SOLUTIONS TO LYAPUNOV STABILITY PROBLEMS:
NONLINEAR SYSTEMS WITH CONTINUOUS MOTIONS

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Abstract. The necessary and sufficient conditions for accurate construction of a Lyapunov function and the necessary and sufficient conditions for a set to be the asymptotic stability domain are algorithmically solved for a nonlinear dynamical system with continuous motions. The conditions are established by utilizing properties of \( o \)-uniquely bounded sets, which are explained in the paper. They allow arbitrary selection of an \( o \)-uniquely bounded set to generate a Lyapunov function.

Simple examples illustrate the theory and its applications.


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1. INTRODUCTION

In his fundamental dissertation [1] Lyapunov referred to papers by Poincaré [2], [3] as those inspiring him to establish a method that has become fundamental for qualitative and stability analysis of motions of a very general class of nonlinear systems.

The promising methodological effectiveness of the Lyapunov method has not been fully achieved due to the need to construct a system Lyapunov function. Significant results on a Lyapunov function generation were initiated by Zubov [14]. The literature on the Lyapunov method is too vast [9]-[11],[13],[14] to be referred to herein.

The problem of the necessary and sufficient conditions for constructing a Lyapunov function and the problem of the necessary and sufficient conditions for a set to be the asymptotic stability domain have not yet been solved. Solutions to these problems will be established by using properties of \( o \)-uniquely bounded sets. Their features will be explained briefly by referring to [7],[8], where they were discovered and studied.

2. NOTATION

\[ A, R^* \supseteq A \]
\[ B_\delta = \{ x : \| x \| < \delta \}, R^* \supseteq B_\delta, \]
\[ \overline{B}_\delta = \{ x : \| x \| \leq \delta \}, R^* \supseteq \overline{B}_\delta, \]
\[ \partial B_\delta = \{ x : \| x \| = \delta \}, R^* \supseteq \partial B_\delta, \]
\[ C(S) \]

- an open connected neighborhood of \( x = 0 \),
- an open hyperball,
- the closure of \( B_\delta \),
- the boundary of both \( B_\delta \) and \( \overline{B}_\delta \),
- the set of all functions of \( x \) continuous on \( S \),
Systems to be analyzed are described by the following equation

$$\frac{dx}{dt} = f(x).$$

(3.1)

They are assumed to possess either of the following two features:

**Weak Smoothness Property:**

(i) There is an open neighborhood $S$ of $x = 0$, $R^n \supseteq S$, such that for every $x_0 \in S$

(a) the system (1) has the unique solution $x(t; x_0)$ through $x_0$ at $t = 0$, and

(b) the motion $x(t; x_0)$ is defined and continuous in $(t, x_0) \in I_0 \times S$.

(ii) For every $x_0 \in (R^n - S)$ every motion $x(t; x_0)$ of the system (1) is continuous in $t \in I_0$.

**Strong Smoothness Property:**

(i) The system (1) has Weak Smoothness Property.

(ii) If the boundary $\partial S$ of $S$ is non-empty then every motion of the system (1) passing through $x_0 \in \partial S$ at $t = 0$ obeys \( \inf \| x(t; x_0) \| : t \in I_0 > 0 \) for every $x_0 \in \partial S$.

4. **DEFINITIONS**

4.1 **ON THE DEFINITIONS OF STABILITY DOMAINS**

For the definitions of the attraction domain $D_\alpha$ see [4]-[6],[9],[11],[14]. The stability domain $D_\alpha$ and...
the asymptotic stability domain $D$ of $x = 0$ are defined in [5],[6]. We shall refer to those definitions in the sequel.

For the system (1) with Weak Smoothness Property, the stability domains are mutually related as follows:

**Lemma 1.** If the state $x = 0$ of the system (1) possessing Weak Smoothness Property has both the domain of attraction $D_o$, $S \supseteq D_o$, and the domain of stability $D$, then they and the asymptotic stability domain $D$ are interrelated by

$$D_o \supseteq D,\quad D = D_o.$$  

**Proof.** Let $x = 0$ have $D_o, S \supseteq D_o$, and $D$. Then it has also $D$ because $D = D_o \cap D$, and both $D_o$ and $D$ are neighborhoods of $x = 0$ [5],[6]. Let $x_0 \in D_o$. Then $x(t;x_0) \rightarrow 0$ as $t \rightarrow +\infty$. This and continuity of $x(t;x_0)$ in $t \in I_o$ (Weak Smoothness Property) imply $\max \| x(t;x_0) \|: t \in I_o \rightarrow \alpha + \infty$. Let $\epsilon = 2\alpha$. Hence, $\| x(t;x_0) \| < \epsilon, \forall t \in I_o$, which yields [5],[6] $x_0 \in D$, so that $D_o \supseteq D_o$, and $D = D_o \cap D$, [5],[6].

