Internat. J. Math. & Math. Sci.
VOL. 16 NO. 3 (1993) 493-502

THICKNESS IN TOPOLOGICAL TRANSFORMATION SEMIGROUPS

TYLER HAYNES

Mathematics Department
Saginaw Valley State University
University Center, Michigan 48710-0001

(Received June 27, 1989 and in revised form February 5, 1993)

ABSTRACT. This article deals with thickness in topological transformation semigroups (τ-semigroups). Thickness is used to establish conditions guaranteeing an invariant mean on a function space defined on a τ-semigroup if there exists an invariant mean on its functions restricted to a sub-τ-semigroup of the original τ-semigroup. We sketch earlier results, then give many equivalent conditions for thickness on τ-semigroups, and finally present theorems giving conditions for an invariant mean to exist on a function space.

KEY WORDS AND PHRASES. Thickness, topological transformation semigroup, transformation semigroup, invariant mean
1980 AMS SUBJECT CLASSIFICATION CODES. 22A20, 43A07

1. *Left-Thickness in Semigroups*

Mitchell introduced the concept of left-thickness in a semigroup [Mitchell, 1965]: a subset T of semigroup S is *left-thick* in S $\leftrightarrow$ $\forall$ finite $U \subseteq S$, $\exists t \in S$: $Ut \subseteq T$.

Any left ideal of a semigroup is left-thick, but not conversely. The complete relationship between left ideals and left-thick subsets is this: Let $\beta(S)$ be the Stone-Čech compactification of semigroup S endowed with the discrete topology, and let $T \subseteq S$. Then T is left-thick in S $\leftrightarrow$ the closure of T in $\beta(S)$ contains a left ideal of $\beta(S)$ [Wilde & Witz, 1967, lemma 5.1]. (See Theorem 4.3.g *infra* for a more general formulation of this result.)

It can be shown that in the definition t can be taken in T or U can be a singleton.

Let B(S) = the set of all bounded complex- or real-valued functions on semigroup S. For any $s \in S$ and $f \in B(S)$, $T_s f$ denotes the function in B(S) defined by $T_s f(t) = f(st)$ ($\forall t \in S$).

A *mean* on B(S) is a member of the dual space $B(S)^*$ of B(S) which satisfies $\mu(1) = 1 = \|\mu\|$. Mean μ is *invariant* $\leftrightarrow \mu(T_s f) = \mu f$ ($\forall s \in S, f \in B(S)$).

The importance of left-thickness for our subject is because of this theorem [Mitchell, 1965, theorem 9].

Theorem. *Let T be a left-thick subsemigroup of semigroup S. Then B(S) has a left-invariant mean $\leftrightarrow$ B(T) has a left-invariant mean.*

H. D. Junghenn generalized Mitchell's concept of left-thickness [Junghenn, 1979, p. 38]. First it is necessary to define more terms.

Subspace F of B(S) is *left-translation invariant* $\leftrightarrow$ $T_s f \in F$ ($\forall s \in S, f \in F$). Let $\mu \in F^*$, the dual space of F; define $T_\mu f$ ($\forall f \in F$) by $T_\mu f(s) = \mu(T_s f)$ ($\forall s \in S$). Then T_μ: $F \to B(S)$. F is *left-introverted* $\leftrightarrow$ $T_\mu(F) \subseteq F$ ($\forall \mu \in F^*$).

Definition. Let S be a semigroup; $F \subseteq B(S)$ be a left-translation invariant, left-introverted, norm-closed subalgebra containing the constant functions; $T \subseteq S$ be non-empty;
$F(T) = \{g \in F \mid \chi_T \le g \le 1\}$. Then
T is *F-left thick* in S $\leftrightarrow$ $\forall \epsilon > 0, g \in F(T)$, and finite $U = \{s_1, s_2, \ldots, s_n\} \subseteq S$ $\exists s \in S$: $g(s_i s) > 1 - \epsilon$ $(i = 1, \ldots, n)$

If $\chi_T \in F$, then Junghenn's definition of F-left thickness reduces to Mitchell's definition of left-thickness: let $g = \chi_T$, then for $0 < \epsilon < 1$, $1 - \epsilon < g(s_i s) = \chi_T(s_i s) \to s_i s \in T$ $(i = 1, \ldots, n)$.

Junghenn generalizes Mitchell's theorem thus:

Theorem. *If T is a left-thick subsemigroup of S, then F has a left-invariant mean* $\Rightarrow$ $F|_T$ *has a left-invariant mean.*

2. *Transformation Semigroups*

Thickness can be defined in the more general setting of a transformation semigroup. This section defines such semigroups and other necessary terms.

Definition 2.1. A *transformation semigroup* is a system $\langle S,X,\pi\rangle$ consisting of a semigroup S, a set X, and a mapping π: $S\times X \to X$ which satisfies

1. $\pi(s,\pi(t,x)) = \pi(st,x)$ $(\forall s,t\in S, x\in X)$;
2. $\pi(e,x) = x$ $(\forall x\in X)$ whenever S has two-sided identity e.

If $\pi(s,x)=sx$ expresses the image of (s,x) under π, then condition (1) becomes $s(tx)=(st)x$ and condition (2) becomes $ex=x$.

The abbreviated notion $\langle S,X\rangle$ will denote a transformation semigroup whenever the meaning of π is clear or whenever π is generic.

$\langle T,Y\rangle$ is a *subtransformation semigroup* of $\langle S,X\rangle$ $\Rightarrow$ T is a subsemigroup of S, $Y\subseteq X$, and $TY\subseteq Y$.

Definition 2.2. Let semigroup S and set X both be endowed with Hausdorff topologies. Transformation semigroup $\langle S,X,\pi\rangle$ is a *topological transformation semigroup*, or *τ-semigroup* $\Rightarrow$ π is separately continuous in the variables s and x.

Again, a τ-semigroup will be denoted briefly by $\langle S,X\rangle$.

Let C(X) denote the set of continuous and bounded complex- or real-valued functions on X.

Definition 2.3. Let $\langle S,X\rangle$ be a τ-semigroup. T_sf denotes, for any $s\in S$ and $f\in C(X)$, the function in C(X) defined by $T_sf(x)=f(sx)$ $(\forall x\in X)$. If F is a linear subspace of C(X), then F is *S-invariant* $\Rightarrow$ $T_sf\in F$ $(\forall s\in S, f\in F)$. Notation: $T_S = \{T_s | s\in S\}$ and $T_sF = \{T_sf | f\in F\}$.

