

## HYPERSURFACES IN A CONFORMALLY FLAT SPACE WITH CURVATURE COLLINEATION

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**ABSTRACT.** We classify the shape operators of Einstein and pseudo Einstein hypersurfaces in a conformally flat space with a symmetry called curvature collineation. We solve the fundamental problem of finding all possible forms of non-diagonalizable shape operators. A physical example of space-time with matter is presented to show that the energy condition has direct relation with the diagonalizability of shape operator.

**KEY WORDS AND PHRASES:** Shape operators, Proper and improper Hypersurfaces, Einstein and pseudo-Einstein hypersurfaces, curvature collineation, matter tensor.

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**1. INTRODUCTION.** The eigenvectors of the shape operator (second fundamental form operator) of a hypersurface of a semi-Riemannian space need not be all real. They are real for positive definite hypersurfaces. For indefinite hypersurfaces some of them may be complex and zero length (null). In the latter case the real eigenvectors (principal directions) may not span the tangent space of the hypersurface at every point. If the eigenvalues (principal normal curvatures) are real and no eigenvectors are null at every point, then the hypersurface is called proper in the terminology used by Fialkow [1]. A hypersurface which is not proper, is called improper.

Usually one prefers to embed a hypersurface in a flat (Euclidean or Minkowskian) space. But, as pointed out by Goenner [2], there is no specific reason for choosing the ambient space flat, one can consider other ambient spaces such as a space form, a Ricci-flat space, an Einstein space and a conformally flat space. Fialkow [1] provided a complete classification of proper Einstein hypersurfaces in an indefinite space form. Magid [8] has algebraically classified improper Einstein hypersurfaces in an indefinite space form.

The aim of this paper is to present an algebraic classification of proper/improper Einstein and pseudo Einstein hypersurfaces in a conformally flat space. An example of this space-time is considered to show that the energy condition has direct relation with the diagonalizability of the shape operator.

**DEFINITION 1.1.** (Katzin et al [4]). A vector field  $V$  in a semi-Riemannian space is said to generate a 1-parameter group of curvature collineations if it satisfies:  $L_V R = 0$ , where  $L_V$  and  $R$  denote the Lie-derivative operator along  $V$  and the Riemann curvature tensor, respectively.

Curvature collineation ( $CC$ ) is a fundamental symmetry [4] property of semi-Riemannian spaces. Indeed, it is known [4] that, for Ricci-flat spaces, more familiar symmetries such as projective and conformal collineations (including affine collineations, motions, conformal and homothetic motions) are subcases of  $CC$ . Physically, the well-known Komar's covariant identity [8] (which acts as a conservation law generator in general relativity) follows naturally as a necessary condition for a  $CC$ . Thus  $CC$ 's are necessarily embraced by the group of general curvilinear coordinate transformations.

We, therefore, choose a conformally flat embedding space  $\bar{M}$  admitting a 1-parameter group of  $CC$ 's. Our choice of the embedded space is a class of semi-Riemannian spaces defined as follows:

**DEFINITION 1.2.** A semi-Riemannian space is said to be pseudo-Einstein if there exists a 1-form  $u$  such that

$$Ric = \chi g + \eta u \otimes u \quad (1.1)$$

and  $g(U, U) = \epsilon(\epsilon^2 = 1)$ ; where  $g(U, X) = u(X)$ , Ric denotes the Ricci tensor and  $\chi, \eta$  are scalar functions. For  $\eta = 0$ , it reduces to an Einstein space.

The above definition is motivated by

- (1) Yano's definition [11] of a pseudo-Einstein hypersurface of a Kaehlerian space, given by equation (1.1) with  $\chi, \eta$  as constants.
- (2) Einstein's field equations in the framework of general relativity [4]

$$Ric + \left\{ \Lambda - \frac{1}{2}r \right\} g = pg + (\mu + p)u \otimes u,$$

where  $p$  and  $\mu$  are the pressure and energy densities of a perfect fluid,  $u$  is the 1-form metrically equivalent to the velocity 4-vector  $U$  and  $\Lambda$  stands for the cosmological constant.