**4.2 ON THE DEFINITION OF A POSITIVE DEFINITE FUNCTION**

The notion of a positive definite function is used in a broader Lyapunov sense [1].

**Definition 1.** A function $\nu: \mathbb{R}^n \rightarrow \mathbb{R}$ is a positive definite if and only if there is an open connected neighborhood $A$ of $x = 0$, $\mathbb{R}^n \supset A$, such that

1) $\nu(x)$ is uniquely determined by $x \in A$ and $\nu$ is continuous on $A: \nu(x) \in C(A)$,
2) $\nu(0) = 0$, and
3) $\nu(x) > 0$ for every $(x = 0) \in A$.

**4.3 DEFINITIONS AND PROPERTIES OF O-UNIQUELY BOUNDED SETS**

O-uniquely bounded sets were introduced, defined and studied in [7],[8].

**Definition 2.** A set $U, \mathbb{R}^n \supset U$, is o-uniquely bounded if and only if it is bounded and for every $(x = 0) \in \mathbb{R}^n$ there is exactly one positive number $\lambda_o, \lambda = \lambda(x;U)$, such that $(\lambda_o x) \in \partial U$.

**Definition 3.** A function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is radially increasing on an open neighborhood $N$ of $x = 0$ if and only if for every $(x = 0) \in N$ and any $\mu_i, i = 1,2$, obeying both $0 < \mu_i < \mu_2$ and $\mu_i x \in N$ it satisfies $u(\mu_i x) < u(\mu_2 x)$.

**Property U.** Let $N$ be an open neighborhood of $x = 0$ and $U, N \supset U$, be a given bounded set. There is a function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ that obeys the following:

(a) $u$ is continuous on $N: u(x) \in C(N)$,
(b) if $N = \mathbb{R}^n$ then $u(x) \rightarrow +\infty$ as $\| x \| \rightarrow +\infty$,
(c) $u(0) = 0$,
(d) $u(x) > 0$ for all $(x = 0) \in N$,
(e) there is positive number $\xi$, $\xi = \xi(U)$, such that both 1. and 2. hold:

1. $u(x) \leq \xi$ for $x \in N$ if and only if $x \in \overline{U}$,
2. $u(x) > \xi$ for $x \in N$ if and only if $x \in \partial U$,
(f) $u(\lambda_i x) = \xi, i = 1,2$, holds for any $(x = 0) \in N$ if and only if $\lambda_1 = \lambda_2 = \lambda(x;U) \in [0, +\infty]$,
(g) $u$ is radially increasing on $N$.

Definition 2 implies the next result due to Definition 2, Corollary 1 and Proposition 4 in [8].

**Lemma 2.** For a bounded subset $U$ of an open neighborhood $N$ of $x = 0$ to be o-uniquely bounded it is both necessary and sufficient that it possesses Property $U$.

**Definition 4.** i) A function $u$ is the generating function on $N$ of an o-uniquely bounded set $U$ if and only if they have Property $U$. 

(ii) The function $u$ is the generating function of the uniquely bounded set $U$ if and only if they obey (i) for $N = \mathbb{R}^n$.

Lemma 2 and Definition 4 imply the following corollary [8].

**COROLLARY 1.** If a function $u$ is the generating function on $N$ of an o-uniquely bounded set $U$ then for any $\zeta > 0$ for which $N \supseteq N_\zeta$ the subset $U_\zeta$ of $N$ is a connected open neighborhood of $x = 0$ that is also an o-uniquely bounded set with the generating function $u$ on $N$.

5. **SOLUTIONS VIA O-UNIQUELY BOUNDED SETS**

We shall make use of the family $E(S; f)$ defined as follows.

**DEFINITION 5.** A function $u: \mathbb{R}^n \to \mathbb{R}$ belongs to the family $E(S; f)$ if and only if

1) $u$ is continuous on $S$, $u(x) \in C(S)$, and

2) the following equations along the motions of the system (3.1),

$$Dv(x) = -u(x), \quad (5.1a)$$

$$v(0) = 0, \quad (5.1b)$$

have a solution $v$ that is well defined in $R$ and continuous for every $x \in \overline{B}_\mu$ for some $\mu \in [0, +\infty]$, $\mu = \mu(u, f)$.

