ be the sections of $\mathcal{O}^r$ on U, then

$$\Gamma_p(U,\mathcal{O}^r) := \{f = (f_1,\dots, f_r) \in \Gamma(U,\mathcal{O}^r): \|f_1\|_{L^p(U)} +\dots+ \|f_r\|_{L^p(U)} < \infty\}$$

If $\mathfrak{F}$ is a coherent analytic sheaf on a neighborhood of the closure $\overline{\Omega}$ of a polycylinder Ω, then by Cartan's theorem A there is an exact sequence

$$\mathcal{O}^m \xrightarrow{\ \lambda\ } \mathfrak{F} \longrightarrow 0$$

of $\mathcal{O}$–homomorphisms in a neighborhood of $\overline{\Omega}$, where m is a positive integer. The L^p–bounded section of $\mathfrak{F}$ over Ω, $\Gamma_p(\Omega,\mathfrak{F})$ is defined by

$$\Gamma_p(\Omega,\mathfrak{F}) = \lambda(\Gamma_p(\Omega,\mathcal{O}^m)).$$

It can be shown easily that the definition of $\Gamma_p(\Omega,\mathfrak{F})$ does not depend on λ and m.

Let X be an open set in $\mathbb{C}^n$, $\mathfrak{U} = \{U_i\}_{i\in I}$ a locally finite covering of X by polycylinders each of which is relatively compact in X. We define the L^p–bounded alternate q–cochain group $C^q_p(\mathfrak{U},\mathfrak{F})$ of the covering $\mathfrak{U}$ with values in $\mathfrak{F}$ by

$$C^q_p(\mathfrak{U},\mathfrak{F}) := \{c = (c_\alpha) \in C^q(\mathfrak{U},\mathfrak{F}): c_\alpha \in \Gamma_p(\mathfrak{U}_\alpha,\mathfrak{F}), \forall\alpha = (\alpha_0,\dots, \alpha_q) \in I^{q+1}\},$$

where $U_\alpha = U_{\alpha_0} \cap\dots\cap U_{\alpha_q}$ and $C^q(\mathfrak{U},\mathfrak{F})$ is the alternate q–cochain group of the covering $\mathfrak{U}$ with values in $\mathfrak{F}$.

The coboundary operator

$$\delta: C^q(\mathfrak{U},\mathfrak{F}) \longrightarrow C^{q+1}(\mathfrak{U},\mathfrak{F})$$

maps $C^q_p(\mathfrak{U},\mathfrak{F})$ into $C^{q+1}_p(\mathfrak{U},\mathfrak{F})$, hence we have a complex

$$C^0_p(\mathfrak{U},\mathfrak{F}) \xrightarrow{\ \delta\ } C^1_p(\mathfrak{U},\mathfrak{F}) \xrightarrow{\ \delta\ } \dots \longrightarrow C^q_p(\mathfrak{U},\mathfrak{F}) \xrightarrow{\ \delta\ } C^{q+1}_p(\mathfrak{U},\mathfrak{F}) \longrightarrow \dots$$

and $H^q_p(\mathfrak{U},\mathfrak{F})$ is the qth cohomology group of this complex. We then have the following:

THEOREM 5: The natural map

$$H^q_p(\mathfrak{U},\mathfrak{F}) \longrightarrow H^q(X,\mathfrak{F})$$

is an isomorphism for $q \geq 0$ and $1 \leq p \leq \infty$.

§2. DOLBEAULT–GROTHENDIECK LEMMA WITH L^p–BOUNDS.

We establish Theorem 1 in this section. The proof parallels completely that of the L^∞–version in [3], but we give a detailed proof because there are lots of misprints in [3].