Observe that $T_tT_s = T_{st}$ $(\forall s,t\in S)$.

Definition 2.4. Let $\langle S,X\rangle$ be a τ-semigroup; F be a linear space $\subseteq C(X)$ which is norm-closed, conjugate-closed, S-invariant, and contains the constant functions; $G\subseteq C(S)$ a linear space, and let $\mu\in F^*$. Define $T_\mu f$ $(\forall f\in F)$ by $T_\mu f(s) = \mu(T_sf)$ $(\forall s\in S)$. Then T_μ: $F\to B(S)$. F is *G-introverted* $\Rightarrow$ $T_\mu(F)\subseteq G$ $(\forall \mu\in F^*)$.

In the preceding definition F^* may be replaced by $C(X)^*$ since every functional in F^* can be extended to a functional in $C(X)^*$. Also it can be shown that F^* can be replaced by M(F), the set of all means on F.

Definition 2.5. Let F be G-introverted, $\mu\in F^*$, and $\lambda\in G^*$. The *evolution product* of λ and μ, denoted $\lambda\mu$, is defined by $\lambda\mu f = \lambda(T_\mu f)$ $(\forall f\in F)$.

Note that $\lambda\mu\in F^*$ and that if G is norm-closed, conjugate-closed, and contains the constant functions, then $\lambda\in M(G)$ and $\mu\in M(F)$ imply $\lambda\mu\in M(F)$.

A *mean* on $F\subseteq C(X)$ is defined in the same way as a mean on B(S) was defined in section 1. If F is an algebra under pointwise multiplication, then mean μ is *multiplicative* $\Rightarrow$ $\mu(fg) = \mu(f)\mu(g)$ $(\forall f,g\in F)$.

Let M(F) = set of all means on F, and MM(F) = set of all multiplicative means on F. M(F) and MM(F) are both w*-compact, being closed subsets of the unit ball in F^*.

Mean $\mu\in M(F)$ is *invariant* $\Rightarrow$ $\mu(T_sf) = \mu(f)$ $(\forall f\in F, s\in S)$. Note that μ is invariant $\Rightarrow$ $e(s)T_\mu = T_\mu$ $(\forall s\in S)$.

An *evaluation* at $x\in X$ is defined by $e(x)f = f(x)$ $(\forall f\in F)$; clearly an evaluation is a mean. A *finite mean* on F is a convex combination of evaluations.

A mean is multiplicative if and only if it is the w*-limit of evaluations.

A special case of transformation semigroup is furnished by letting $X = S$ and $\pi = \lambda(\bullet)$ where λ_s: $S \to S$ is defined for any fixed $s \in S$ by $\lambda_s(t) = st$ ($\forall t \in S$). If $G \subseteq C(S)$ is a linear space, then $L_s g(t) = g(st)$ ($\forall s,t \in S, g \in G$); also, $\lambda, \mu \in M(G) \to \lambda\mu \in M(G)$. If $F \subseteq C(X)$ is a linear space then $L_s T_\mu = T_\mu T_s$ ($\forall s \in S, \mu \in M(F)$). Mean $\mu \in M(G)$ is *left-invariant* $\leftrightarrow$ $\mu(L_s g) = \mu(g)$ ($\forall g \in G$).

3. Thickness in Transformation Semigroups

Junghenn's generalization of F-left thickness carries over in a straightforward way to transformation semigroups. The corresponding concept is defined in Definition 3.1, and a plethora of alternative characterizations is given by Theorem 3.3.

Assumptions:

$\langle S,X \rangle$ is a transformation semigroup;

$G \subseteq C(S)$ is a subalgebra;

$F \subseteq C(X)$ is an algebra which is norm-closed, S-invariant, G-introverted, and contains the constant functions;

$Y \subseteq X$.

Notation:

$F(Y) = \{g \in F \mid \chi_Y \leq g \leq 1\} = \{g \in F \mid 0 \leq g \leq 1, g \equiv 1 \text{ on } Y\}$

$Z(Y) = \{g \in F \mid g \equiv 0 \text{ on } Y\}$.

Definition 3.1. Y is *F,S-thick* in X $\leftrightarrow$ $\forall \epsilon > 0, g \in F(Y)$, and finite $U = \{s_1, s_2, \ldots, s_n\} \subseteq S$, $\exists x \in X$: $g(s_k x) > 1 - \epsilon$ $(k = 1, \ldots, n)$.

Remark 3.2. If $X = S$ and the action is left multiplication, then the definition is identical to Junghenn's.

Relative to Theorem 3.3 b,h,i,j *infra* it is necessary to recall that a norm-closed subalgebra F of C(X) is also a closed lattice, so that, in particular, $f \in F \to |f| \in F$ [Simmons, p. 159, lemma].

Theorem 3.3. The following statements are equivalent:

a. Y is F,S-thick in X;

b. $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \ldots, g_m\} \subseteq F(Y)$,
finite $U = \{s_1, s_2, \ldots, s_n\} \subseteq S$
$\exists x \in X$: $\inf\{g_i(s_k x) \mid g_i \in D, s_k \in U\} > 1 - \epsilon$;

c. $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \ldots, g_m\} \subseteq F(Y)$,
finite $U = \{s_1, s_2, \ldots, s_n\} \subseteq S$
$$\exists x \in X: \frac{1}{n} \sum_{k=1}^{n} g_i(s_k x) > 1 - \epsilon \ (i = 1, \ldots, m) \text{ and } \frac{1}{m} \sum_{i=1}^{m} g_i(s_k x) > 1 - \epsilon \ (k = 1, \ldots, n);$$

d. $\exists \lambda \in MM(F)$, $\forall s \in S, g \in F(y)$: $\lambda(T_s g) = 1$ and $\lambda(g) = 1$;

e. $\exists \mu \in M(F)$, $\forall s \in S, g \in F(Y)$: $\mu(T_s g) = 1$ and $\mu(g) = 1$;

f. $\exists \mu \in M(F)$, $\forall \nu \in M(G), g \in F(Y)$: $\nu\mu(g) = 1$;

g. Cle(Y) contains a compact MM(G)-invariant set;

h. $\forall \epsilon > 0, g \in Z(Y)$, finite $U = \{s_1, s_2, \ldots, s_n\} \subseteq S$ $\exists x \in X$: $|g(s_k x)| < \epsilon$ $(k = 1, \ldots, n)$;