**THEOREM 1.** In order for the state $x = 0$ of the system (1) with Strong Smoothness Property to have the domain $D$ of asymptotic stability and for a set $N$, $\mathbb{R}^n \supseteq N$, to be the domain of its asymptotic stability, $N = D$, it is both necessary and sufficient that

1) the set $N$ is an open connected neighborhood of $x = 0$ and $S \supseteq N$,

2) $f(x) = 0$ for $x \in N$ if and only if $x = 0$, and

3) for arbitrarily selected o-uniquely bounded set $U$, $S \supseteq U$, with the generating function $u$ on $S$ obeying $u \in E(S; f)$, the equations (5.1) have a unique solution function $v$ on $N$ with the following properties:

(i) $v$ is positive definite on $N$, and

(ii) if the boundary $\partial N$ of $N$ is non-empty then $v(x) \to +\infty$ as $x \to \partial N, x \in N$.

**PROOF.** Necessity. Let $x = 0$ of the system (3.1) with Strong Smoothness Property have the asymptotic stability domain $D$. Definitions of $D_\psi$, $D_\phi$, and $D_\delta$ [5], [6] show that it has also the attraction domain $D_\psi, D_\phi \supseteq D$. It is a neighborhood of $x = 0$ due to Definition of $D_\psi$, and $S$ is a neighborhood of $x = 0$ in view of the smoothness property. Hence, $D_\psi \cap S \neq \emptyset$. Let us prove $S \supseteq D_\psi$. If $\partial S = \emptyset$, then $S = \mathbb{R}^n$ and $S \supseteq D_\phi$ due to $\mathbb{R}^n \supseteq D_\phi$. If $\partial S \neq \emptyset$, then we shall consider both $x_0 \in \partial S$ and $x_0' \in (\mathbb{R}^n - \overline{S})$. If $x_0 \in \partial S$, then $x_0 \notin D_\psi$ due to (ii) of Strong Smoothness Property. Therefore, $\partial S \cap D = \emptyset$. If $x_0' \in (\mathbb{R}^n - \overline{S})$, then for $x(t; x_0') \to 0$ as $t \to +\infty$ it is necessary that there is $t' \in \mathbb{R}$ such that $x(t'; x_0') \in \partial S$, because $D$ and $S$ are neighborhoods of $x = 0, x_0' \notin \overline{S}$ and the motion $x(t; x_0)$ is continuous in $t \in \mathbb{R}$ due to (ii) of Weak Smoothness Property ensured by (i) of Strong Smoothness Property. However, $x(t'; x_0') \in \partial S$ implies that $x(t; x_0)$ does not converge to $x = 0$ because of (ii) of Strong Smoothness Property. This yields $x_0' \notin D$ and $(\mathbb{R}^n - \overline{S}) \cap D = \emptyset$. By connecting the above results, that is $D_\psi \cap S = \emptyset, D_\phi \cap \partial S = \emptyset$ and $D_\psi \cap (\mathbb{R}^n - \overline{S}) = \emptyset$, we conclude that $S \supseteq D_\psi$. Therefore, $D = D_\psi$ (Lemma 1) and $S \supseteq D$. Let $N = D$ so that $S \supseteq N$. Hence, $N$ is open connected neighborhood of $x = 0$ due to (i-b) of Weak Smoothness Property, $N = D = D_\psi$, and invariance of $D_\psi$ with respect to system motions (Theorem 1.5.14 by Bhatia and Szegő [4], Theorem 33.3 by Hahn [9]). This proves necessity of the condition 1). From $N = D = D_\psi, D_\phi \supseteq D_\psi$, and Definitions of $D_\psi$ and $D$ it results that $x = 0$ is the unique equilibrium state of the system (1) in $N$, which implies $f(x) = 0$ for $x \in N = D$ if and only if $x = 0$ (Proposition 7 in [6]) and proves necessity of the condition 2).
From $N = D$ it follows that the interval $I_0$ of existence of $x(t; x_0)$ equals $R$, $I_0 = R$, for every $x_0 \in N$, due to Definitions of $D_*, D_+$ and $D [5],[6]$. Let $U$ be arbitrarily selected open $\alpha$-uniquely bounded set such that $N \supseteq \overline{U}$ and its generating function $u$ on $S$ obeys $u \in E(S; f)$. Such a set $U$ exists because $S$ is open neighborhood of $x = 0$ (Lemma 2). Definition 3, Property $U$, and Lemma 2 show that the function $u$ is also positive definite on $S$. Since $S \supseteq N = D$ then the function $u$ is the positive definite generating function on $N$, too. The property of $u \in E(S; f)$ ensures existence of $\mu > 0$ such that there exists a solution function $v$ to the equations (5.1), which is well defined in $R$ and continuous for every $x \in \overline{B}_\mu$, that is that

$$|v(x)| < +\infty \quad \text{for every } x \in \overline{B}_\mu \quad \text{and} \quad v(x) \in C(\overline{B}_\mu).$$