The proof is by induction, the inductive statement being that the theorem is true if f does not involve $d\bar{z}_{k+1}, d\bar{z}_{k+2},\ldots, d\bar{z}_n$. When $k = 0$, there is nothing to prove because then f must be zero. If $k = n$, then the statement is the theorem. We assume therefore that the theorem is true if f does not involve $d\bar{z}_k, d\bar{z}_{k+1},\ldots, d\bar{z}_n$ and assume that

$$f = d\bar{z}_k \wedge g + h$$

where g is of type $(0,q)$ and h is of type $(0,q + 1)$, and g and h are independent of $d\bar{z}_k,\ldots, d\bar{z}_n$.

$$g = \sum_I{}' g_I d\bar{z}^I$$

$$h = \sum_J{}' h_J d\bar{z}^J$$

If I is an increasing multi–index and j is a positive integer not in I, (I,j) is the increasing multi–index obtained by adding j to the integers in I and $(I,j_1,j_2) = ((I,j_1),j_2)$ when j_1 is not in I and j_2 is not (I,j_1). Now on Ω

$$0 = \bar{\partial} f = d\bar{z}_k \wedge \Big(\sum_{j=1}^n d\bar{z}_j \wedge \big(\sum_I{}' \frac{\partial g_I}{\partial \bar{z}_j} d\bar{z}^I\big)\Big) + \sum_{j=1}^n d\bar{z}_j \wedge \big(\sum_J{}' \frac{\partial h_J}{\partial \bar{z}_j} d\bar{z}^J\big),$$

hence if I_0 is an increasing multi–index of length q, $1 \leq j_0 < k$ and j_0 is not in I_0 the coefficient of $d\bar{z}_k \wedge d\bar{z}^{(I_0,j_0)}$ in $\bar{\partial} f$ is

$$0 = \sum_{\substack{1 \leq j < k \\ (I,j)=(I_0,j_0)}} \epsilon\,(I,j) \frac{\partial g_I}{\partial \bar{z}_j} \pm \frac{\partial h_{J_0}}{\partial \bar{z}_k} \tag{1}$$

where $(J_0,k) = (I_0,j_0,k)$, $\epsilon(I,j) = \pm 1$ and the summation is over $1 \leq j < k$ and $(I,j) = (I_0,j_0)$; because

$$\frac{\partial g_I}{\partial \bar{z}_j} = 0,\ j > k,$$

this apart from a factor of ± 1, being the coefficient of $d\bar{z}_k \wedge d\bar{z}_j \wedge d\bar{z}^I$ in $\bar{\partial} f = 0$.

In Ω let

$$G_I(z) = \frac{1}{2\pi i} \int_{\Omega_k} (\tau_k - z_k)^{-1} g_I(z_1,\ldots, z_{k-1},\tau_k,z_{k+1},\ldots, z_n) d\bar{\tau}_k \wedge d\tau_k \tag{2}$$

Then clearly $G_I \in C^\infty(\Omega)$ and $\|G_I\|^{(n)}_{L^p(\Omega)} \leq K_1^* \|g_I\|^{(n)}_{L^p(\Omega)}$ for some constant K_1^*.

$$\frac{\partial G_I}{\partial \bar{z}_k} = g_I \text{ and } \frac{\partial G_I}{\partial \bar{z}_j} = 0 \text{ for } j > k. \tag{3}$$

Let $G = \sum_I{}' G_I dz^I$. Then $\|G\|^{(n)}_{L^p_{(0,q)}(\Omega)} \leq K_1^* \|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)}$
and

$$\bar\partial G = \sum_I{}' \sum_{j=1}^{n} \frac{\partial G_I}{\partial \bar z_j} d\bar z_j \wedge d\bar z^I = d\bar z_k \wedge g + h_1 \tag{4}$$

where h_1 is the sum when j runs from 1 to $k-1$ and it is independent of $d\bar z_k, \ldots, d\bar z_n$. Hence $h - h_1 = f - \bar\partial G$ does not involve $d\bar z_k, \ldots, d\bar z_n$. If I_0 is an increasing multi-index of length q, $1 \leq j_0 < k$ and j_0 is not in I_0 the coefficient of $d\bar z^{(I_0,j_0)}$ in h_1 is

$$H_{(I,j_0)} = \sum_{\substack{1 \leq j < k \\ (I,j) = (I_0,j_0)}} \epsilon(I,j) \frac{\partial G_I}{\partial \bar z_j}, \tag{5}$$

the meaning of the symbols being as in (1). From (1) it follows that

$$H_{(I_0,j_0)}(z) = \frac{\pm 1}{2\pi i} \int_{\Omega_k} (\tau_k - z_k)^{-1} \frac{\partial h_{J_0}}{\partial \bar z_k}(z_1, \ldots, \tau_k, \ldots, z_n) d\bar\tau_k \wedge d\tau_k \tag{6}$$

where $(I_0,j_0,k) = (J_0,k)$.