i. $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \ldots, g_m\} \subseteq Z(Y)$, finite $U = \{s_1, s_2, \ldots, s_n\} \subseteq S$;
$\exists x \in X$: $\sup\{|g_j(s_k x)| \mid g_j \in D, s_k \in U\} < \epsilon$;

j. $\forall \epsilon > 0$, finite $D = \{g_1, g_2, \ldots, g_m\} \subseteq Z(Y)$, finite $U = \{s_1, s_2, \ldots, s_n\} \subseteq S$;
$$\exists x \in X: \frac{1}{n} \sum_{k=1}^{n} |g_i(s_k x)| < \epsilon \ (i = 1, \ldots, m) \text{ and } \frac{1}{m} \sum_{i=1}^{m} |g_i(s_k x)| < \epsilon \ (k = 1, \ldots, n);$$

k. $\exists \lambda \in MM(F)$, $\forall s \in S, g \in Z(Y)$: $\lambda(T_s g) = 0$ and $\lambda(g) = 0$;

l. $\exists \mu \in M(F)$, $\forall s \in S, g \in Z(Y)$: $\mu(T_s g) = 0$ and $\mu(g) = 0$;

m. $\exists \mu \in M(F)$, $\forall \nu \in M(G), g \in Z(Y)$: $\nu\mu(g) = 0$.

PROOF: $a \to b$: $f(x) = \inf\{g_i(x) | g_i \in D\}$ is in $F(Y)$ because $0 \le g_i \le 1$, $g_i \equiv 1$ on Y ($i=1,\dots,m$). By (a) $\exists x \in X$: $f(s_k x) > 1-\epsilon$ ($k=1,\dots,n$). Because U is finite, $\inf\{f(s_k x) | s_k \in U\} > 1-\epsilon$.

$b \to c$: $\inf\{g_i(s_k x) | g_i \in D, s_k \in U\} > 1-\epsilon \to \sum_{k=1}^{n} g_i(s_k x) \ge n\,[\inf\{g_i(s_k x)\}] > n(1-\epsilon)$

and $\sum_{i=1}^{m} g_i(s_k x) \ge m\,[\inf\{g_i(s_k x)\}] > m(1-\epsilon)$.

$c \to d$: For each (ϵ, U, D) in (c) choose $x = x(\epsilon, U, D)$ so that $\frac{1}{n}\sum_{k=1}^{n} g(s_k x) > 1 - \frac{1}{n}\epsilon$ ($\forall g \in D$). Let $r \in U, g \in D$. Then $g(s_k x) \le 1$ ($k=1,\dots,n$) $\to \sum_{s_k \ne r} g(s_k x) \le n-1 = -\sum_{s_k \ne r} g(s_k x) \ge -n+1 \to g(rx) = \sum_{k=1}^{n} g(s_k x) - \sum_{s_k \ne r} g(s_k x) > 1-\epsilon$. Define $(\epsilon, U, D) \le (\epsilon', U', D') \to \epsilon \ge \epsilon', U \subseteq U', D \subseteq D'$. The net $\langle e(x(\epsilon, U, D))\rangle \subset MM(F)$ has a subnet $\langle e(x_m)\rangle$ which w*-converges to some $\lambda' \in MM(F)$, since $MM(F)$ is compact. For $\delta > 0$ and $(\epsilon, U, D) \ge (\delta, \{s\}, \{g\})$ it follows that $1-\delta \le 1-\epsilon < g(sx(\epsilon, U, D)) = e(x(\epsilon, U, D))\, T_s g$ by the earlier inequality. Therefore, $1-\delta \le \lim_m [e(x_m)(T_s g)] = [\lim_m e(x_m)]\,(T_s g) = \lambda'(T_s g)$. Since δ was arbitrary, $1 \le \lambda' T_s g$. Because $0 \le g \le 1$, $T_s g \le 1$, and so $\lambda'(T_s g) \le 1$. Thus, the first part of (d) is proven. Let $v \in MM(G)$; then $\lambda = v\lambda' \in MM(F)$ and $(T_{\lambda'} T_s g)(t) = \lambda'\,[T_t T_s g] = \lambda'(T_{st} g) = 1 \to \lambda(T_s g) = v\lambda'(T_s g) = v[T_{\lambda'} T_s g] = v1 = 1$; also $v\lambda'(g) = v[T_{\lambda'} g] = v1 = 1$.

$d \to e$: $MM(F) \subseteq M(F)$.

$e \to f$: Let $v \in M(G)$ and μ be as in (e), so that $(T_\mu g)(s) = (\mu T_s g) = 1$; then $v\mu(g) = v(T_\mu g) = v(1) = 1$.

$f \to a$: We prove (not (a)) $\to$ (not (f)). Suppose $\exists \epsilon > 0$, $h \in F(Y)$, $U = \{s_1, s_2, \dots, s_n\} \subseteq S$ such that $\forall x \in X$, $\exists s_x \in U$: $h(s_x x) \le 1-\epsilon$. Define $v = \frac{1}{n}\sum_{k=1}^{n} e(s_k)$. Then ($\forall x \in X$)

$[ve(x)]h = \frac{1}{n}\sum_{k=1}^{n} h(s_k x) \le 1-\epsilon/n$ because $0 \le h \le 1$ and, for some $s_k = s_x$, $h(s_k x) \le 1-\epsilon$. This inequality, valid for all evaluations $e(x)$, also holds for all finite means, and so for all limits $\mu \in M(F)$ of finite means: $v\mu(h) \le 1 - \frac{\epsilon}{n}$. Therefore (f) is impossible.

$d \to g$: Choose $\lambda \in MM(F)$ as in (d). $MM(G)\lambda$ is then an $MM(G)$-invariant set. Since $Cl[e(Y)]$ is closed, it suffices to show that $e(s)\lambda \in Cl[e(Y)]$ for $\forall s \in S$. Suppose that $\exists s_0$: $e(s_0)\lambda \notin Cl[e(Y)]$. Then, since $MM(F)$ is compact Hausdorff and so completely regular, $\exists h \in C(MM(F))$: $0 \le h \le 1$, $h(e(s_0)\lambda) = 0$, and $h(Cl[e(Y)]) = 1$. $g = h \circ e \in F(Y)$ because for $y \in Y$ $g(y) = h(e(y)) = 1$. Then $\lambda(T_{s_0} g) = [e(s_0)\lambda]g = h(e(s_0)\lambda) = 0$, contradicting (d).