(5.2)

Let $\xi \in [0, +\infty[$ be such that

$$\overline{B}_\mu \cap U \supseteq \overline{U}_\xi.$$ (5.3)

The existence of such $\xi$ is assured by Corollary 1. Let $\tau \in [0, +\infty[$, $\tau = \tau(x_0; f; u; \xi)$, be such that for any $x_0 \in N$ the following condition holds,

$$x(t; x_0) \in U_\xi \quad \text{for every} \quad t \in [\tau, +\infty[.$$ (5.4)

Such $\tau$ exists in view of Definitions of $D_*$ and $D$, $D_+, D, N = D$ and $x_0 \in N$. Notice that $x_0 \in N$ implies also

$$x(+\infty; x_0) = 0.$$ (5.5)

After integrating (5.1a) from $t \in R$, to $+\infty$ we derive

$$v[x(+\infty; x_0)] - v[x(t; x_0)] = - \int_t^{+\infty} u[x(\sigma; x_0)] d\sigma \quad \text{for every} \quad (t, x_0) \in R \times N.$$ (5.6)

Since $u \in E(S; f)$ then the following holds,

$$v(0) = 0.$$ (5.7)

Now, (5.5)-(5.7) yield

$$v[x(t; x_0)] = \int_t^{+\infty} u[x(\sigma; x_0)] d\sigma \quad \text{for every} \quad (t, x_0) \in R \times N.$$ (5.8)

This can be written in the following form,

$$v[x(t; x_0)] = \int_t^{+\infty} u[x(\sigma; x_0)] d\sigma \quad \text{for every} \quad (t, x_0) \in R \times N.$$ (5.9)

Positive invariance of $D$ with respect to system motions, $N = D$, continuity of the motions $x$ due to the smoothness property, continuity of $u$ on $N$, the definition of $\tau$ (5.4) and (5.2), and compactness of $[\tau, t]$ for any $t \in R$, prove

$$\left| \int_t^\tau u[x(\sigma; x_0)] d\sigma \right| < +\infty \quad \text{for every} \quad (t, x_0) \in R \times N.$$ (5.10)

From (5.2)-(5.4) we obtain

$$\left| \int_\tau^{+\infty} u[x(\sigma; x_0)] d\sigma \right| < +\infty \quad \text{for every} \quad x_0 \in N.$$ (5.11)

(5.9)-(5.11) together prove boundedness of $v[x(t; x_0)]$ expressed as

$$|v[x(t; x_0)]| < +\infty \quad \text{for every} \quad (t, x_0) \in R \times N.$$ (5.12)

Hence, by setting $t = 0$ and $x_0 = x$ in (5.12) we derive

$$|v(x)| < +\infty \quad \text{for every} \quad x \in N.$$ (5.13)
Continuity of the motion $x$ in $x_0 \in N$, continuity of $u$ in $x \in N$, and of $v$ in $x \in B$, $B \supset \overline{U}$, positive invariance of $N = D$ with respect to system motions, (5.4), (5.9) and (5.12) prove continuity of $v$ in $x \in N$

$$v(x) \in C(N).$$

Positive invariance of $N$ with respect to system motions, positive definiteness of $u$ on $N$ and (5.8) imply

$$v(x) > 0 \text{ for all } (x \neq 0) \in N.$$  

Now, (5.7), (5.14) and (5.15) prove necessity of the positive definiteness of $v$ on $N$.

To prove uniqueness of the solution $v$ to (5.1),(ab) we shall suppose that there are two solutions $v_1$ and $v_2$ to (5.1). Hence,

$$v_1(x_0) - v_2(x_0) = \int_0^\infty \{u[x_1(\sigma;x_0)] - u[x_2(\sigma;x_0)]\}d\sigma \text{ for every } x_0 \in N.$$  

Since $u(x)$ is uniquely determined by $x \in N$, due to (a) of Property U and Definition 4, and the motion of the system is unique through $x_0$, $x_1(\sigma;x_0) = x_2(\sigma;x_0)$ and $u[x_1(\sigma;x_0)] = u[x_2(\sigma;x_0)]$ so that $v_1(x_0) - v_2(x_0) = 0$ for every $x_0 \in N$. This proves uniqueness of the solution $v$ to (5.1) and completes the proof of 3(i).