From (3), (5) and (6) it follows that

$$\|h_1\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} \leq K_2^* \|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)}$$

for some constant K_2^*, hence

$$\|f - \bar\partial G\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} \leq \|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} + K_2^* \|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} < \infty.$$

By the induction hypothesis, since $f - \bar\partial G$ does not involve $d\bar z_k, \ldots, d\bar z_n$ and $\bar\partial(f - \bar\partial G) = 0$ on Ω, there is a smooth $(0,q)$ – form v such that $\bar\partial v = f - \bar\partial G$ on Ω and

$$\|v\|^{(n)}_{L^p_{(0,q)}(\Omega)} \leq K_3^* \|f - \bar\partial G\|^{(n)}_{L^p_{(0,q+1)}(\Omega)} \leq K_3^*(1 + K_2^*) \|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)}$$

for some constant K_3^*.

Now let $u = v + G$, then $\bar\partial u = f$ on Ω and

$$\|u\|^{(n)}_{L^p_{(0,q)}(\Omega)} \leq (K_1^* + K_3^*(1 + K_2^*)) \|f\|^{(n)}_{L^p_{(0,q+1)}(\Omega)}$$

which completes the proof of Theorem 1 with $K_* = (K_1^* + K_3^*(1 + K_2^*))$.

§3. VANISHING THEOREMS:

1. To prove Theorem 2, let E_r be the sheaf of germs of smooth $(0,r)$ forms in $\mathbb{C}^n$ and F_r the sheaf of germs of smooth $\bar\partial$–closed $(0,r)$ forms, $r \geq 0$. If U is open in $\mathbb{C}^n$, define $\Pi_p(U,F_r)$ and $\Lambda_p(U,E_r)$ by

$$\Pi_p(U,F_r) := \{f \in \Gamma(U,F_r): \|f\|^{(n)}_{L^p_{(0,r)}(\Omega)} < \infty\}$$

$$\Lambda_p(U,E_r) := \{f \in \Gamma(U,E_r): \|f\|^{(n)}_{L^p_{(0,r)}(\Omega)} < \infty, \|\bar{\partial} f\|^{(n)}_{L^p_{(0,r+1)}(\Omega)} < \infty\}.$$

Let $\mathfrak{U} = \{U_i\}_{i\epsilon I}$ be a finite covering of $\overline{\Omega}$ by polycylinders where Ω is a polycylinder in $\mathbb{C}^n$ For each $\alpha = (i_0,\dots, i_q) \epsilon I^{q+1}$ let $\Omega_\alpha = \Omega \cap U_{i_0} \cap \dots \cap U_{i_q}$ and let $\mathfrak{U}_\Omega = \{\Omega \cap U_i\}_{i\epsilon I}$. Let $C^q_p(\mathfrak{U}_\Omega,F_r)$ and $D^q_p(\mathfrak{U}_\Omega,E_r)$ defined by $C^q_p(\mathfrak{U}_\Omega,F_r) := \{c = (c_\alpha) \epsilon C^p(\mathfrak{U}_\Omega,F_r): c_\alpha \epsilon \Pi_p(\Omega_\alpha,F_r)$, for each $\alpha = (i_0,\dots, i_q) \epsilon I^{q+1}\}$ $D^q_p(\mathfrak{U}_\Omega,E_r) = \{c = (c_\alpha) \epsilon C^p(\mathfrak{U}_\Omega,E_r): c_\alpha \epsilon \Lambda_p(\Omega_\alpha,E_r)$, for each $\alpha = (i_0,\dots, i_q) \epsilon I^{q+1}\}$.