$g \to d$: Let I be an $MM(G)$-invariant set $\subseteq Cl(e(Y))$. If $\lambda \in I$, then $e(s)\lambda \in I \subseteq Cl(e(Y))$ ($\forall s \in S$). Therefore, $\lambda(T_s g) = [e(s)\lambda]\, g = 1$ ($\forall g \in F(Y)$). Clearly $\lambda(g) = 1$ ($\forall g \in F(Y)$).

a $\to$ h: Assume Y is F,S-thick in X. Let $\epsilon>0$, $g\epsilon Z(Y)$, finite $U\subseteq S$. If $g=0$, result is trivial; hence, assume that $g\neq 0$. Then $1-\frac{1}{||g||}|g|\epsilon F(Y)$. Consequently, $\exists x\epsilon X$:

$$1-\frac{1}{||g||}|g(s_kx)| \geq 1-\frac{\epsilon}{||g||}, \text{ whence } |g(s_kx)|<\epsilon \ (k=1,\dots,n).$$

h $\to$ a: Assume (h). Let $\epsilon>0$, $g\epsilon F(Y)$, finite $U\subseteq S$. Then $1-g\epsilon Z(Y)$. Therefore, $\exists x\epsilon X: |1-g(s_kx)|<\epsilon \to -\epsilon<1-g(s_kx)<\epsilon \to -g(s_kx)< -1+\epsilon \to g(s_kx)> 1-\epsilon$ $(k=1,\dots,n)$.

h $\to$ i: $\sup\{|g_j| \,|\, g_j\epsilon D\}\epsilon Z(Y)$, because $g_j\equiv 0$ on Y $(j=1,\dots,m)$.

i $\to$ k: For each (ϵ,U,D) in (i) choose $x = x(\epsilon,U,D)$. Define $(\epsilon,U,D)\leq(\epsilon',U',D') \to \epsilon\geq\epsilon'$, $U\subseteq U'$, $D\subseteq D'$. The net $<e(x(\epsilon,U,D))>\subseteq MM(F)$ has a subnet $\langle e(x_m)\rangle$ which converges to some $\lambda\epsilon MM(F)$ since MM(F) is compact. Let $\delta>0$. If $(\epsilon,U,D) \geq (\delta,\{s\},\{g\})$, then $\delta\geq\epsilon>\sup\{|g_j(s_kx(\epsilon,U,D))| \,|\, g_j\epsilon D, s_k\epsilon U\}\geq|g(sx(\epsilon,U,D))|$. Ergo $\delta\geq \lim_m[e(x_m)|T_sg|] = [\lim_m e(x_m)]|T_sg| = \lambda|T_sg|$. Since δ was arbitrary, the first part of (k) is proven. The second part is shown in the same manner as the second part of (c) $\to$ (d).

i $\to$ j: Trivial.

j $\to$ i: In the first part of (j), replace ϵ by $\frac{\epsilon}{n}$: $\frac{\epsilon}{n} > \frac{1}{n}\sum_{k=1}^{n}|g_j(s_kx)|$ $(j=1,\dots,n) \to$

$$\epsilon > \sum_{k=1}^{n}|g_j(s_kx)| > \sup\{|g_j(s_kx)| \,|\, g_j\epsilon D, s_k\epsilon U\}.$$

k $\to$ l, l $\to$ m: Trivial.

m $\to$ h: We show (not (h)) $\Rightarrow$ (not (m)). Suppose $\exists\epsilon>0$, $h\epsilon Z(Y)$, finite $U\subseteq S$ such that $\forall x\epsilon X, \exists s_x\epsilon U$: $|h(s_xx)|\geq\epsilon$. Define $v=\frac{1}{n}\sum_{k=1}^{n}e(s_k)$. Then $\forall x\epsilon X$: $[v(e(x))]|h| =$

$\frac{1}{n}\sum_{k=1}^{n}|h(s_kx)|\geq\epsilon/n$, because $|h| \geq 0$ and for some $s_k = s_x$, $|h(s_kx)|\geq\epsilon$. Hence, replacing e(x) by any finite mean, then for any $\mu\epsilon M(F)$, $v\mu|h|\geq\epsilon/n$. Therefore (m) is impossible. QED

Remark 3.4. Parts d., e., k., and l., of Theorem 3.3 suggest that S behaves with regard to thickness as though it contained an identity. In fact, if S^1 denotes the semigroup S with a discrete identity 1 adjoined, then Y is F,S-thick in X $\to$ Y is F,S^1-thick in X where S^1 acts on X in the natural way.

Corollary 3.5. If the characteristic function $\chi_Y\epsilon F$, then the following statements are equivalent:

a. Y is F,S-thick in X;
b. $\forall$ finite $U = \{s_1,s_2,\dots,s_n\}\subseteq S, \exists x\epsilon X$: $s_kx\epsilon Y$ $(k=1,\dots,n)$;
c. $\forall$ finite $U = \{s_1,s_2,\dots,s_n\}\subseteq S, \exists y\epsilon Y$: $s_ky\epsilon Y$ $(k=1,\dots,n)$;
d. The family $\{s^{-1}Y|s\epsilon S\}$ has the finite intersection property;
e. $\bigcap_{s\epsilon S}\mathrm{Cl}\ e(s^{-1}Y) \neq\emptyset$ where $e(s^{-1}Y) = \{e(x)|sx\epsilon Y\}$.

PROOF: e $\to$ a: Let $\mu\epsilon\bigcap_{s\epsilon S}\mathrm{Cl}\ e(s^{-1})Y$; also let $s\epsilon S$, $g\epsilon F(Y)$. Then $\mu\epsilon\mathrm{Cl}\ e(s^{-1}Y)$, so $\exists$ net $\langle x_n\rangle$ such that $\mu=w^*-\lim e(x_n)$ and $sx_n\epsilon Y$ $(\forall n)$; whence $\mu T_sg = [w^*-\lim_n e(x_n)]\, T_sg = \lim_n [g(sx_n)] = \lim 1_n = 1$. Now let $\lambda\epsilon M(G)$. Then $\lambda\mu\epsilon M(F)$ and $\lambda\mu T_sg = \lambda[T_\mu(T_sg)] =$

$\lambda[L_sT_\mu g] = \lambda[L_s 1] = 1$; also $\lambda\mu(g) = \lambda[T_\mu g] = \lambda[\mu T_{(\cdot)}g] = \lambda[1] = 1$. Therefore by 3.3.e Y is F,S-thick. QED

Results for transformation semigroups comparable to the theorems of section 1 can be generalized in the same way as in [Junghenn 1979, p. 40, theorem 2].

