

## ACOUSTIC-GRAVITY WAVES IN A VISCOUS AND THERMALLY CONDUCTING ISOTHERMAL ATMOSPHERE

(Part II: For Small Prandtl Number)

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**ABSTRACT:** In part one of these series we investigated the effect of Newtonian cooling on acoustic-gravity waves in an isothermal atmosphere for large Prandtl number. It was shown that the atmosphere can be divided into two regions connected by an absorbing and reflecting layer, created by the exponential increase of the kinematic viscosity with height, and if Newtonian cooling coefficient goes to infinity the temperature perturbation associated with the wave will be eliminated. In addition all linear relations among the perturbation quantities will be modified. In this paper we will consider the effect of Newtonian cooling on acoustic-gravity waves for small Prandtl number in an isothermal atmosphere. It is shown that if the Newtonian cooling coefficient is small compared to the adiabatic cutoff frequency the atmosphere may be divided into three distinct regions. In the lower region the motion is adiabatic and the effect of the kinematic viscosity and thermal diffusivity are negligible, while the effect of these diffusivities is more pronounced in the upper region. In the middle region the effect of the thermal diffusivity is large, while that of the kinematic viscosity is still negligible. The two lower regions are connected by a semitransparent reflecting layer as a result of the exponential increase of the thermal diffusivity with height. The two upper regions are joined by an absorbing and reflecting barrier created but the exponential increase of the kinematic viscosity. If the Newtonian cooling coefficient is large compared to the adiabatic cutoff frequency, the wavelengths below and above the lower reflecting layer will be equalized. Consequently the reflection produced by the thermal conduction is eliminated completely. This indicates that in the solar photosphere the temperature fluctuations may be smoothed by the transfer of radiation between any two regions with different temperatures. Also the heat transfer by radiation is more dominant than the conduction process.

**KEY WORDS:** Acoustic-Gravity Waves, Atmospheric Waves, Wave Propagation

**AMS SUBJECT CLASSIFICATION CODES.** 76N,76Q

### 1 INTRODUCTION

Upward propagating small amplitude acoustic-gravity waves in an isothermal atmosphere will be reflected downward if the gas is viscous or thermally conducting. This type of reflection is most significant when the wavelength is large compared to the density scale height ( Alkahby and Yanowitch [1989, 1991], Campos [1983a, 1983b], Cally [1984], Lindzen [1968, 1970], Webb and Roberts [1980], Lyons and Yanowitch [1974], Priest [1984], Yanowitch [1967a, 1967b, 1979], Zhugzhda and Dzhallov [1986]).

In part one of this series we considered the the effect of Newtonian cooling on acoustic-gravity waves in an isothermal atmosphere for large Prandtl number. It was shown that for an arbitrary value of the Newtonian cooling coefficient the atmosphere may be divided into two distinct regions, which are connected by an absorbing and reflecting layer produced by the exponential increase of the kinematic viscosity with

height. In the lower region the motion is adiabatic if the Newtonian cooling coefficient is small compared to the adiabatic cutoff frequency. On the other hand, if the Newtonian cooling coefficient is sufficiently large the motion will be driven towards an isothermal one. Consequently all linear relations among perturbation quantities will be modified. In particular, it decreases the amplitude of the wave and thereby the energy flux as well. In the upper region the solution will decay exponentially with altitude before it is influenced by the effect of the thermal conduction.

In this paper we will study the effect of Newtonian cooling on upward propagating acoustic gravity waves in an isothermal atmosphere for small Prandtl number. It is shown that, if the Newtonian cooling coefficient is small compared to the adiabatic cutoff frequency, the atmosphere may be divided into three different regions. In the lower region the effect of thermal diffusivity and kinematic viscosity is negligible, the oscillatory process is adiabatic and for frequencies greater than the adiabatic cutoff frequency the solution can be written as a linear combination of an upward and a downward travelling wave. In the middle region the effect of the thermal diffusivity is large while that of the kinematic viscosity is still negligible. Consequently, the motion in the middle region is isothermal and these two regions are connected by a semitransparent reflecting layer, allowing part of the energy to be transmitted upward, while the remaining part is reflected downward. The reflecting layer now separates two regions with different sound speeds, and therefore different wavelengths, which account for the reflection process. In the upper region, where the kinematic viscosity and thermal diffusivity are large, the amplitude of the velocity oscillations approaches a constant value. The two upper regions are connected by an absorbing and reflecting layer, through which the kinematic viscosity changes from small to large value. The existence of two reflecting layers will influence the reflection process in the lower region and the final conclusion depends on their relative locations.