2. The coboundary operator δ: $C^q(\mathfrak{U}_\Omega,L_r) \longrightarrow C^{q+1}(\mathfrak{U}_\Omega,L_r)$, where L_r is E_r or F_r, maps $C^q_p(\mathfrak{U}_\Omega,F_r)$ into $C^{q+1}_p(\mathfrak{U}_\Omega,F_r)$ and $D^q_p(\mathfrak{U}_\Omega,E_r)$ into $D^{q+1}_p(\mathfrak{U}_\Omega,E_r)$. We then define $H^q(\mathfrak{U}_\Omega,E_r)$ as the qth cohomology group of the complex:

$$C^0_p(\mathfrak{U}_\Omega,F_r) \to \dots \to C^q_p(\mathfrak{U}_\Omega,F_r) \xrightarrow{\ \delta\ } C^{q+1}_p(\mathfrak{U}_\Omega,F_r) \to \dots\dots$$

and $G^q_p(\mathfrak{U}_\Omega,E_r)$ as the qth cohomology group of the complex:

$$D^0_p(\mathfrak{U}_\Omega,E_r) \to \dots \to D^q_p(\mathfrak{U}_\Omega,E_r) \xrightarrow{\ \delta\ } D^{q+1}_p(\mathfrak{U}_\Omega,E_r) \to \dots.$$

LEMMA 1: $G^q_p(\mathfrak{U}_\Omega,E_r) = 0$ for $q \geq 1$.

The proof of this is easy and does not involve the use of the $\bar{\partial}$–operator.

LEMMA 2: $H^1_p(\mathfrak{U}_\Omega,F_r) = 0$ for $r \geq 0$.

PROOF: Let $\{\varphi_i\}_{i\epsilon I}$ be a C^∞ – partition of unity subordinate to the covering $\mathfrak{U}$ of $\overline{\Omega}$, so that $0 \leq \varphi_i \leq 1$, supp $\varphi_i \subset U_i$ and $\sum_i \varphi_i \equiv 1$ on $\overline{\Omega}$. If $c \epsilon C^1_p(\mathfrak{U}_\Omega,F_r)$ and $\delta c = 0$, $c = (c_{ij})$, define outside $\Omega \cap U_i \cap U_j$ as zero and set

$$c_i = \sum_{j\epsilon I} \varphi_j c_{ij}.$$

Then for each $i \epsilon I$, c_i' is a $C^\infty - (0,r)$ – form and $\|c_i'\|^{(n)}_{L^p_{(0,r)}(\Omega_i)} < \infty$. Since $\delta c = 0$,

$$c_i' - c_j' = \sum_{k\epsilon I} \varphi_k(c_{kj} + c_{ik}) = \sum_{k\epsilon I} \varphi_k c_{ij} = c_{ij}$$

and so $\bar{\partial} c_j' = \bar{\partial} c_i'$ on $\Omega \cap U_i \cap U_j$. Hence there is a C^∞ $(0, r + 1)$ – form f on Ω such that $f|U_i \cap \Omega = \bar{\partial} c_i'$. Hence on $U_i \cap \Omega$, $-f = \sum_{j\epsilon I} (\bar{\partial}\varphi_j) \wedge c_{ji}$ and $\|f\|^{(n)}_{L^p_{(0,r+1)}(\Omega \cap U_i)} < \infty$ which implies $\|f\|^{(n)}_{L^p_{(0,r+1)}(\Omega)} < \infty$. From Theorem 1, there is a smooth $(0,r)$ – form u on Ω such that $\|u\|^{(n)}_{L^p_{(0,r)}(\Omega)} < \infty$ and $\bar{\partial} u = f$. Define $c_i'' = c_i' - u$ on $U_i \cap \Omega$ for each $i \epsilon I$. Then

$\bar{\partial} c'' = \bar{\partial} c_i' - \bar{\partial} u = \bar{\partial} c_i' - f = \partial c_i' - \bar{\partial} c_i' = 0$ on $\Omega \cap U_i$ and

$$\|c''_i\|^{(n)}_{L^p_{(0,r)}(\Omega\cap U_i)} < \infty.$$

Therefore if $c'' = (c''_i)$, then $c'' \in C^0_p(\mathfrak{U}_\Omega,F_r)$ and

$$(\delta c'')_{ij} = c''_j - c''_i = c'_j - c'_i = c_{ij}.$$

Therefore $\delta c'' = c$ and $H^1_p(\mathfrak{U}_\Omega,F_r) = 0$.