Theorem 3.6. Let $\langle S,X\rangle$ be a transformation semigroup;
$\langle T,Y\rangle$ be a subtransformation semigroup of $\langle S,X\rangle$; and
$F\subseteq B(X)$ be a translation invariant, conjugate-closed, norm-closed subalgebra which contains the constant functions.

If F has invariant mean μ with respect to $\langle T,X\rangle$ such that $\inf\{\mu(g)\,|\,g\in F(Y)\}>0$, then $F|_Y$ has invariant mean with respect to $\langle T,Y\rangle$.

PROOF: X is embedded in the compact set MM(F) by $e(\bullet)$, and $F\text{-}C(MM(F))$ by the Gelfand representation theorem. Also $Cl\ e(Y)\subseteq MM(F)$. By the Riesz representation theorem, the invariant mean μ defines a regular Borel probability measure $\hat\mu$ on MM(F) such that $\mu(f) = \int_{MM(F)}\hat f d\hat\mu\ (\forall f\in F)$. Invariance of μ is reflected in $\hat\mu$ as follows:

$$\int_{MM(F)}T_{e(t)}\hat f d\hat\mu = \int_{MM(F)}\widehat{T_t f}\, d\hat\mu = \mu(T_t f) = \mu(f) = \int_{MM(F)}\hat f d\hat\mu\ (\forall t\in T).$$

Since μ is regular, $\hat\mu(Cl\ e(Y)) = \inf\{\hat\mu(U)\,|\,U \text{ open, } Cl\ e(Y)\subseteq U\}$. Now let U be any open set such that $Cl\ e(Y)\subseteq U$. Because MM(F) is normal, by Urysohn's lemma, $\exists\,\hat g\in C(MM(F))\text{-}F$ such that $\hat g(Cl\ e(Y))\equiv 1$, $\hat g(U^c)\equiv 0$, and $0\le \hat g\le 1$; thus $\hat g\le\chi_U$ and g, the correlative of $\hat g$, is in F(Y). $\mu(g) = \int_{MM(F)}\hat g d\hat\mu \le \int_{MM(F)}\chi_U d\hat\mu = \hat\mu(U)$. Therefore by hypothesis $0 < \inf\{\mu(g)\,|\,g\in F(Y)\} \le \inf\{\hat\mu(U)\,|\,U \text{ open, } Cl\ e(Y)\subseteq U\} = \hat\mu(Cl\ e(Y))$. Ergo,

$v(f) = \dfrac{1}{\hat\mu(Cl\ e(Y))}\int_{Cl\ e(Y)}\hat f d\hat\mu$ is a mean on F.

Define v_0 on $F|_Y$ by $v_0(f|_Y) = v(f)$. v_0 is well-defined because $f|_Y \equiv g|_Y \rightarrow f-g\in Z(Y) \rightarrow \widehat{(f-g)} \equiv 0$ on $Cl\ e(Y) \rightarrow 0 = v(f-g) = v(f)-v(g)$. Also $v_0\in M(F|_Y)$.

To show that v_0 is invariant it suffices to prove that $\int_{Cl\,e(Y)}T_{e(t)}\hat f d\hat\mu = \int_{Cl\,e(Y)}\hat f d\hat\mu\ (\forall t\in T)$.

Fix $t\in T$. Define $E_1 = e(t)^{-1}(Cl\ e(Y))\backslash Cl\ e(Y)$, $E_n = e(t)^{-1}(E_{n-1})$ $(n\ge 2)$.
The E_n are pairwise disjoint: $\mu\in E_2 \rightarrow e(t)\mu\in E_1 \rightarrow e(t)\mu\notin Cl\ e(Y) \rightarrow \mu\notin E_1$, so $E_1\cap E_2 = \emptyset$. Assume that E_m and E_n are pairwise disjoint $(1\le m<n)$. Then $\mu\in E_{n+1} \rightarrow e(t)\mu\in E_n \rightarrow e(t)\mu\notin E_m$ $(1\le m<n)$ $\rightarrow \mu\notin e(t)^{-1}E_m = E_{m+1} = E_p$ $(2\le p=m+1<n+1)$, so $E_{n+1}\cap E_p = \emptyset$. Also $\mu\in E_{n+1} \rightarrow e(t)^n\mu\in E_1$ (by induction) $\rightarrow e(t)^n\mu\notin Cl\ e(Y)$, but $\mu\in E_1 \rightarrow e(t)\mu\in Cl\ e(Y) \rightarrow e(t)^n\mu\in CL\ e(Y)$ (by invariance of Y), so $E_{n+1}\cap E_1 = \emptyset$. The E_n are Borel sets since $\mu \rightarrow e(t)\mu$ is w*=continuous for $\forall\mu\in MM(F)$.

Because $(\forall n\ge 2)$ $T_{e(t)}\chi_{E_{n-1}}(\mu) = \chi_{E_{n-1}}(e(t)\mu) = \chi_{e(t)^{-1}E_{n-1}}(\mu)$, it follows that

$$\hat\mu(E_n) = \hat\mu(e(t)^{-1}E_{n-1}) = \int_{MM(F)}\chi_{e(t)^{-1}E_{n-1}}d\hat\mu = \int_{MM(F)}T_{e(t)}\chi_{E_{n-1}}d\hat\mu = \int_{MM(F)}\chi_{E_{n-1}}d\hat\mu = \hat\mu(E_{n-1}).$$

Therefore, $1\ge\hat\mu(E_1\cup E_2\cup\ldots\cup E_n) = \sum_{j=1}^{n}\hat\mu(E_j) = n\hat\mu(E_1)$. Since this holds for arbitrary n,

$\hat\mu(E_1) = 0$.