When Newtonian cooling is large compared to the adiabatic cutoff frequency the oscillatory process in the adiabatic region is driven towards an isothermal one and this will decrease the wavelength from the adiabatic to the isothermal values. Thus, the wavelengths below and above the lower reflecting layer are equalized. As a result, the reflection, produced by the thermal conduction, is eliminated altogether. This indicates that Newtonian cooling influences only the adiabatic regions in the atmosphere and if the heat exchange, due to radiation, is intense the temperature perturbation associated with the wave will be eliminated in a time small compared to the period of oscillation. Consequently, the effect of thermal conduction will be excluded.

From the above discussion we may conclude that in the solar photosphere the temperature fluctuations associated with vertically propagating acoustic-gravity waves may be evened out by the transfer of radiation between any two regions with different temperatures. Also, high emissivities in the presence of an open boundary allow rapid loss of radiation to space. In addition this result indicates that the heat transfer by radiation is more dominant than that of the conduction process which is the case in the hot regions of the solar atmosphere.

We conclude by reconsidering the case of the effect of Newtonian cooling alone in section (3). Three ranges for the frequency are identified, above the adiabatic cutoff frequency, below the isothermal cutoff frequency and in between. The results of section (3) are used in section (4). Finally the problem in section (4) is described by a fourth-order differential equation which is solved by matching procedure, in which inner and outer expansions are matched in an overlapping domain.

## 2 MATHEMATICAL FORMULATION OF THE PROBLEM

We will consider an isothermal atmosphere, which is viscous and thermally conducting, and occupies the upper half-space  $z > 0$ . We will investigate the problem of small vertical oscillations about equilibrium, i.e. oscillations which depend only on the time  $t$  and on the vertical coordinate  $z$ .

Let the equilibrium pressure, density and temperature be denoted by  $P_0$ ,  $\rho_0$ , and  $T_0$ , where  $P_0$  and  $T_0$  satisfy the gas law  $P_0 = RT_0\rho_0$  and the hydrostatic equation  $P_0' + g\rho_0 = 0$ . Here  $R$  is the gas constant,  $g$  is the gravitational acceleration and the prime denotes differentiation with respect to  $z$ . The equilibrium pressure and density,

$$P_0(z) = P_0(0)\exp(-z/H), \quad \rho_0(z) = \rho_0(0)\exp(-z/H),$$

where  $H = RT_0/g$  is the density scale height

Let  $p$ ,  $\rho$ ,  $w$ , and  $T$  be the perturbations in the pressure, density, vertical velocity, and temperature. The linearized equations of motion are:

$$\rho_0 w_t + p_z + g\rho = (4/3)\mu w_{zz}, \quad (2.1)$$

$$\rho_t + (\rho_0 w)_z = 0, \quad (2.2)$$

$$\rho_0(c_V(T_t + qT) + gHw_z) = \kappa T_{zz}, \quad (2.3)$$

$$p = R(\rho_0 T + T_0 \rho). \quad (2.4)$$

These are, respectively, the equation for the change in the vertical momentum, the mass conservation equation, the equation for the rate change of the x-component of the magnetic field, the heat flow equation and the gas law. Here  $c_V$  is the specific heat at constant volume,  $q$  is the Newtonian cooling coefficient which refers to the heat exchange and  $\kappa$  is the thermal conductivity, all assumed to be constants. The subscript  $z$  and  $t$  denote differentiation with respect to  $z$  and  $t$  respectively. Equation (4) includes the heat flux term  $c_V \rho_0 q T$ , which comes from the linearized form of the Stefan-Boltzman law. We will consider solutions which are harmonic in time i.e.  $w(z, t) = W(z)\exp(-i\omega t)$  and  $T(z, t) = T(z)\exp(-i\omega t)$ , where  $\omega$  denotes the frequency of the wave.

It is more convenient to rewrite the equations in dimensionless form;  $z' = z/H$ ,  $\omega_a = c/2H$ ,  $W' = w/c$ ,  $\omega' = \omega/\omega_a$ ,  $t' = t\omega_a$ ,  $\kappa' = 2\kappa/c_V c H \rho_0(0)$ ,  $T' = T/2\gamma T_0$ ,  $q' = q/\omega_a$ , where  $c = \sqrt{\gamma R T_0} = \sqrt{\gamma g H}$  is the adiabatic sound speed, and  $\omega_a$  is the adiabatic cutoff frequency. The primes can be omitted, since all variables will be written in dimensionless form from now on.