**2. SHAPE OPERATOR OF EINSTEIN HYPERSURFACES.** First we state the following result of Katzin et al [5].

**LEMMA 2.1.** Let a vector field  $V$  generate  $CC$  in an  $m$ -dimensional conformally flat space  $\bar{M}$  with metric  $g$ . If  $\bar{M}$  is a space form then  $V$  defines a motion (isometry) for  $m \geq 3$  and a conformal motion for  $m = 2$ . If  $\bar{M}$  is not a space form then  $L_V g = 2\sigma g + \tau \bar{Ric}$ , where  $\sigma$  and  $\tau$  are scalar functions and  $\bar{Ric}$  is the Ricci tensor of  $\bar{M}$ .

Katzin et al [5,6] have shown that there are essentially only two types of conformally flat spaces admitting proper  $CC$  viz. reducible and irreducible. In the reducible case  $K_1 \times K_{n-1}$



As  $\bar{M}$  is conformally flat, we have

$$g(\bar{R}(\bar{X}, \bar{Y})\bar{Z}, \bar{W}) = [\bar{\text{Ric}}(\bar{Y}, \bar{Z})g(\bar{X}, \bar{W}) - \bar{\text{Ric}}(\bar{X}, \bar{Z})g(\bar{Y}, \bar{W}) \\ + g(\bar{Y}, \bar{Z})\bar{\text{Ric}}(\bar{X}, \bar{W}) - g(\bar{X}, \bar{Z})\bar{\text{Ric}}(\bar{Y}, \bar{W})]/(n-1) \\ - \{\bar{r}/n(n-1)\}[g(\bar{Y}, \bar{Z})g(\bar{X}, \bar{Z})g(\bar{Y}, \bar{W})] \quad (2.6)$$

The Gauss equation for  $M$  is

$$g(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) \\ - \epsilon\{g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W)\} \quad (2.7)$$

where  $A$  is the shape operator of  $M$ . Using it in (2.6) gives

$$g(R(X, Y)Z, W) = [\bar{\text{Ric}}(Y, Z)g(X, W) - \bar{\text{Ric}}(X, A)g(Y, W) \\ + g(Y, Z)\bar{\text{Ric}}(X, W) - g(X, Z)\bar{\text{Ric}}(Y, W)]/(n-1) \\ - \{\bar{r}/n(n-1)\}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ + \epsilon[g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W)] \quad (2.8)$$

Now, if we substitute  $\bar{X} = \bar{W} = N$  and  $\bar{Y} = Y, \bar{Z} = z$  in (2.6), then

$$g(\bar{R}(N, Y)Z, N) = \{[\bar{\text{Ric}}(N, N) - (\epsilon\bar{r}/n)]/(n-1)\}g(Y, Z) \\ + (\epsilon/(n-1))\bar{\text{Ric}}(Y, Z).$$

By the substitution  $X = W = e_i$ ; where  $\{e_i\}$  is an orthonormal frame in  $M$  in eqn. (2.7), multiplying by  $\epsilon_i = g(e_i, e_i)$  and finally summing over  $i$ , we find

$$\bar{\text{Ric}}(Y, Z) - \epsilon g(\bar{R}(N, Y)Z, N) = \bar{\text{Ric}}(Y, Z) - \epsilon\{\text{tr}.A\}g(AY, Z) - g(AY, AZ),$$

keeping in mind that  $\{e_i, N\}$  constitutes an orthonormal frame in  $\bar{M}$ . Eliminating  $g(\bar{R}(N, Y)Z, N)$  from the last two eqns. gets

$$\bar{\text{Ric}}(Y, Z) = \{(n-1)/(n-2)\}[\text{Ric}(Y, Z) \\ + \{(\epsilon n \bar{\text{Ric}}(N, N) - \bar{r})/n(n-1)\}g(y, z) \\ + \epsilon\{\text{tr}.A\}g(AY, Z) - g(AY, AZ)] \quad (2.9)$$

Again, substituting  $Y = Z = e_i$ , multiplying  $\epsilon_i$  and then summing over  $i$ ; in the above eqn. yields

$$\bar{r} - r = 2\epsilon \bar{\text{Ric}}(N, N) - \epsilon\{(\text{tr}.A)^2 - \text{tr}.A^2\}. \quad (2.10)$$

