FOURIER TRANSFORMS OF DINI-LIPSCHITZ FUNCTIONS ON VILENKIN GROUPS

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ABSTRACT. In [4] we proved some theorems on the Fourier Transforms of functions satisfying conditions related to the Dini-Lipschitz conditions on the n-dimensional Euclidean space $R^n$ and the torus group $T^n$. In this paper we extend those theorems for functions with Fourier series on Vilenkin groups.

KEY WORDS AND PHRASES. Dini-Lipschitz Functions, Vilenkin Fourier Series.
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1. INTRODUCTION.

Let $f(z)$ belong to the Lebesgue space $L^p, 1 < p \leq 2$ of functions on the real line $R$ or on the circle group $T$ with its usual norm $\| \cdot \|_p$. The $p$th integral modulus of continuity $\omega_p(f, h)$ is defined as

$$\omega_p(f, h) = \sup \| f(x + h) - f(x) \|_p .$$

In [4] (Theorem 3.3) we proved that if $f(x)$ belongs to $L^p(R)$ such that

$$\omega_p(f, h) < \theta (\log h)^{\alpha}, \quad 0 < \alpha \leq 1$$

then the Fourier Transform $\hat{f}$ belongs to $L^\beta(R)$ where

$$p/(p + \alpha p - 1) < \beta \leq p'/p - 1, \quad \beta > 1.$$
known that
\[ \{e\} = V_0 \subset V_1, \ldots \subset \hat{G}, \text{ and that } \bigcup_{n \geq 0} V_n = \hat{G}. \]

If all \( V_n \) are finite, the inclusion is proper. We introduce the numbers \( m_0, m_1, m_2, \ldots, m_k \) such that
\[ m_0 = 1, m_{k+1} = p_k m_k; \quad k \in \mathbb{N}, \]
p_\text{\( k \)} being a prime \( \geq 2 \). Then every \( V_n \) has \( m_n \) as its measure and the quotient subgroup \( V_n/V_{n-1} \) has \( P_n \) for its measure.

DEFINITION 2.1. For \( x \in G \), let \( (n,x) \) denote the continuous character of \( x \), i.e. \( (n,x) \in \hat{G} \). The Fourier transform \( \hat{f}(n) \) of \( f(x) \in L^p(G) \) is defined by
\[ \hat{f}(n) = \int_G f(x) \overline{(n,x)} dx \]
where \( \overline{(n,x)} \) is the complex conjugate of \( (n,x) \).

DEFINITION 2.2. Let \( f \in L^p(G) \). The \( p \)th modulus of continuity \( \omega_p(f,k) \) is defined by
\[ \omega_p(f,k) = \sup_{h \in G} \| f(x+h) - f(x) \|_p \]
The Lipschitz and the Dini-Lipschitz classes in \( L^p(G) \) are those for which \( \omega_p(f,k) = 0 \) \( (m_k^{-\alpha}) \)
and \( 'o |\log m_k|^{-1} \) respectively.

DEFINITION 2.3 If every \( P_k \) is finite as \( k \to \infty \) we say that \( G \) has the boundedness property \((P)\).

3. MAIN RESULTS.

With the previous definitions and notations in hand, we now prove the analogue of Theorem 3.3 in [4]. Thus we state the following

THEOREM 3.1. Le \( f \in L^p(G) \), \( 1 < p \leq 2 \), such that
\[ \omega_p(f,k) = 0(m_k^{-\alpha}/(\log m_k)^\gamma), \quad 0 < \alpha \leq 1. \]
Then \( \hat{f}(n) \in L^q(\hat{G}) \) for \( q = P/(p-1) \geq \beta > \max(p/(p+\alpha p-1), 1/\gamma) \).

PROOF. Since the Fourier transform of \( f(x+h) - f(x) \) is given by \( \hat{f}(n)(n,h)-1 \), the Hausdorff-Young theorem yields
\[ \sum_{n \neq n_k} |\hat{f}(n)|^q |(n,h)|^{-1} < M \omega_p(f,k)^q = 0(m_k^{-\alpha q}/(\log m_k)^{\gamma q}). \]
The boundedness property \((P)\) for \( G \) gives (see Onneweer [1], (2)).
\[ \sum_{n=m_k}^{m_k+1} |\hat{f}(n)|^q = 0(m_k^{-\alpha q}/(\log m_k)^{\gamma q}) \]
Applying the Holder’s inequality with \( \beta \leq \gamma \) for the last estimate one arrives at
\[ \sum_{n=m_k}^{m_k+1} \int_{\hat{G}} |f(n)|^\beta = 0(m_k^{-\alpha \beta}/(\log m_k)^{\gamma \beta}) (m_k^{1-\beta/\gamma}) \]
and this leads to the final estimate
\[ \sum_{n \in \hat{G}} |\hat{f}(n)|^\beta = 0 \left( \sum_{k=0}^{\infty} (m_k^{1-\alpha \beta - \beta + \beta/p}) (\log m_k^{-\gamma \beta}) \right). \]
If $1 - \alpha \beta - \beta / p < 0$ and $-\gamma \beta < -1$ the series is convergent since $m_k \geq 2^k$, and this proves the theorem.