Let $\partial N$ be non-empty, $x_1, x_2, \ldots, x_n, \ldots$ be a sequence converging to $x'$, $x_k \to x'$ as $k \to +\infty$, where $x_k \in N$, for all $k = 1, 2, \ldots$, and $x' \in \partial N$. Let $\xi \in [0, +\infty[$ be arbitrarily chosen so that $U_\xi = \{x: u(x) < \xi\}$, $U \supset U_\xi$. Such $\xi$ exists because the set $U$ is $\omega$-uniquely bounded and the function $u$ is its generating function on $N$ (Definitions 2 and 3, Property U, Lemma 2 and Definition 4). The set $U_\xi$ is a connected open neighborhood of $x = 0$ (Corollary 1). Let $T_1, T_2 = T(x_1, \xi)\in [0, +\infty[$ be the first instant obeying the following

$$x(t;x_1) \in U_\xi \text{ for all } t \in [T_1, +\infty[.$$  

The existence of such $T_1$ is guaranteed by $x_1 \in N$ and $N = D$ (Definitions of $D_1$ and $D$ [5], [6]). Continuity of the motions $x$ in $(t, x_0) \in R \times N$ due to Strong Smoothness Property and $N = N = D$ (Theorem 33.1 by Hahn [9]) and $S \supset D$ imply $T_k \to +\infty$ as $k \to +\infty$ (Theorem 33.2 by Hahn [9]). Let $m$ be a natural number such that $x_k \in (N - \overline{U})$ for all $k = m, m + 1, \ldots$. Such $m$ exists because $N$ is open, $N \supset \overline{U}$ and $x_k \to \partial N$ as $k \to +\infty$. Let $\alpha'$ be defined by (18),

$$\alpha' = \min[\alpha(x); x \in (N - U_\xi)].$$  

(5.18)

The $\omega$-unique boundedness of the set $U_\xi$, the fact that the function $u$ is its generating function on $N$ (Corollary 1), $N \supset \overline{U}$, and $U \supset U_\xi$ guarantee (Property U and Lemma 2) that $\alpha'$ defined by (5.18) satisfies

$$\alpha' \in \xi \in [0, +\infty[.$$  

(5.19)

From (5.9) we get, after replacing $\tau$ by $T_k$,

$$v(x(t;x_1)) = \int_0^{T_k} u[x(\sigma;x_1)]d\sigma + \int_{T_k}^{\infty} u[x(\sigma;x_1)]d\sigma \text{ for every } (t, x_1) \in R \times N,$$

and for $k = m, m + 1, \ldots$.  

(5.20)

Setting $t = 0$ in (5.20) and using (5.18) and (5.19) we derive

$$v(x_k) \geq \int_0^{T_k} \xi d\sigma + \int_{T_k}^{\infty} u[x(\sigma;x_1)]d\sigma \text{ for } x_k \in N \text{ and all } k = m, m + 1, \ldots.$$  

(5.21)

Positive invariance of $N = D$ with respect to system motions, positive definiteness of $u$ on $N$, and (5.21) imply

$$v(x_k) \geq \xi T_k \text{ for } x_k \in N \text{ and all } k = m, m + 1, \ldots.$$  

(5.22)

Since $T_k \to +\infty$ as $k \to +\infty$, the last inequality, the definitions of $T_k, T_k = T(x_k, \xi)$, and of $x_k$, and $\alpha > 0$ imply

$$v(x_k) \to +\infty \text{ as } x_k \to \partial N \text{ due to } k \to +\infty, x_k \in N,$$

which proves necessity of the condition (3-ii).
Sufficiency. Let all the conditions of Theorem 1 hold. Then, $S \supseteq N$. Two possible cases will be considered separately: a) $N$ is a bounded set, b) $N$ is an unbounded set.

a) Let $N$ be a bounded set. Then, under the conditions of the theorem to be proved all the conditions of Theorem 1 by Vanelli and Vidyasagar [12] are satisfied, which proves $N = D$. Since $D = D$ (in view of Weak Smoothness Property implied by Strong Smoothness Property and Lemma 1), $N = D$.

b) Let $N$ be an unbounded set. Under the conditions of the theorem to be proved the zero state $x = 0$ of the system (1) is asymptotically stable (cf. Yoshizawa [13]). Hence, it has the domain of asymptotic stability $D$. Since both $N$ and $D$ are open connected neighborhoods of $x = 0$,

$$N \cap D = \emptyset. \quad (5.23)$$

Since $S \supseteq N$, $S$ is also unbounded. If $dS$ is empty, then $S = R^n$, which implies $S \supseteq D$. If $dS$ is non-empty, then $dS \cap D = \emptyset$ due to (ii) of Strong Smoothness Property and Definitions of $D_\epsilon$, $D$, and $D$ [5],[6]. This result implies $S \supseteq D$ because both $D$ and $S$ are neighborhoods of $x = 0$ and $D$ is also connected. Altogether, in both cases $S \supseteq D$. We shall treat separately the cases of non-empty $dD$ and of empty $dD$. The definition of the function $\nu$, $S \supseteq D$, and the proof of the necessity part prove continuity of $\nu$ on $D$ and $\nu(x) \rightarrow +\infty$ as $x \rightarrow dD$, which together with continuity of $\nu$ also on $N$, $S \supseteq N$ and $\nu(x) \rightarrow +\infty$ as $x \rightarrow dN$ [the condition 3(ii)] imply both