One can eliminate  $p$ , and  $\rho$  from equation (1) by differentiating it with respect to  $t$  and substituting equations (2.2-2.5) to obtain a system of differential equations for  $W(z)$  and  $T(z)$ ,

$$(D^2 - D + \gamma\omega^2/4)W(z) + \gamma\mu\epsilon^2 D^2 W(z) + i\gamma(D - 1)T(z) = 0, \quad (2.5)$$

$$(\gamma - 1)DW(z) = \gamma(i\omega - q)T(z) + \gamma\kappa\epsilon^2 D^2 T(z), \quad (2.6)$$

where  $D = d/dz$ . If, furthermore,  $W(z)$  is eliminated from the differential equation (2.6) by applying  $D$  to it and substituting for  $DW(z)$  from differential equation (2.7) one obtains a single fourth-order differential equation for  $T(z)$ :

$$\begin{aligned} & [\gamma\omega(D^2 - D + \omega^2/4) + i\gamma(D^2 - D + \gamma\omega^2/4) - i\kappa\epsilon^2 D^2(D^2 + D + \gamma\omega^2/4) \\ & - i\gamma(\omega + iq)\mu\epsilon^2(D^2 + D) - \gamma\mu\kappa\epsilon^2 D^2(D + 1)(D + 2)]T(z) = 0 \end{aligned} \quad (2.7)$$

In addition it is convenient to introduce the dimensionless Prandtl number  $P_r = \mu/\kappa$ , which measures the relative strength of the the viscosity with respect to the thermal conduction. Consequently the differential equation (2.8) becomes

$$\begin{aligned} & \gamma\omega(D^2 - D + \omega^2/4) + iq(D^2 - D + \gamma\omega^2/4) - i\kappa e^z D^2(D^2 + D + \gamma\omega^2/4) \\ & - iP_r\gamma(\omega + iq)\kappa e^z D(D+1) - \gamma P_r(\kappa e^z)^2 D^2(D+1)(D+2)T(z) = 0 \end{aligned} \quad (2.8)$$

Finally the first two terms of equation (2.9) may be combined to give the following equation,

$$\begin{aligned} & i(D^2 - D + \tau\omega^2/4) - i(\kappa/m)e^z D^2(D^2 + D + \gamma\omega^2/4) \\ & - iP_r m\tau(\kappa/m)e^z D(D+1) - (\gamma P_r m)(\kappa e^z/m)^2 D^2(D+1)(D+2)T(z) = 0 \end{aligned} \quad (2.9)$$

where  $\tau = \gamma(\omega + iq)/(\gamma\omega + iq) = \gamma(\omega + iq)/m$  and  $m = \gamma\omega + iq$ .

*Boundary Conditions* : To complete the formulation of the problem certain boundary conditions must be imposed to ensure a unique solution. The exact nature of the exciting force need not be specified since our object is to investigate the reflection and dissipation of the waves which, for small  $\kappa$  and  $Pr$ , take place at a high altitude.

Boundary conditions are required at  $z = 0$ , and we shall adopt the lower boundary condition (LBC): In a fixed interval  $0 < z < z_0$ , the solution of the differential equation (2.10) should approach some solution of the limiting differential equation ( $\kappa \rightarrow 0$  and  $\mu \rightarrow 0$ ),

$$[D^2 - D + \tau\omega^2/4]T(z) = 0. \quad (2.10)$$

Considering the lower boundary condition is simpler than prescribing  $T(z)$  and  $W(z)$  at  $z = 0$ . To first order the boundary layer has no effect on the reflection and dissipation process, which takes place at a high altitude.

Two further conditions which refer to the behaviour of the solutions for large  $z$  are required and we shall call these conditions the upper boundary conditions. The first one is the Entropy Condition (EC), which is determined by the equation for the rate of change of the entropy (see Alkahby and Yanowitch[1991], Lyons and Yanowitch[1974]). From which it follows that

$$\kappa \int_0^\infty |T_z|^2 dz < \infty. \quad (2.11)$$

The second condition is the Dissipation Condition (DC), which requires the finiteness of the rate of change of the energy dissipation in an infinite column of fluid of unit cross-section ( Alkahby and Yanowitch[1991], Campos[1983a,1983b], Lyons and Yanowitch[1974]). Since the dissipation function depends on the squares of the velocity gradients, the dissipation condition is equivalent to

$$\mu \int_0^\infty |W_z|^2 dz < \infty, \quad (2.12)$$

Both of the upper boundary conditions are necessary and sufficient as an upper boundary condition if  $\mu, \kappa > 0$ . Finally if  $\kappa = \mu = 0$  the Radiation Condition is sufficient to ensure a unique solution.