By feeding eqn. (2.10) into (2.9) we obtain

$$\bar{\text{Ric}}(Y, Z) = \{(n-1)/(n-2)\}[\bar{\text{Ric}}(Y, Z) - \epsilon\{\text{tr}.A\}g(AY, Z) - g(AY, AZ)] \\ + \frac{g(X, Y)}{2(n-1)}[(n-2)(\bar{r}/n) - r + \epsilon\{(\text{tr}.A)^2 - \text{tr}.A^2\}]$$

Through eqn. (2.5) and the hypothesis that  $\xi$  generates motion in  $M$ , we find

$$\overline{\text{Ric}}(Y, Z) = -(2f/\tau)g(AY, Z) + \rho g(Y, Z)$$

where  $\rho$  is a scalar function on  $M$ . A cumbersome computation using the last two equations shows

$$\varepsilon Q = \phi I + \theta A - A^2$$

$Q$  being defined by  $g(QX, Y) = \overline{\text{Ric}}(X, Y)$ , and

$$\varepsilon n\phi = r + \{2f(n-2)/(n-1)\tau\}tr.A - \varepsilon\{(tr.A)^2 - tr.A^2\} \tag{2.11}$$

$$\theta = tr.A - 2\varepsilon f(n-2)/(n-1)\tau \tag{2.13}$$

Now, using the Petrov classification scheme [10] for symmetric tensors,  $A$  can be cast into the form:

$$A = \begin{bmatrix} B_1 & & & & & \\ & \ddots & & & & \\ & & B_k & & & \\ & & & C_1 & & \\ & & & & \ddots & \\ & & & & & C_t \end{bmatrix}$$

where

$$B_i = \begin{bmatrix} d_i\lambda_i & d_i & & & \\ 0 & d_i\lambda_i & d_i & & \\ & & \ddots & & \\ & & & \ddots & d_i \\ & & & & d_i\lambda_i \end{bmatrix} \quad (d_i = \pm 1)$$

is an  $(s_i \times s_i)$ -matrix and

$$C_j = \begin{bmatrix} \alpha_j & \beta_j & 1 & 0 & & & & \\ -\beta_j & \alpha_j & 0 & 1 & & & & \\ & & \alpha_j & \beta_j & 1 & 0 & & \\ & & -\beta_j & \alpha_j & 0 & 1 & & \\ & & & & \ddots & & & \\ & & & & & & \ddots & \\ & & & & & & & 1 & 0 \\ & & & & & & & 0 & 1 \\ & & & & & & & \alpha_j & \beta_j \\ & & & & & & & -\beta_j & \alpha_j \end{bmatrix} \quad (\beta_j \neq 0)$$

is a  $(2t_j \times 2t_j)$ -matrix. As per our hypothesis,  $M$  is Einstein, i.e.  $Q = (r/n)I$ . Thus, eqn. (2.11) implies that the orders of block matrices  $B_i$  and  $C_j$  are  $\leq 2$ . Consequently,  $A$  has blocks of type:

$$[\nu_i] \text{ or } \begin{bmatrix} d_j\lambda_j & d_j \\ 0 & d_j\lambda_j \end{bmatrix} \text{ or } \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}$$





where  $a, b, \nu$  and  $\eta$  are scalar function on  $M^4$  and  $e^2 = \epsilon^2 = 1$ , with respect to some specially chosen basis.

**PROOF:** With respect to an orthonormal basis formed by  $U$  and three orthonormal vectors orthogonal to  $U$ , we observe that the Ricci map can be represented by

$$Q = \begin{bmatrix} x + e\eta & & & \\ & x & & \\ & & x & \\ & & & x \end{bmatrix}$$

where  $e = g(U, U) = \pm 1$ . Proceeding exactly as in the proof of theorem 2.1, up to eqn. (2.11) we find that  $A$  has blocks of type

$$[\nu_i] \text{ or } \begin{bmatrix} \lambda_j & 1 \\ 0 & \lambda_j \end{bmatrix} \text{ or } \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}$$