$$dD \cap N = \emptyset \quad \text{and} \quad D \cap dN = \emptyset.$$

These equations and (5.23) prove both $dD = dN$ and $D = N$ due to the fact that both $D$ and $N$ are open connected neighborhoods of $x = 0$. Let now $dD$ be empty. Then $D = R^n$. Hence, $\nu$ is positive definite on $R^n$ (see the proof of the necessity part). Thus, it is continuous on $R^n$, which implies $\nu(x) < +\infty$ for every $x \in R^n$. Therefore, $\partial N \cap R^n = \emptyset$ due to the conditions 3(ii), which yields $N = R^n$ so that $N = D$. Finally, $N = D$ in all the cases, which completes the proof.

The conditions slightly change if the system (3.1) possesses Weak Smoothness Property rather than Strong Smoothness Property.

**Theorem 2.** For the state $x = 0$ of the system (1) possessing Weak Smoothness Property to have the domain $D$ of asymptotic stability and that a subset $N$ of $S$, $S \supseteq N$, equals $D$, it is both necessary and sufficient that

1) the set $N$ is an open connected neighborhood of $x = 0$,
2) $f(x) = 0$ for $x \in N$ if and only if $x = 0$, and
3) for arbitrarily selected o-uniquely bounded set $U, S \supseteq U$, with the generating function $u$ on $R^n$ obeying $u \in E(S; f)$, the equations (5.1) have a unique solution function $\nu$ on $N$ with the following properties:
   (i) $\nu$ is positive definite on $N$, and
   (ii) if the boundary $\partial N$ of $N$ is non-empty then $\nu(x) \rightarrow +\infty$ as $x \rightarrow \partial N$, $x \in N$.

**Proof. Necessity.** Let the system (3.1) possess Weak Smoothness Property. Let $x = 0$ have the asymptotic stability domain $D$, $S \supseteq D$, and let $N, S \supseteq N$, be equal to $D$. Let an o-uniquely bounded set $U, S \supseteq U$, with the generating function $u$ obeying $u \in E(S; f)$, be arbitrarily selected. From this point on we have to repeat the proof of the necessity part of Theorem 1 to show that the conditions 1)-3) of Theorem 2 hold. In that way we complete the proof of the necessity part.

**Sufficiency.** Let the system (3.1) possess Weak Smoothness Property and the conditions 1)-3) be valid. Then $x = 0$ of the system (3.1) is asymptotically stable [1]. Therefore, $x = 0$ has the domain of asymptotic stability (Definitions of $D_\epsilon$, $D$, and $D$ [5],[6]). Let $x_0 \in (R^n - N)$. Since $x(t; x_0)$ is continuous in $t \in I_p$, then it can enter $N$ iff it passes through $\partial N$. But $\nu(x) \rightarrow +\infty$ as $x \rightarrow \partial N$, $x \in N$ [the condition 3(ii)]. This and $D \nu(x) < 0$ for $x \in (R^n - N)$ in view of positive definiteness of $u$ on $R^n$ and (5.1a), show that $x(t; x_0)$ cannot reach $\partial N$. Hence, $x(t; x_0) \in (R^n - N)$ for all $t \in I_p$. Therefore, $N \supseteq D$. Furthermore,
(5.1a) and positive definiteness of \( u \) on \( R^* \) imply (see the proof of the necessity part of Theorem 1) \( v(x) \to +\infty \) as \( x \to \partial D \), \( x \in D \), which together with the condition 3(i) proves \( \partial D \cap N = \emptyset \). This result, \( \bar{N} \supseteq D \), and the fact that \( D \) and \( N \) are non-empty open connected neighborhoods of \( x = 0 \) imply \( D = N \) and complete the proof.  

The properties of the generating function \( u \) of an \( o \)-uniquely bounded set \( U \) are essential for the accurate one-shot determination of the asymptotic stability domain. However, such properties are not needed for asymptotic stability of \( x = 0 \) only. This is clarified by the next result.

**THEOREM 3.** For the state \( x = 0 \) of the system (3.1) possessing Weak Smoothness Property to be asymptotically stable it is both necessary and sufficient that for any positive definite function \( p \in E(S;f) \) there exists a unique solution function \( v \) to (5.24) with (5.24a) determined along system motions,

\[
D^* v(x) = -p(x), \quad (5.24a)
\]

\[
v(0) = 0, \quad (5.24b)
\]

which is also positive definite.