### 3 THE EFFECT OF NEWTONIAN COOLING ALONE

In this section we will consider the effect of Newtonian cooling alone on acoustic-gravity waves in an ideal atmosphere. The results will be used in section (4). For this case, the differential equation can be obtained by setting  $\kappa = \mu = 0$  in the differential equation (2.10). The resulting differential equation is

$$[D^2 - D + \tau\omega^2/4]T(z) = 0. \quad (3.1)$$

where  $\tau = \gamma(\omega + iq)/m$ , and  $m = \gamma\omega + iq$ . The solution of the above differential equation can be written in the following form

$$T(z) = c_1 \exp[(1 + \sqrt{1 - \tau\omega^2})z/2] + c_2 \exp[(1 - \sqrt{1 - \tau\omega^2})z/2], \tag{3.2}$$

where  $c_1$  and  $c_2$  are constants and they will be determined from the boundary condition. To investigate the nature of the effect of Newtonian cooling on the wave propagation in an isothermal atmosphere, it is convenient to consider the following two limiting cases

**CASE A:** when  $q = 0$ , the parameter  $\tau$  reduces to 1 and for frequencies greater than the adiabatic cutoff frequency  $\omega_a = 1$ , equation (14) has the following form

$$T(z) = c_1 \exp[(1/2 + ik_a)z] + c_2 \exp[(1/2 - ik_a)z], \tag{3.3}$$

where  $2k_a = \sqrt{\omega^2 - 1}$  is the adiabatic wave number.

**CASE B:** when  $q \rightarrow \infty$ , the parameter  $\tau \rightarrow \gamma$  and for frequencies greater than the isothermal cutoff frequency  $\omega_i = 1/\sqrt{\gamma}$ , the solution of the differential equation (13) can be written like

$$T(z) = c_1 \exp[(1/2 + ik_i)z] + c_2 \exp[(1/2 - ik_i)z], \tag{3.4}$$

where  $2k_i = \sqrt{\gamma\omega^2 - 1}$  is the isothermal wave number.

To investigate the effect of Newtonian cooling on the behaviour of the wave propagation for an arbitrary value of  $q$ , it is convenient to write  $\tau$  in the following form

$$\tau = [(\gamma + 1) - (\gamma - 1)(\cos 2\theta_q - i \sin 2\theta_q)]/2 \tag{3.5}$$

where  $\theta_q = \arctan(q/\omega)$ ,  $0 < \theta_q < \pi/2$ . For  $\omega > \omega_a$  we have

$$\sqrt{1 - \tau\omega^2}/2 = \pm(-s(q) + ik_a), \tag{3.6}$$

where  $s(q)$  is the attenuation factor. Consequently the solution in equation (3.2) becomes

$$T(z) = c_1 \exp[(1/2 - s(q) + ik_i)z] + c_2 \exp[(1/2 + s(q) - ik_i)z], \tag{3.7}$$

To obtain the behaviour of  $s(q)$ , it is convenient to write

$$1 - \tau\omega^2 = [(1 - (\gamma + 1)\omega^2/2) + [(\gamma - 1)\omega^2(\cos 2\theta_q - i \sin 2\theta_q)]/2], \tag{3.8}$$

It is clear that equation (3.8) represents a semicircle in the complex plane with center at  $1 - (\gamma + 1)\omega^2/2$  and radius  $(\gamma - 1)\omega^2/2$  as  $\theta_q$  varies from 0 to  $\pi/2$ .

It follows from equation (3.7) that the solution can be described in the following way: the first term on the right represents an upward travelling wave, its amplitude decaying with altitude like  $\exp(-s(q)z)$ , while the second term is a downward travelling wave decaying at the same rate. We have to indicate the the upper boundary conditions (2.11) and (2.12) cannot be applied because  $\mu = \kappa = 0$ . A unique solution can be determined by the radiation condition which requires  $c_2 = 0$ . Also there are three ranges for the frequency  $\omega$ . The first one is for  $\omega > \omega_a = 1$ , the second one for  $1/\sqrt{\gamma} = \omega_i < \omega < \omega_a$ , and the third one for  $\omega < \omega_i$ . They are denoted, respectively, by  $R_1$ ,  $R_2$  and  $R_3$ .

From this observation and the three ranges of the frequency we have the following conclusions.

(A) When  $\omega$  belongs to  $R_1$  the attenuation factor  $s(q)$  is positive and equals to zero at the extreme limits  $q = 0$  and  $q \rightarrow \infty$ . It increases to its maximum value,  $s(q) = 0.1$ , when  $(q/\omega) = O(1)$  and decays to zero as  $q \rightarrow \infty$ .

( B ) When the frequency in  $R_2$  and for small value of  $q$ , a decaying wave exists and changes to undamped travelling wave as  $q \rightarrow \infty$

( C ) If  $\omega$  belongs to  $R_3$  and for small value of  $q$ , a weak damped wave exists. As  $q \rightarrow \infty$  the travelling wave changes to a standing one

( D ) In (A), (B), and (C) the wave number increases monotonically from its adiabatic value  $k_a$  to the isothermal one  $k_i$ , as  $q \rightarrow \infty$ , because of the change of the sound speed from its adiabatic value to the isothermal one. Thus the attenuation factor remains positive,  $s(q) \geq 0$ , for all values of  $q$ . At the same time the oscillatory process is transformed from the adiabatic form to the isothermal one.