Because Y is invariant, $e(T)Cl\ e(Y)\subseteq Cl\ e(Y)$, whence $Cl\ e(Y)\setminus e(t)^{-1}Cl\ e(Y) = \emptyset$. Since $Cl\ e(Y)\Delta e(t)^{-1}Cl\ e(Y) = [Cl\ e(Y)\setminus e(t)^{-1}Cl\ e(y)]\cup E_1 = E_1$, $\hat\mu[Cl\ e(Y)\Delta e(t)^{-1}Cl\ e(Y)] = 0$, so

$$\int_{Cl\ e(Y)}T_{e(t)}\hat f d\hat\mu = \int_{e(t)^{-1}Cl\ e(Y)}T_{e(t)}\hat f d\hat\mu = \int_{MM(F)}T_{e(t)}[\hat f\chi_{Cl\ e(Y)}]d\hat\mu = \int_{Cl\ e(Y)}\hat f d\hat\mu.$$ QED

Theorem 3.7. Let $\langle S,X\rangle$ be a τ-semigroup;
$\langle T,Y\rangle$ be a sub τ-semigroup of $\langle S,X\rangle$;
$F\subseteq B(X)$ be a translation invariant, norm-closed, G-introverted subalgebra which contains the constant functions.

1. If $F|_Y$ has an invariant mean with respect to $\langle T,Y\rangle$ and T is G-thick in S, then F has an invariant mean with respect to $\langle S,X\rangle$.
2. If G has a left-invariant mean and Y is F,S-thick in X, then $F|_Y$ has an invariant mean with respect to $\langle T,Y\rangle$.

PROOF: 1. Functional $\bar{\mu}$ in $F|_Y{}^*$ defines a functional μ in F^* by $\mu f = \bar{\mu} f|_Y$ $(\forall f\in F)$, thus $\mu T_t f = \bar{\mu} T_t f|_Y$ $(\forall f\in F, t\in T)$. Therefore, because F is G-introverted, $F|_Y$ is $G|_T$-introverted.

Relative to the algebra $F|_Y$ defined on $\langle T,Y\rangle$: Let $\bar{\mu}$ be an invariant mean of $F|_Y$; then $e(t)\bar{\mu}=\bar{\mu}(T_s\bullet)=\bar{\mu}$ $(\forall t\in T)$ where $e(t)\in MM(G|_T)$. Let $\bar{\lambda}\in \mathrm{Cl}\ e(T) = MM(G|_T)$, and let $\langle e(t_\alpha)\rangle\subseteq e(T)\subseteq MM(G|_T)$ be a net such that $\bar{\lambda} = w^*-\lim e(t_\alpha)$. Ergo,

$\bar{\lambda}\bar{\mu} = [w*-\lim_\alpha e(t_\alpha)]\bar{\mu} = \lim_\alpha [e(t_\alpha)\bar{\mu}] = \lim_\alpha \bar{\mu} = \bar{\mu}$. That is, $\bar{\lambda}\bar{\mu} = \bar{\mu}$ $(\forall\bar{\lambda}\in \mathrm{Cl}\ e(T))$.

Relative to the algebra F defined on $\langle S,X\rangle$: $\exists$ left-ideal K of Cl e(S) in Cl $e(T)\subseteq MM(G)$ [Wilde & Witz, 1967, lemma 5.1]. Choose $\lambda_0\in K$. Then $e(s)\lambda_0\in K\subseteq \mathrm{Cl}\ e(T)\subseteq MM(G)$ $(\forall s\in S)$.

Any $\lambda\in$ Cl $e(T)\subseteq MM(G)$ gives rise to a $\bar{\lambda}\in$ Cl $e(T)\subseteq MM(G|_T)$ in the following way: $\lambda'=w^*-\lim_\alpha e(t_\alpha)\in MM(G)$. Now $\langle e(t_\alpha)\rangle$ is a net in $e(T)\subseteq MM(G|_T)$ so has a convergent subnet $\langle e(t_\beta)\rangle$ with $\bar{\lambda}=w^*-\lim e(t_\beta)\in MM(G|_T)$. $\bar{\lambda}$ may not be unique. For $\bar{\mu}\in F|_Y{}^*$ define $\mu\in F^*$ by $\mu f=\bar{\mu} f|_Y$ $(\forall f\in F)$ as we have done earlier. Then for all $f\in F$ $\bar{\lambda}\bar{\mu} f|_Y = \bar{\lambda}(T_{\bar{\mu}} f|_Y) =$ $\lim_\beta [e(t_\beta)T_{\bar{\mu}} f|_Y] = \lim_\beta [\bar{\mu} T_{t_\beta} f|_Y]$; also. $\lambda\mu f = \lambda(T_\mu f) = \lim_\alpha [\bar{\mu} T_{t_\alpha} f|_Y]$; ergo $\lambda\mu(f) =$ $\bar{\lambda}\bar{\mu}(f|_Y)$, regardless of the choice of $\bar{\lambda}$ which is associated with λ.

Finally, choose $\bar{\mu}$ to be an invariant mean of $F|_Y$, and define $\mu\in M(F)$ as before. Then $\lambda\mu(f) = \bar{\lambda}\bar{\mu}(f|_Y) = \bar{\mu}(f|_Y) = \mu(f)$, that is, $\lambda\mu=\mu$ $(\forall\lambda\in \mathrm{Cl}\ e(T)\subseteq MM(G))$. In particular, $e(s)\lambda_0\mu = \mu$ $(\forall s\in S)$, so that $\lambda_0\mu$ is invariant.

2. Because Y is F,S-thick in X, then by Theorem 3.3.f $\exists\mu\in M(F)$ such that $\nu\mu(f) = 1$ $(\forall\nu\in M(G), f\in F(Y))$. Let ν be an invariant mean of G. Then $\nu\mu$ is an invariant mean of F such that $\nu\mu(f) = 1$ $(\forall f\in F(Y))$. By Theorem 3.6 $F|_Y$ has an invariant mean with respect to $\langle T,Y\rangle$. QED

In the preceding theorem the thickness condition on T in (1) implies the thickness condition on Y in (2) according to the following lemma:

Lemma 3.8. Let $\langle S,X\rangle$ be a τ-semigroup;
$\langle T,Y\rangle$ be a sub τ-semigroup of $\langle S,X\rangle$;
$F\subseteq B(X)$ be a translation-invariant, norm-closed, G-introverted subalgebra which contains the constant functions.

If T is G-thick in S, then Y is F,S-thick in X.

PROOF: Let $f\in F(Y)$: $0\le f\le 1$, $f\equiv 1$ on Y. Then $T_{e(y)}f\in F(T)$ $(\forall y\in Y)$. By Theorem 3.3.e applied to L(S,G) $\exists\mu\in M(G)$ such that $1 = \mu(L_s T_{e(y)}f) = \mu(T_{e(y)}T_s f) = \mu e(y)T_s f$ and $1 = \mu T_{e(y)}f = \mu e(y)f$. Then $\mu e(y)\in M(F)$ has the properties required by Theorem 3.3.e for Y to be F,S-thick. QED

Junghenn's theorem of section 1 is obtained from Theorem 3.7 and Lemma 3.8 by letting $X = S$, $Y = T$, and the action be left multiplication.