**PROOF. Necessity.** Let the system (3.1) possess the Weak Smoothness Property. Let \( x = 0 \) be asymptotically stable. Then it has \( D_a, D, D_i, D_s \) and \( D_s \cap S = \emptyset, D_i \cap N = \emptyset \) and \( D \cap S = \emptyset \), because \( D_a, D_s, D_i, D_s, D_i, S \) and \( S \) are neighborhoods of \( x = 0 \). Let \( p \in E(S;f) \) be an arbitrarily selected positive definite function (Definition 1). Such properties of \( p \) and its membership to \( E(S;f) \) guarantee existence of a solution \( v \) to the equations (5.24), which is well defined in \( R \) and continuous (see the proof of the necessity part of Theorem 1) on the set \( \mathcal{A} \) determined in Definition 1. The set \( \mathcal{L} \supseteq \mathcal{A} \cap D, D \supseteq \mathcal{L} \), is also an open connected neighborhood of \( x = 0 \) (see the proof of Theorem 1 for such a property of \( D \)). Let \( \varepsilon \) satisfying \( \mathcal{L} \supseteq \mathcal{D}_\varepsilon \) be arbitrarily selected. Then \( D \supseteq \mathcal{D}_\varepsilon \). Let \( \rho \in [0,\varepsilon] \) obeying \( D_i(\rho) \supseteq \mathcal{D}_\rho \) be also arbitrarily selected, where \( D_i(\rho) \) is defined [5],[6] as the neighborhood of \( x = 0 \) such that \( \|x(t;x_0)\| < \varepsilon \) for all \( t \in R \), holds iff \( x_0 \in D_i(\rho) \). By following the proofs of (5.13) and (5.14), we prove that \( v \), defined by (5.24), has the following properties since \( \mathcal{A} \supseteq \mathcal{L} \supseteq \mathcal{D}_\rho \supseteq D_i(\rho) \supseteq \mathcal{D}_\rho \),

\[
|v(x)| < +\infty \text{ for every } x \in \mathcal{D}_\rho, \quad (5.25a)
\]

\[
v(x) \in C(\mathcal{D}_\rho). \quad (5.25b)
\]

Notice that \( D_i(\rho) \supseteq \mathcal{D}_\rho \) and the definitions of \( D_i(\rho) \) and \( D \) guarantee [5],[6] \( x(t;x_0) \in \mathcal{D}_\rho \) for every \( (t,x_0) \in R \times \mathcal{D}_\rho \). This result, \( \mathcal{A} \supseteq \mathcal{D}_\rho \), positive definiteness of the function \( p \) on \( \mathcal{A} \), \( \mathcal{X}(+\infty;x_0) = 0 \) for every \( x_0 \in \mathcal{D}_\rho \) (because \( D \supseteq \mathcal{D}_\rho \)) and (5.24a), integrated from \( t = 0 \) to \( t = +\infty \), together with (5.24b) prove (5.26),

\[
v(x_0) > 0 \text{ for every } (x_0,0) \in \mathcal{D}_\rho. \quad (5.26)
\]

Now, (5.24b) through (5.26) prove positive definiteness of the solution \( v \) to (5.24) on \( \mathcal{D}_\rho \). Its uniqueness is proved in the same way as in the proof of Theorem 1, which completes the proof of the necessity part.

**Sufficiency.** Sufficiency of the conditions of Theorem 3 for asymptotic stability of \( x = 0 \) of the system (3.1) with Weak Smoothness Property is well known [13]. This completes the proof of Theorem 3.  

6. EXAMPLES

**Example 1.** Let \( n = 1 \),

\[
\frac{dx}{dt} = -x + h(x), \quad h(x) = \begin{cases} x & \text{for } |x| \in [0,1], \\ x(|x|)^{1/2} & \text{for } |x| \in [1, +\infty[ \end{cases} \quad (6.1)
\]

The system possesses Strong Smoothness Property because \( f(x) = -x + h(x) \) is Lipschitzian on \( R^1 \). The equilibrium states are \( x_1 = -1 \), \( x_2 = 0 \) and \( x_3 = +1 \). They suggest \( S = [-1, +1] \) and \( U = \{x: x \in R^1, |x| < \alpha \} = -\alpha, +\alpha \}, \alpha \in ]0, 1[. \) The generating function \( u \) on \( N \), \( u(x) = |x| \), of the
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\(o\)-uniquely bounded set \(U\) and (5.1a) yield

\[
D'v(x) = -|x|, \quad x \in S.
\]