#### 4 EFFECTS OF THERMAL CONDUCTION, VISCOSITY AND NEWTONIAN COOLING

In this section we will investigate the singular perturbation boundary value problem for the following differential equation

$$\begin{aligned} & [(D^2 - D + \tau\omega^2/4) - \nu(\kappa/m)e^z D^2(D^2 + D + \gamma\omega^2/4) \\ & - \nu P_r m \tau (\kappa/m) e^z D(D+1) - (\gamma P_r m)(\kappa e^z/m)^2 D^2(D+1)(D+2)]T(z) = 0, \end{aligned} \quad (4.1)$$

where  $\tau = \gamma(\omega + \nu q)/(\gamma\omega + \nu q) = \gamma(\omega + \nu q)/m$  and  $m = \gamma\omega + \nu q$ , subjected to the boundary condition (2.11), (2.12), and the lower boundary condition. At the outset we have to indicate that the parameters  $\mu$  and  $\kappa$  are sufficiently small and proportional to the values at  $z=0$  of the kinematic viscosity and thermal diffusivity. Prandtl number  $P_r$  can be written like

$$P_r = \mu/\kappa = (\mu/\rho_0)/(\kappa/\rho_0) = |(\mu/m\rho_0)(\kappa/m\rho_0)| \quad (4.2)$$

Thus for small  $P_r$  we have  $\mu e^z \ll \kappa e^z$ . As a result, for  $|\kappa/m|e^z \ll 1$  and small values of  $q$  the atmosphere may be divided into three distinct regions connected by two different transition layers in which the reflection and the wave modification take place. In the lower region,  $0 < z \ll z_1 = -\log|m\kappa|$ , the motion is adiabatic, because the effect of thermal diffusivity is negligible, and the solution of the differential equation (4.1) can be approximated by the solution of the following differential equation

$$[D^2 - D + \tau\omega^2/4]T(z) = 0, \quad (4.3)$$

the solution of which is investigated in section (3). In the middle region,  $z_1 < z < z_2 = -\log\mu$ , the oscillatory process is an isothermal, because the influence of the thermal diffusivity is large, and the solution of the differential equation (4.1) can be approximated by the solution of

$$[D^2(D^2 + D + \gamma\omega^2/4)]T(z) = 0. \quad (4.4)$$

The two lower regions are connected by a semitransparent reflecting layer in the vicinity of  $z_1$ . In the upper region the oscillatory process is influenced by the combined effect of the thermal conduction and the viscosity and the solution of the differential equation (4.1) can be approximated by the solution of the following differential equation

$$[D^2(D+1)(D+2)]T(\xi) = 0. \quad (4.5)$$

The two upper regions are joined by an absorbing and reflecting transition region in the vicinity of  $z_2 = -\log(\gamma\mu)$ , above it the solution which satisfies the upper boundary conditions must behave as a constant as  $z \rightarrow \infty$ .

To obtain the solution of the differential equation (4.1) it is convenient to introduce a new dimensionless

variable  $\xi$  defined by

$$\xi = \exp(-z)/(\kappa/m) = \exp[-z - \log|\kappa/m| + i\theta_m + 3\pi i/2], \tag{4.6}$$

where  $\theta_m = \arg(m)$ , which transforms the differential equation (4.1) into

$$\begin{aligned} & \{\xi^2(\theta^2 + \theta + \tau\omega^2/4) - \xi\theta^2(\theta^2 - \theta + \gamma\omega^2/4) \\ & - mP_r\tau\xi(\theta^2 - \theta) + \gamma P_r m\theta^2(\theta - 1)(\theta - 2)\}T(\xi) = 0. \end{aligned} \tag{4.7}$$

where  $\theta = \xi d/d\xi$ . It is clear that the point  $\xi = 0$  corresponds to  $\lim z = \infty$ , the point  $\xi_0 = \exp[-\log|\kappa/m| + i(\theta_m + 3\pi/2)]$  to  $z = 0$  and the segment connecting these points in the complex  $\xi$ -plane to  $z \geq 0$ . As  $|\kappa/m| \rightarrow 0$  the point  $\xi_0$  tends to  $\infty$ .