4. *Multiplicative Means and Thickness*

Several results connect multiplicative means with thickness. F is assumed to be an S-invariant, norm-closed algebra $\subseteq C(X)$ which contains the constant functions.

Theorem 4.1 If F has an invariant multiplicative mean, then for any finite partition $\{A_i\}_1^n$ of X $\exists k$ such that A_k is F,S-thick.

PROOF: Let $v \in MM(F)$ be invariant. v induces a regular Borel probability measure $\hat{v}$ defined on MM(F), and $\sum_1^n \hat{v}(\mathrm{Cl}\, e(A_i)) \ge 1$. Because v is multiplicative, for each i $\hat{v}(\mathrm{Cl}\, e(A_i)) = 0$ or $\hat{v}(\mathrm{Cl}\, e(A_i)) = 1$. Hence, $\exists k$ such that $\hat{v}(\mathrm{Cl}\, e(A_k)) = 1$. Therefore, $v(f) = 1$ $(\forall f \in F(A_k))$ because $\chi_{A_k} \le f \le 1 \rightarrow \chi_{\mathrm{Cl}\, e(A_k)} \le \hat{f} \le 1$ and $1 = \hat{v}(\mathrm{Cl}\, e(A_k)) = \int \chi_{\mathrm{Cl}\, e(A_k)} d\hat{v} \le \int \hat{f}\, d\hat{v} = v(f) \le 1$. Then, by Theorem 3.3.d A_k is F,S-thick. QED

Definition 4.2. $K(f,s) = \{\mu \in MM(F) \mid \mu(T_s f - f) = 0\}$

Theorem 4.3. The following are equivalent:

a. F has an invariant multiplicative mean;

b. It is not the case that $MM(F) \subseteq \bigcup_{\substack{f \in F \\ s \in S}} K^c(f,s)$;

c. It is not the case that $\exists f_1,\dots,f_n \in F$; $\exists s_1,\dots,s_n \in S$: $MM(F) \subseteq \bigcup_{i=1}^n K^c(f_i,s_i)$;

d. $\forall f_1,\dots,f_n \in F$; $\forall s_1,\dots,s_n \in S$; $\forall \delta>0$; $\exists x_\delta$: $e(x_\delta) \sum_{i=1}^n | T_{s_i} f_i - f_i | < \delta$;

e. $\forall f_1,\dots,f_n \in F$; $\forall s_1,\dots,s_n \in S$; $\forall \delta>0$, $\exists x_\delta$: $T_{s_i} f_i(x_\delta) - f_i(x_\delta)| < \delta$ $(i=1,\dots,n)$;

f. $\forall f_1,\dots,f_n \in F$; $\forall s_1,\dots,s_n \in S$; $\exists \lambda \in MM(F)$: $\lambda | T_{s_i} f_i - f_i | = 0$ $(i=1,\dots,n)$;

g. $\forall f_1,\dots,f_n \in F$; $\forall s_1,\dots,s_n \in S$; $\exists \lambda \in MM(F)$: $\lambda(T_{s_i} f_i - f_i) = 0$ $(i=1,\dots,n)$;

h. $\forall \epsilon>0$; $\forall f_1,\dots,f_n \in F$; $\forall s_1,\dots,s_n \in S$: $\exists c_1,\dots,c_n \in C$; $\exists Y \subseteq X$: $|f_k - c_k| < \epsilon$ and $|T_{s_k} f_k - c_k| < \epsilon$ on Y $(k=1,\dots,n)$ and Y is F,S-thick in X.

PROOF: a $\rightarrow$ b: F has an invariant multiplicative mean $\rightarrow \exists \lambda \in MM(F)$: $\lambda \in K(f,s)$ $(\forall f \in F, s \in S)$ $\rightarrow$ the $K^c(f,s)$ do not cover all of MM(F).

~b $\rightarrow$ ~c: MM(F) is compact and the $K^c(f,s)$ are open.

~c $\rightarrow$ ~d: Let $f_1,\dots,f_n \in F$ and $s_1,\dots,s_n \in S$ be as in the negation of (c). If for any $\delta>0$ $\exists x_\delta \in X$ such that $e(x_\delta) \sum | T_{s_k} f_k - f_k | = \sum | T_{s_k} f_k(x_\delta) - f(x_\delta) | < \delta$, then the net $\langle e(x_\delta) \rangle_{\delta>0} \subseteq MM(F)$ contains a convergent subnet $\langle e(x_{\delta_\alpha}) \rangle_{\alpha \in A}$ of $\langle e(x_\delta) \rangle$ and $w*-\lim_\alpha e(x_{\delta_\alpha}) = \lambda \in MM(F)$; thus, for any $\epsilon>0$ $\exists \alpha_0 \in A$: $\alpha \ge \alpha_0 \rightarrow | \lambda \sum | T_{s_k} f_k - f_k | - e(x_{\delta_\alpha}) \sum | T_{s_k} f_k - f_k | | < \frac{\epsilon}{2}$. Let $\alpha_1 \in A$ be $\ge \alpha_0$ and such that $\delta_{\alpha_1} < \frac{\epsilon}{2}$, so that $e(x_{\delta_{\alpha_1}}) \sum | T_{s_k} f_k - f_k | < \frac{\epsilon}{2}$. Then $0 \le \lambda \sum | T_{s_k} f_k - f_k | < e(x_{\delta_{\alpha_1}}) \cdot \sum | T_{s_k} f_k - f_k | + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ was arbitrary, $\lambda \sum | T_{s_k} f_k - f_k | = 0 \rightarrow | T_{s_k} f_k - f_k | = 0$ $(\forall k) \rightarrow \lambda(T_{s_k} f_k - f_k) = 0$ $(\forall k)$. The last equation contradicts that $\lambda \in \bigcup_{1=1}^n K^c(f_i, s_i)$.

~d → ~c: Suppose that $\exists\, f_1,\dots,f_n \in F$ and $s_1,\dots,s_n \in S$ and $\delta > 0$ such that $(\forall x)\ e(x) \sum |T_{s_k} f_k - f_k| \geq \delta$. Let $\lambda \in MM(F)$, so that $\lambda = w^*\text{-}\lim e(x_\nu)$ with $x_\nu \in X$ $(\forall \nu)$.