The solution \(v\) to this equation is

\[
v(x) = -\ln(1 - |x|), \quad x \in S.
\] (6.2)

The function \(v(27)\) and the set \(N = S = ]-1, +1[\) satisfy all the requirements of Theorem 1, that is that,

1) \(N = ]-1, +1[\) is an open connected neighborhood of \(x = 0\) and \(N = S\),

2) \(f(x) = -x + h(x) = 0\) for \(x \in N\) iff \(x = 0\),

3) \(v(x) = 0\) for \(x \in N\) iff \(x = 0\), \(v(x) \in C(N)\), and \(v(x) > 0\) for every \((x \neq 0) \in N\), which prove positive definiteness of \(v\) on \(N\),

4) \(v(x) \to +\infty\) as \(x \to aN = \{-1, +1\}, x \in N\).

Hence \(N = ]-1, +1[\) is the domain \(D\) of asymptotic stability of \(x = 0\),

\[D = ]-1, +1[.\]

Notice that \(|f(x)| = |x| |1 - |x||\), \(x \in N\), is not a generating function on \(N\) of any \(o\)-uniquely bounded set because it is not radially increasing on \(N\).

**Example 2.** Let the function \(h\) be defined as in Example 1 and

\[
dx = x - h(x).
\] (6.3)

It is clear that the system possesses Strong Smoothness Property on \(R^1\) and has the equilibrium states \(x_1 = -1, x_2 = 0\) and \(x_3 = +1\) (see Example 1). Let, again, \(U = \{x: x \in R^1, |x| < \alpha\} = ]-\alpha, +\alpha[\) so that \(u(x) = |x|\). From (5.1a) we get

\[
D'v(x) = -|x|, \quad x \in N.
\]

Integrating this equation along motions of the system (6.3) we derive

\[
v(x) = \ln(1 - |x|), \quad x \in N,
\]

which is negative definite on \(N\) and, thus, does not satisfy the necessary and sufficient conditions for asymptotic stability of \(x = 0\) of the system (6.3). Hence, \(x = 0\) of the system (6.3) is not asymptotically stable and does not have the asymptotic stability domain.

**Example 3.** Let \(n = 2\) and

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
x_1(1 + |x_1| |x_2|)(1 - |x_1|) \\
x_2 - x_2(1 - |x_1| |x_2|)(1 - |x_2|)
\end{bmatrix} f(x).
\] (6.4)

The function \(f\) is globally Lipschitz continuous. The system has Strong Smoothness Property on \(R^2\). The set \(S_\gamma\) of its equilibrium states is determined by

\[S_\gamma = \{x: x \in R^2, (x = 0) \text{ or } (|x_1| < 1, |x_2| < 1)\}.
\]

This suggests \(S = \{x: x \in R^2, |x_1| < 1, |x_2| < 1\}\). The system (6.4) has Weak Smoothness Property on \(S\). Let \(U = \{x: x \in R^2, |x_1| + |x_2| < \alpha\}, \alpha \in ]0, 1[\), so that \(U\) is \(o\)-uniquely bounded set with the generating function \(u\) on \(R^2\) defined by \(u(x) = |x_1| + |x_2|\), which together with (5.1) and (6.4) yields

\[
v(x) = -\ln((1 - |x_1|)(1 - |x_2|)).
\]

The function \(v\) and the set \(N = S\) obey all the conditions of Theorem 2. Therefore, \(x = 0\) of the system (6.4) is asymptotically stable with the domain \(D\) of its asymptotic stability obtained as \(D = N = S\), that is that

\[D = \{x: x \in R^2, |x_1| < 1, |x_2| < 1\}.
\]
7. CONCLUSION

The necessary and sufficient conditions for asymptotic stability of the zero equilibrium state and for a set to be the domain of its asymptotic stability are proved in an algorithmic form that enables accurate construction of a system Lyapunov function. If a function \( v \) obtained from \( D^tv = -u \) for an arbitrarily chosen \( u \), which is a generating function of an \( \alpha \)-uniquely bounded set, is not positive definite then the zero state is not asymptotically stable. There is no sense to try with another function \( u \). However, if so derived function \( v \) is positive definite then the zero state is asymptotically stable. In this way the problem of an algorithm to construct accurately and directly a system Lyapunov function has been solved. However, it imposes other very complex mathematical problems: the problem of finding conditions on \( u \) guaranteeing existence of well defined and continuous \( v \) satisfying (5.1) on anyhow small neighborhood \( \bar{B}_\rho \) of \( x = 0 \), and the problem of solving (5.1). These problems have not been solved.

Theorems of the paper open and initiate new directions in the Lyapunov stability analysis.

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REFERENCES


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