It is clear that the point  $\xi = 0$  is a regular singular point of this differential equation (4.7). Consequently, there are four linearly independent solutions, which in the neighbourhood of  $\xi = 0$  can be written in the following form

$$\begin{aligned} T_1(\xi) &= \sum a_n(e_1)\xi^{n+e_1}, & T_2(\xi) &= \sum a'_n(e_2)\xi^{n+e_2} + T_1(\xi)\log(\xi), \\ T_3(\xi) &= \sum a''_n(e_3)\xi^{n+e_3}, & T_4(\xi) &= \sum a'''_n(e_4)\xi^{n+e_4} + T_3(\xi)\log(\xi), \end{aligned} \tag{4.8}$$

where  $e_1 = 2, e_2 = 1, e_3 = e_4 = 0$ . The prime denotes differentiation of  $a_n$  and the sums are taken from  $n = 0$  to  $n = \infty$ . The coefficients  $a_n(e_i)$  are determined from the following three term recursion formula

$$p_0(n + 2 + e)a_{n+2} + p_1(n + 1 + e)a_{n+1} + p_2(n + e)a_n = 0, \tag{4.9}$$

where

$$\begin{aligned} p_0(x) &= \gamma P_r m x^2(x - 1)(x - 2), \\ p_1(x) &= -m\tau P_r x(x - 1) - x^2(x^2 - x + \gamma\omega^2/4), \\ p_2(x) &= (x^2 + x + \tau\omega^2/4). \end{aligned}$$

To determine which of the solutions defined in equation (4.8) satisfies the upper boundary conditions (2.11) and (2.12) for large  $z$ , the solutions must be transformed to the variable  $z$  by means of (4.6). Thus for  $|\kappa/m| > 0$  and  $P_r > 0$  we have

$$T_1(z) = O(e^{-2z}), \quad T_2(z) = O(e^{-z}), \quad T_3(z) = O(1), \quad T_4(z) = O(z). \tag{4.10}$$

It is clear that  $T_4(z)$  is the only solution which does not satisfy the entropy condition (2.11).

To apply the dissipation condition (2.12), equation (2.7) must be used to determine the the amplitudes of the velocity corresponding to the solutions defined in (4.8). As a result we have

$$DW_1(z) = O(e^{-z}), \quad DW_2(z) = O(1), \quad DW_3(z) = O(e^{-z}), \quad DW_4(z) = O(z). \tag{4.11}$$

It is clear that  $W_2(z)$  and  $DW_4(z)$  do not satisfy the boundary condition (2.12). As a result, we obtain

$$T(z) = c_1 T_1(z) + c_2 T_3(z) \quad W(z) = c'_1 W_1(z) + c'_2 W_3(z). \tag{4.12}$$

To determine the linear combination of  $T(z)$  in equation(4.8), the behaviour of  $T_1(z)$  and  $T_3(z)$  for small  $z$  must be found. Recall that small  $z$  corresponds to large  $|\xi|$  with  $\arg(\xi) = 3\pi/2 + \theta_m$ . Thus the asymptotic expansions of  $T_1(\xi)$  and  $T_3(\xi)$  about infinity should be found.

The point  $\xi = \infty$  is an irregular singular point of the differential equation(4.7), and there are four linearly independent solutions whose asymptotic behaviour, to the first order, is governed by

$$T_1'(\xi) \sim \xi^{\alpha_1}[1 + h_{11}\xi^{-1} + h_{12}\xi^{-2} + \dots], \tag{4.13}$$

$$T_2'(\xi) \sim \xi^{\alpha_2}[1 + h_{21}\xi^{-1} + h_{22}\xi^{-2} + \dots], \tag{4.14}$$

$$T_3'(\xi) \sim \xi^{-1/4}[1 + h_{31}\xi^{-1/2} + \dots]\exp(-\xi^{1/2}), \tag{4.15}$$

$$T_4'(\xi) \sim \xi^{-1/4}[1 + h_{41}\xi^{-1/2} + \dots]\exp(\xi^{1/2}), \tag{4.16}$$

where  $\alpha_1 = -1/2 - s(q) + ik_a$ ,  $\alpha_2 = -1/2 + s(q) - ik_a$ . Reintroducing the dimensionless variable  $z$  by means of (23) we have

$$T_1'(z) \sim \exp[(1/2 + s(q) - ik_a)z], \quad T_2'(z) \sim \exp[(1/2 - s(q) + ik_a)z],$$

$$T_3'(z) \sim \exp[-(|m/2\kappa|)^{1/2}z], \quad T_4'(z) \sim \exp[(|m/2\kappa|)^{1/2}z].$$