REMARK 3.2. We remark here that for special choice of $\alpha, \gamma,$ and $P$ like $\alpha = 1, \gamma = 1, P = 2$, the previous theorem gives special interesting cases. This is quite obvious and we shall not deal with it any further. However, the special case $P = 2$ and $o < \alpha < 1$ is particularly important and deserves special consideration.

4. FUNCTIONS IN $L^2(G)$. The origin of this section is a theorem proved in Titchmarsh ([3] Theorem 85, p. 117) for functions belonging to $\operatorname{Lip}(\alpha, 2)$ on the real line $R$. For further reference we state it as.

THEOREM 4.1. Let $f(x) \in \operatorname{Lip} (\alpha, 2)$ on $R$. Then the conditions

$$\omega_2(f, h) = o(h^\alpha), \quad 0 < \alpha < 1, \quad h \to 0$$

and

$$\left[ \int_{-\infty}^{\infty} + \int_{\infty}^{X} \right] |\hat{f}|^2 \, du = o(X^{-2\alpha}), \text{ as } X \to \infty$$

are equivalent.

This theorem was studied rather extensively in [5] and [6] for functions in $L^2(R^2)$, and $L^2(T^2)$ respectively, where several conditions of the order of magnitude for the Fourier transforms $\hat{f}$ of $f$ proved to be equivalent to one another.

In [4] (Theorems 5.1, 5.2) we proved an analogue of Theorem 4.1 for the Dini-Lipschitz functions in $L^2(R)$. In this section we shall prove Theorem 5.2 in [4] for functions in $L^2(G)$.

THEOREM 4.2. Let $f(x) \in L^2(G)$. Then the conditions

$$\omega_2(f, k) = o(h^\alpha/(\log h)^\gamma), \quad h \in G_k$$

and

$$\sum_{n \equiv m_k}^{\infty} |\hat{f}(n)|^2 = o(m_k^{-2\alpha}/(\log m_k^\gamma))^\gamma$$

are equivalent. Here $h = m_k^{-1}$.

PROOF. That the first implication is true follows from Theorem 3.1 where it is proved that

$$\sum_{n \equiv m_k}^{\infty} |\hat{f}(n)|^q = o((\omega_p(f, k))^q)$$

Taking $p = q = 2$ and substituting for $h = m_k^{-1}$ we obtain (4.2). We also hint that an argument based on the Parseval’s identity similar to that of Titchmarsh’s leads independently to the same result. To prove the converse let (4.2) hold. Then

$$\sum_{n \equiv m_k}^{m_k+1-1} |\hat{f}(n)|^2 = o(m_k^{-2\alpha}/(\log m_k)^\gamma - 0(m_k^{-2\alpha}/(\log m_k+1)^\gamma)^\gamma)$$

Since $G$ has the boundedness property ($P$); hence every $P_k = m_k+1/m_k$ is finite for all $k \in N$, the same is true of $\log P_k$. Thus the right hand sides of (4.2) and (4.3) are the same. This applies to estimates of the form

$$\sum_{n \equiv m_k}^{m_k+1-1} |\hat{f}(n)|^2 \quad (n, h) - 1 |^2$$
To sum up, by the Parseval's identity one obtains

\[ \int_G |f(x+h) - f(x)|^2 dx = \sum_{n} |\hat{f}(n)|^2 |(n,h) - 1|^2 \]

\[ = \sum_{m_k} \frac{m_k}{\alpha} + \sum_{m_k} \frac{1}{\alpha} = \frac{m_k - 2\alpha}{(\log m_k)^{2\gamma}} \]

This is equivalent to (4.1) upon substituting for \( h = m_k^{-1} \), \( h \in G_k \) and the proof is complete.

REMARK 4.2. We conclude by indicating that Theorem 5.1 in [4] is true for Vilenkin Fourier series, since, it can be deduced as a special case of Theorem 4.1. We also add that for \( 0 < \alpha < 1 \), Theorems 3.1 and 4.1 of the present paper can be proved for higher differences of \( f(x) \in L^p(G) \). The statements and the proofs are almost straightforward and will not be given.

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REFERENCES

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