3. To continue with the proof of Theorem 2, for each $\alpha = (i_0,\ldots, i_q)$ if $\gamma: \Pi_p(\Omega_\alpha,F_r) \to \Lambda_p(\Omega_\alpha,E_r)$ is the inclusion map, since each Ω_α is a polycylinder, there is, from Theorem 1, the following exact sequence

$$0 \longrightarrow \Pi_p(\Omega_\alpha,F_r) \xrightarrow{\gamma} \Lambda_p(\Omega_\alpha,E_r) \xrightarrow{\bar\partial} \Pi_p(\Omega_\alpha,F_{r+1}) \longrightarrow 0$$

for each α. From this we get the following exact sequence

$$0 \longrightarrow C^q_p(\mathfrak{U}_\Omega,F_r) \xrightarrow{\gamma} D^q_p(\mathfrak{U}_\Omega,E_r) \xrightarrow{\bar\partial} C^q_p(\mathfrak{U}_\Omega,F_{r+1}) \longrightarrow 0. \qquad (7)$$

And from (7) we have the following long exact sequence

$$\cdots \longrightarrow H^q_p(\mathfrak{U}_\Omega,F_r) \longrightarrow G^q_p(\mathfrak{U}_\Omega,E_r) \longrightarrow H^q_p(\mathfrak{U}_\Omega,F_{r+1}) \longrightarrow \qquad (8)$$
$$\longrightarrow H^{q+1}_p(\mathfrak{U}_\Omega,F_r) \longrightarrow G^{q+1}_p(\mathfrak{U}_\Omega,E_r) \longrightarrow H^{q+1}_p(\mathfrak{U}_\Omega,F_{r+1}) \longrightarrow \cdots$$

Since $G^q_p(\mathfrak{U}_\Omega,E_r) = 0$ for every $q > 0$, it follows that

$$H^q_p(\mathfrak{U}_\Omega,F_{r+1}) \cong H^{q+1}_p(\mathfrak{U}_\Omega,F_r) \quad \text{for } q \geq 1.$$

In particular

$$H^q_p(\mathfrak{U}_\Omega,\mathcal{O}) = H^q_p(\mathfrak{U}_\Omega,F_0) \cong H^{q-1}_p(\mathfrak{U}_\Omega,F_1) \cong\cdots\cong H^1_p(\mathfrak{U}_\Omega,F_{q-1}) \qquad (9)$$

Therefore from lemma 2

$$H^q_p(\mathfrak{U}_\Omega,\mathcal{O}) = 0 \quad \text{for } q \geq 1. \qquad (10)$$

Since every finite open covering of $\overline{\Omega}$ has a refinement whose members are polycylinders, if B^p_0 is the presheaf of L^p – holomorphic functions on $\overline{\Omega}$, (10) implies that

$$H^q(\overline{\Omega}, B^p_0) = 0$$

Therefore, since $H^q(\overline{\Omega}, B_p) \cong H^q(\overline{\Omega},B^p_0)$ we get

$$H^q(\overline{\Omega},B_p) = 0 \quad \text{for } q \geq 1 \text{ and } 1 \leq p \leq \infty.$$

§4. LERAY'S THEOREM WITH L^p – BOUNDS

1. To prove Theorem 5, let X be an open set in $\mathbb{C}^n$ and $\mathfrak{U} = \{U_i\}_{i\in I}$ be a locally finite covering of X by polycylinders each of which is relatively compact in X, and $\mathfrak{F}$ a coherent analytic sheaf on X. Let $\sigma = \{U_{i_0},\ldots,U_{i_r}\}$ be in the nerve of $\mathfrak{U}$ so that the support $|\sigma| = U_{i_0}\cap\ldots\cap U_{i_r}$ of σ is not empty and let $\mathfrak{U}_{|\sigma|} = \{|\sigma| \cap U_i\}_{i\in I}$, then $\mathfrak{U}_{|\sigma|}$ is a finite covering of the closure of $|\sigma|$ by polycylinders. First we show that $H^q_p(\mathfrak{U}_{|\sigma|}, \mathfrak{F}) = 0$ for all $q \geq 1$ and $1 \leq p \leq \infty$:

As in [4] and [5], there is a terminating chain of syzygies

$$0 \longrightarrow \mathfrak{O}^{P_r} \xrightarrow{N_r} \mathfrak{O}^{P_{r-1}} \longrightarrow \cdots \longrightarrow \mathfrak{O}^{P_0} \xrightarrow{N_0} \mathfrak{F} \longrightarrow 0 \qquad (11)$$

in a neighborhood of the closure of $|\sigma|$, where $\mathfrak{O}$ is the structure sheaf on $\mathbb{C}^n$ and r is a natural number. We use induction on the length r of the terminating chain of syzygies. When r = 0 the exact sequence (11) reduces to

$$0 \longrightarrow \mathfrak{O}^{P_0} \xrightarrow{N_0} \mathfrak{F} \longrightarrow 0$$

Thus, in this case we need only show that $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^m) = 0$ for $q \geq 1$, $m \geq 1$. This is done by induction on m. When m = 1, we know from (10) that $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}) = 0$. When m > 1 from the exact sequence of sheaves

$$0 \longrightarrow \mathfrak{O} \longrightarrow \mathfrak{O}^m \longrightarrow \mathfrak{O}^{m-1} \longrightarrow 0 \qquad (12)$$

we get for each $\beta = (\beta_0, \ldots, \beta_q)$ where $U'_\beta = |\sigma| \cap U_{\beta_0} \cap \ldots \cap U_{\beta_q}$ an exact sequence

$$0 \longrightarrow \Gamma_p(U'_\beta, \mathfrak{O}) \longrightarrow \Gamma_p(U'_\beta, \mathfrak{O}^m) \longrightarrow \Gamma_p(U'_\beta, \mathfrak{O}^{m-1}) \longrightarrow 0, \qquad (13)$$

having used the fact that $\Gamma_p(U'_\beta, \mathfrak{O}^{m-1})$ defined by using the exact sequence

$$\mathfrak{O}^m \longrightarrow \mathfrak{O}^{m-1} \longrightarrow 0$$

coincides with the original definition.

From (13) there is the exact sequence

$$0 \longrightarrow C_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}) \longrightarrow C_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^m) \longrightarrow C_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^{m-1}) \longrightarrow 0. \qquad (14)$$

From (14) there is a long exact sequence of L^p – bounded cohomology groups

$$\ldots \longrightarrow H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}) \longrightarrow H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^m) \longrightarrow H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^{m-1}) \longrightarrow \qquad (15)$$
$$\longrightarrow H_p^{q+1}(\mathfrak{U}_{|\sigma|}, \mathfrak{O}) \longrightarrow \ldots$$

$H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}) = H_p^{q+1}(\mathfrak{U}_{|\sigma|}, \mathfrak{O}) = 0$, hence $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^m) \cong H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^{m-1})$;

Thus by induction, $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^m) = 0$ for all $q \geq 1$, $m \geq 1$, $1 \leq p \leq \infty$.

To conclude the proof of $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{F}) = 0$ for all $q \geq 1$ assume that $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{G}) = 0$ for all $q \geq 1$, when $\mathfrak{G}$ is a coherent analytic sheaf which has a terminating chain of syzygies of length $\leq r - 1$. The exact sequence (11) can be reduced to the two shorter exact sequences

$$0 \longrightarrow \mathfrak{O}^{P_r} \xrightarrow{N_r} \mathfrak{O}^{P_{r-1}} \longrightarrow \cdots \longrightarrow \mathfrak{O}^{P_1} \xrightarrow{N_1} \mathfrak{R} \longrightarrow 0 \qquad (16)$$
$$0 \longrightarrow \mathfrak{R} \longrightarrow \mathfrak{O}^{P_0} \xrightarrow{N_0} \mathfrak{F} \longrightarrow 0,$$

where $\mathfrak{R}$ is the kernel of N_0. By the induction hypothesis $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{R}) = 0$ for $q \geq 1$. From the short exact sequence in (16) it is easy to see that we have a long exact sequence

$$\ldots \longrightarrow H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^{P_0}) \longrightarrow H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{F}) \longrightarrow H_p^{q+1}(\mathfrak{U}_{|\sigma|}, \mathfrak{R}) \longrightarrow \ldots \qquad (17)$$

since, also $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{O}^{P_0}) = 0$ for all $q \geq 1$, we get the desired result that $H_p^q(\mathfrak{U}_{|\sigma|}, \mathfrak{F}) = 0$.