As a result of that  $T_1'(z)$  represents a downward propagating wave, its amplitude decaying like  $\exp(-s(q)z)$ , while  $T_2'(z)$  is an upward propagating wave decaying at the same rate. Also  $T_3'(z)$  corresponds to the boundary layer term. It is important only near  $z = 0$  for any small fixed value of  $z$  and it decays with  $z$  like  $\exp[-(|m|/4\kappa)^{1/2}z]$ . It follows that, for fixed value of  $z$ , Newtonian cooling reduces the width of the boundary layer. Thus it is more convenient to use the lower boundary condition, from which it follows that the solution of the differential equation (4.7), when  $\kappa \rightarrow 0$  and small value of  $q$  should behave asymptotically like a linear combination of  $T_1'(\xi)$  and  $T_2'(\xi)$ , i.e

The determination of the asymptotic behaviour of  $T_1(\xi)$  and  $T_3(\xi)$  from the solution (4.8) is difficult because the coefficients  $a_n(e_i)$  are determined from three terms recursion formula (4.9). Instead we will do it by matching procedure, in which the inner and the outer expansions for the solutions will be matched in an overlapping domain.

To find the inner approximation, assume there exists a regular perturbation expansion of the form

$$T(\xi) = I(\xi) + P_r I_1(\xi) + O(P_r^2). \tag{4.17}$$

Substituting (4.17) into the differential equation (4.7) and setting  $P_r = 0$ . We obtain the following differential equation

$$[(\xi^2 + \theta + \tau\omega^2/4) - \theta^2(\theta^2 - \theta + \gamma\omega^2/4)]I(\xi) = 0, \tag{4.18}$$

where  $\theta = \xi d/d\xi$ . The second step of the matching procedure begins with the stretching transformation,

$$\xi = P_r \psi, \tag{4.19}$$

of the complete differential equation (4.7). The resulting differential equation is:

$$[\gamma m \theta^2(\theta - 1)(\theta - 2) - \psi \theta^2(\theta^2 - \theta + \gamma \omega^2/4) + P_r \psi(\psi \theta^2 + \theta + \tau \omega^2/4) - m \tau(\theta^2 - \theta)]T(\psi) = 0, \tag{4.20}$$

where  $\theta = \psi d/d\psi$ . To obtain the outer approximation, assume that there exists a singular perturbation expansion of the following form

$$T(\psi) = U(\psi) + P_r U_1(\psi) + O(P_r^2), \tag{4.21}$$

substituting this expansion into the differential equation (4.7) and letting  $P_r = 0$ , we have

$$[\gamma m \theta^2(\theta - 1)(\theta - 2) - \psi \theta^2(\theta^2 - \theta + \gamma \omega^2/4)]U(\psi) = 0. \tag{4.22}$$

The solutions of this differential equation will approximate the solutions of the differential equation (4.20) if  $\xi = P_r \psi$  is small. Since  $\psi \rightarrow \infty$  for any fixed  $|\psi|$  as  $P_r \rightarrow 0$ , the asymptotic behaviour of  $U(\psi)$  as  $\psi \rightarrow \infty$  must be matched with the asymptotic behaviour of  $I(\xi)$  as  $\xi \rightarrow 0$ . Hence, the main task is the determination of the family of the solutions of the singular perturbation differential equation (4.22), which satisfy the upper boundary conditions (2.11) and (2.12). The differential equations (4.18) and (4.22) are similar to the differential equation (4.6) and (4.9) in Lyons and Yanowitch [1974]. The procedure for finding the asymptotic behaviour of the solution is the same. However, the physical nature of the solution is quite different in the two problems. Consequently the details need not be repeated, and we merely indicate the results.

Now we will study the behaviour of the solution in the atmosphere for  $z \geq 0$ . It is convenient to start with the upper region.

**UPPER REGION:** In this region the solution of the differential equation (4.1) is approximated by the solution of the differential equation (4.5). It follows from equation (4.10) that the solution which satisfies the upper boundary conditions must behave as a constant as  $z \rightarrow \infty$ .

**MIDDLE REGION:** In the middle the solution of the differential equation (4.1) is described by the solution of the differential equation (4.4), which can be written like

$$T(z) \sim \text{const. } \exp[(-1/2 + ik_i)z] + RC_m \exp[(-1/2 - ik_i)z], \tag{4.23}$$

where  $RC_m$  denotes the reflection coefficient in the middle region and defined by

$$RC_m = \exp[-\pi k_i + 2i(\theta_1 - k_i \log(\gamma\mu))], \tag{4.24}$$

$$\theta_1 = \arg(1/2 - ik_i) + \arg\Gamma(-2ik_i) + 2\arg\Gamma(3/2 - ik_i) \tag{4.25}$$

It is clear that  $|RC_m| = \exp(-\pi k_i)$ . Consequently the two upper regions are connected by an absorbing and reflecting layer in which the kinematic viscosity changes from a small to a large value because of the exponential decrease of the density with height. In addition the middle region will not influenced by the effect of Newtonian cooling.