Then $\lambda \sum |T_{s_k} f_k - f_k| = w^* - \lim e(x_\nu) \sum |T_{s_k} f_k - f_k| \geq \delta \rightarrow \exists k^0$ such that

$\frac{\delta}{n} \leq \lambda |T_{s_{k^0}} f_{k^0} - f_{k^0}| = |\lambda(T_{s_{k^0}} f_{k^0} - f_{k^0})|$ $(\lambda|g| = |\lambda g|$ because λ is multiplicative)

$$\Rightarrow \lambda(T_{s_{k^0}} f_{k^0} - f_{k^0}) \neq 0 \rightarrow \lambda \notin K(f_{k^0}, s_{k^0}) \rightarrow \lambda \in K^c(f_{k^0}, s_{k^0}) \rightarrow \lambda \in \bigcup_{k=1}^{n} K^c(f_k, s_k).$$

c → f: $\langle e(x_\delta)\rangle_{\delta>0}$ is a net in MM(F) so has a convergent subnet $\langle e(x_{\delta_\alpha})\rangle_{\alpha\in A}$. Let λ denote the w^*-limit of $\langle e(x_{\delta_\alpha})\rangle$. Then by the same reasoning as in ~c → ~d, $\exists \alpha_1 \in A$ such that $0 \leq \lambda|T_{s_k} f_k - f_k| < e(x_{\delta_{\alpha_1}})\ |T_{s_k} f_k - f_k| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ is arbitrary, $\lambda|T_{s_k} f_k - f_k| = 0$.

f → e: Since $\lambda \in MM(F)$, $\lambda = w^*\text{-}\lim e(x_\nu)$ for some net $\langle e(x_\nu)\rangle$ with $x_\nu \in X$ $(\forall \nu)$. By the definition of w^*-convergence, for any $\delta > 0$ $\exists\, e(x_\delta) \in \langle e(x_\nu)\rangle$ such that $e(x_\delta)|T_{s_i} f_i - f_i| < \delta$ $(i=1,\dots,n)$.

a → h: Assume (a) and let $f_1,\dots,f_n \in F$; $s_1,\dots,s_n \in S$; and $\epsilon > 0$.

Notation: $L(r_1,\dots,r_n) = f^{-1}(S_\epsilon(r_1)) \cap f_2^{-1}(S_\epsilon(r_2)) \cap \dots \cap f_n^{-1}(S_\epsilon(r_n)) \cap (T_{s_1} f_1)^{-1}(S_\epsilon(r_1)) \cap \dots \cap (T_{s_n} f_n)^{-1}(S_\epsilon(r_n))$ for $r_1,\dots,r_n \in \mathbb{C}$, where $S_\epsilon(r_k) = \{x \in \mathbb{C} \mid |x - r_k| < \epsilon\}$ $(k=1,\dots,n)$. If some $L(r_1,\dots,r_n)$ is F,S-thick, then it suffices for the Y of (h) with $r_1 = c_1,\dots,r_n = c_n$. Assume that no $L(r_1,\dots,r_n)$ is F,S-thick. A contradiction shall be deduced. For each non-empty $L(r_1,\dots,r_n)$ and for each $\lambda \in MM(F)$, $\exists s \in S$, $\exists g \in Z(L(r_1,\dots,r_n))$ such that $\lambda(T_s(g)) \neq 0$ by (k) of Theorem 4.3. In particular, if λ is invariant, then $\lambda(g) = \lambda(T_s(g)) \neq 0$. Let $\langle e(x_\nu)\rangle$ be a net in MM(F) such that $\lambda = w^* - \lim_\nu e(x_\nu)$. Then for $i=1,\dots,n$, $\exists N_i$ such that $\nu \geq N_i \rightarrow |f_i(x_\nu) - \lambda f_i| < \epsilon$ and $|T_{s_i} f_i(x_\nu) - \lambda f_i| < \epsilon$; this entails that $\nu \geq N_1, N_2, \dots, N_n \rightarrow |f_i(x_\nu) - \lambda f_i| < \epsilon$ and $|T_{s_i} f_i(x_\nu) - \lambda f_i| < \epsilon$ $(i=1,\dots,n) \rightarrow x_\nu \in L(\lambda f_1,\dots,\lambda f_n)$. For $L(\lambda f_1,\dots,\lambda f_n)$, $\exists g \in Z(L(\lambda f_1,\dots,\lambda f_n))$ with $\lambda(g) \neq 0$, as previously noted, so $g(x_\nu) = 0$ for all $\nu \geq N_1, N_2, \dots, N_n$. Therefore, $\lambda(g) = \lim_\nu e(x_\nu) g = 0$, a contradiction.

d → e, e → d, f → g, h → e: Easy QED

REFERENCES

1. Berglund, John F., Junghenn, Hugo D., and Milnes, Paul. *Compact Right Topological Semigroups and Generalizations of Almost Periodicity*. Lecture Notes in Mathematics No. 663. Berlin: Springer-Verlag, 1978.

2. Junghenn, Hugo D. "Amenability of Function Spaces on Thick Sub-semigroups." *Proceedings of the American Mathematical Society*, Vol. 75, No. 1 (1979), 37–41.

3. Lau, Anthony To-Ming. "Extremely Amenable Algebras." *Pacific Journal of Mathematics*, Vol. 33, No. 2 (1970), 329–336.

4. ———. "Action of topological semigroups, invariant means, and fixed points." *Studia Mathematica*, Vol. 43 (1972), 139–156.

5. Mitchell, Theodore. "Constant Functions and Left Invariant Means on Semigroups." *Transactions of the American Mathematical Society*, Vol. 119 (1965 August), 244–261.

6. Sakai, Kokichi. "On amenable transformation semigroups."
 I. *Journal of Mathematics, Kyoto Univ.* 16–3 (1976), 555–595.
 II. *Ibid.*, 597–626.
 III. *Scientific Reports, Kagoshima Univ.*, No. 25 (1976 September), 31–51.

7. Simmons, George F. *Introduction to Topology and Modern Analysis*. New York: McGraw-Hill Book Co., 1963.

8. Wilde, Carroll, and Witz, Klaus. "Invariant Means and the Stone-Čech Compactification." *Pacific Journal of Mathematics*, Vol. 21 (1967), 577–586.