2. Since the cover $\mathfrak{U}$ of X is acyclic with respect to the coherent analytic sheaf $\mathfrak{F}$, the canonical alternating resolution of $\mathfrak{F}$ relative to the cover $\mathfrak{U}$.

$$0 \longrightarrow \mathfrak{F} \xrightarrow{i} \mathfrak{S}_0 \xrightarrow{d_0} \mathfrak{S}_1 \xrightarrow{d_1} \ldots \longrightarrow \mathfrak{S}_r \xrightarrow{d_r} \mathfrak{S}_{r+1} \longrightarrow \ldots$$

is acyclic and we use this resolution to compute the cohomology groups of X with values in $\mathfrak{F}$, up to isomorphisms. Also because $\mathfrak{U}$ is locally finite each $\mathfrak{S}_r$, $r \geq 0$ is a coherent analytic sheaf. Now $H_p^q(\mathfrak{U}_{|\sigma|},\mathfrak{F}) = 0$ for $q \geq 1$, $1 \leq p \leq \infty$ and all σ in the nerve of $\mathfrak{U}$, implies that the following two sequences are exact:

$$0 \longrightarrow C_p^q(\mathfrak{U},\mathfrak{F}) \xrightarrow{j^*} C_p^q(\mathfrak{U},\mathfrak{S}_0) \xrightarrow{d_0^*} C_p^q(\mathfrak{U},\mathfrak{S}_1) \xrightarrow{d_1^*} C_p^q(\mathfrak{U},\mathfrak{S}_2) \longrightarrow \ldots \quad (18)$$

$$0 \longrightarrow \Gamma(X,\mathfrak{S}_r) \longrightarrow C_p^0(\mathfrak{U},\mathfrak{S}_r) \xrightarrow{\delta} C_p^1(\mathfrak{U},\mathfrak{S}_r) \xrightarrow{\delta} C_p^2(\mathfrak{U},\mathfrak{S}_r) \longrightarrow \ldots \quad (19)$$

The two sets of sequences (18) and (19) can be written in the following double complex

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Gamma(X,\mathfrak{F}) & \xrightarrow{j^*} & \Gamma(X,\mathfrak{S}_0) & \xrightarrow{d_0^*} & \Gamma(X,\mathfrak{S}_1) & \xrightarrow{d_1^*} & \Gamma(X,\mathfrak{S}_2) & \xrightarrow{d_2^*} \ldots \\
 & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & \\
0 \longrightarrow & C_p^0(\mathfrak{U},\mathfrak{F}) & \xrightarrow{j^*} & C_p^0(\mathfrak{U},\mathfrak{S}_0) & \xrightarrow{d_0^*} & C_p^0(\mathfrak{U},\mathfrak{S}_1) & \xrightarrow{d_1^*} & C_p^0(\mathfrak{U},\mathfrak{S}_2) & \xrightarrow{d_2^*} \ldots \\
 & \downarrow\delta & & \downarrow\delta & & & & \downarrow\delta \quad \downarrow\delta & \\
0 \longrightarrow & C^1(\mathfrak{U},\mathfrak{F}) & \xrightarrow{j^*} & C^1(\mathfrak{U},\mathfrak{S}) & \xrightarrow{d_0^*} & C_p^1(\mathfrak{U},\mathfrak{S}_1) & \xrightarrow{d_1^*} & C_p^1(\mathfrak{U},\mathfrak{S}_2) & \xrightarrow{d_2^*} \ldots \\
 & \downarrow & & \downarrow & & & & \downarrow \quad \downarrow & \\
 & \vdots & & \vdots & & & & \vdots \quad \vdots &
\end{array}$$

In this double complex, all rows except the first are exact and all columns except the first are exact and the whole diagram is commutative. Therefore as it is well known the natural map of $H_p^q(\mathfrak{U},\mathfrak{F})$ into the qth cohomology group of the complex which is the first row is an isomorphism. That is to say, the natural map

$$H_p^q(\mathfrak{U},\mathfrak{F}) \longrightarrow H^q(X,\mathfrak{F})$$

is an isomorphism for $q \geq 0$. ◁

ACKNOWLEDGEMENT: This research was partly sponsored by the NSF grant INT–#8914788.

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