**LOWER REGION:** For  $|\kappa/m|e^z \ll 1, \omega > \omega_a$  and small value of the Newtonian cooling coefficient  $q$ , the solution of the differential equation (4.1) can be written in the following form

$$T(z) \sim \text{const. } \exp[(1/2 - s(q) + ik_a)z] + RC_L \exp[(1/2 + s(q) - ik_a)z], \tag{4.26}$$

where  $RC_L$  denotes the reflection coefficient defined by

$$RC_L = \exp[-\pi k_a + A_* - iB_*] C_* \frac{L_1 - L_2 \exp[-2k_i \log P_r + i\theta^* + 2k_i \theta_m]}{L_3 - L_4 \exp[-2k_i \log P_r + i\theta^* + 2k_i \theta_m]}$$

$$A_* = 2s(q) \log|\kappa/m| - 2k_a \theta_m, \quad B_* = 2k_a \log|\kappa/m| + \pi s(q) + 2s(q) \theta_m,$$

$$C_* = -\frac{\Gamma^2(1/2 + s(q) - ik_a) \Gamma[s(q) - i(k_i + k_a)] \Gamma(-2s(q) + 2ik_a) \Gamma[1 + s(q) + i(k_i - k_a)]}{\Gamma^2(1/2 - s(q) + ik_a) \Gamma[-s(q) + i(k_i + k_a)] \Gamma(2s(q) - 2ik_a) \Gamma[1 - s(q) - i(k_i - k_a)]},$$

$$L_1 = \exp(2\pi k_i) - \exp(2\pi k_a) [\cos(2\pi s(q)) + i \sin(2\pi s(q))],$$

$$L_2 = \exp(2\pi k_a) [\cos(2\pi s(q)) + i \sin(2\pi s(q))] - \exp(-2\pi k_i),$$

$$L_3 = \exp(2\pi k_i) - \exp(-2\pi k_a) [\cos(2\pi s(q)) - i \sin(2\pi s(q))],$$

$$L_4 = \exp(-2\pi k_a) [\cos(2\pi s(q)) - i \sin(2\pi s(q))] - \exp(2\pi k_i),$$

$$D_* = - \frac{(1/2 - \nu k_i) \Gamma^2(2i k_i) \Gamma(1 + s(q) - \nu(k_i + k_a)) \Gamma(1 - s(q) - \nu(k_i - k_a))}{(1/2 + \nu k_i) \Gamma^2(-2i k_i) \Gamma(1 - s(q) + \nu(k_i + k_a)) \Gamma(1 + s(q) + \nu(k_i - k_a))},$$

$$\theta^* = \arg D_* - 2k_i \log \gamma.$$

In addition to the conclusions of part one we have the following observations

( I ) When  $q = 0$ , we have  $s(q) = A_* = \theta_m = 0$  and  $B_* = 2k_a \log(\kappa)$ , and we recover the result obtained in Lyons and Yanowitch [1974]. In this case the reflection process is more complicated because of the existence of two reflecting layers and the final conclusion depends on their relative locations. The magnitude of the reflection coefficient is

$$\exp(-\pi k_a) \leq |RCL| < 1.$$

(II ) For fixed  $\kappa$ , small  $q$  and  $\omega > \omega_a$  the solution is given by equation (4.26) and its behaviour is described in section (3). The atmosphere is divided into three distinct regions. The lower region is approximately adiabatic and the middle one is isothermal. In the upper region the solution is influenced by the combined effects of the thermal diffusivity and the kinematic viscosity.

( III ) When  $q \rightarrow \infty$  and  $\omega > \omega_i$  one obtains  $L_1 = L_4 = s(q) = \theta_m = A_* = 0$ ,  $k_a \rightarrow k_i$ ,  $C_* D_* \rightarrow \theta_i$ ,  $L_2 \rightarrow L_3$  and  $RCL \rightarrow RC_m$ . Consequently the lower reflecting layer will be eliminated because the wavelengths below and above the this layer become equal. In addition we have  $\kappa/m \rightarrow 0$  and the solution of the differential equation (4.1) can be approximated by the solution of the following differential equation

$$[(D^2 - D + \gamma\omega^2/4) - i\gamma\mu e^z(D^2 + D)]T(z) = 0.$$

the solution of which is investigate in ( Compos [1983a, 1983b], Yanowitch [1967]).

( IV ) The above conclusions indicate that, in the solar photosphere the temperature fluctuations could be smoothed by the transfer of the radiation between any two regions , with different temperatures. In addition the heat transfer by radiation is more dominant than the conduction process.

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