Therefore, either  $A$  will be diagonalizable at each point or can be put into one of the following forms:

$$\begin{array}{ll} (1) \begin{bmatrix} \nu_i & & & \\ & \nu_2 & & \\ & & \lambda & 1 \\ & & 0 & \lambda \end{bmatrix} & (2) \begin{bmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \alpha & \beta \\ & & -\beta & \alpha \end{bmatrix} \\ (3) \begin{bmatrix} \lambda_1 & 1 & & \\ 0 & \lambda_1 & & \\ & & \lambda_2 & 1 \\ & & 0 & \lambda_2 \end{bmatrix} & (4) \begin{bmatrix} \alpha_1 & \beta_1 & & \\ -\beta_1 & \alpha_1 & & \\ & & \alpha_2 & \beta_2 \\ & & -\beta_2 & \alpha_2 \end{bmatrix} \end{array}$$

Plugging the above listed forms of  $A$  into eqn. (2.11) shows that the type (2) is not possible whereas the types (3) and (4) hold only if  $M^4$  is Einstein which is taken care of by theorem 2.1. The remaining type (1) leads to:

$$\nu_2 = \lambda = \theta/2, \quad \nu_1 = \nu_2 + \sqrt{(-e\epsilon\eta)}.$$

Thus, in this case

$$A = \begin{bmatrix} \nu \pm \sqrt{(-e\epsilon\nu)} & & & \\ & \nu & & \\ & & \nu & 1 \\ & & 0 & \nu \end{bmatrix}$$

Where  $\nu = \theta/2$  and  $tr.A = (8\epsilon f/3\tau) \mp \sqrt{(-e\epsilon\eta)}$ .

**COROLLARY 2..** Under the hypothesis of theorem 3.1, if  $M^4$  is not Einstein then  $A$  is either diagonalizable at each point or can be put in the form

$$A = \begin{bmatrix} \nu \pm \sqrt{(-e\epsilon\eta)} & & & \\ & \nu & & \\ & & \nu & 1 \\ & & 0 & \nu \end{bmatrix}$$

with respect to some specially chosen basis.

We now interpret the physical significance of a pseudo-Einstein space in the light of the above corollary. Einstein's field equations of general relativity can be written in suitable units as:

$$\text{Ric} + (\Lambda - \frac{1}{2}r)g = T \tag{3.1}$$

where  $T$  is the energy-momentum tensor of matter distribution. A comparison of eqn.(3.1) with the defining eqn.(1.1) of a pseudo Einstein space, shows that

$$T = pg + (\mu + p)u \otimes u \tag{3.2}$$

where  $p$  and  $\mu$  are given by

$$p + \frac{1}{2}r - \Lambda = \chi, \quad \mu + p = \eta \tag{3.3}$$

Eqn. (3.2) represents an isotropic matter of type I [3] with energy density  $\mu$  and pressure  $p$ , provided  $g(U, U) = -1$  (signature of  $M^4$  being  $-+++$ ). It is remarkable to observe that such a space is non-Einstein because otherwise we would get the non-physical state  $\mu + p = 0$ .

The signature of  $\bar{M}^5$  could be  $-++++$  or  $--+++$  only. If  $M^5$  has the former (Lorentzian) signature then  $\epsilon = 1$ . Moreover, since  $g(U, U) = -1$  we have  $e = -1$ . Hence  $\sqrt{-e\epsilon\eta} = \sqrt{(\mu + p)}$  which is always non-zero real (because  $\mu + p > 0$ ). Thus, in this case either  $A$  is diagonalizable at each point or

$$A = \begin{bmatrix} \nu \pm \sqrt{(\mu + p)} & & & \\ & \nu & & \\ & & \nu & 1 \\ & & 0 & \nu \end{bmatrix}$$

On the other hand, if the signature of  $\bar{M}^5$  is  $--+++$ , then we have  $\epsilon = e = -1$ ,  $\sqrt{-e\epsilon} = \sqrt{-(\mu + p)}$  which is always imaginary for physically realistic matter. Consequently,  $M$  is proper. For this later case, the energy condition:  $\mu + p > 0$ , is equivalent to saying that  $M^4$  is proper.

**4. EXAMPLE.** Isometric embeddings of space-times has been used to obtain deeper insight into the geometrical properties of the embedded space-time. This technique has also shed some light on a number of global questions concerning singularities and causality properties. In particular, several physically useful exact solutions have been found by the embedding technique, at least for some cases of low embedding class [the maximum number of extra dimensions is called the embedding class]. In fact, the maximal analytic extension of the Schwarzschild solution [3] was found by the method of embedding.

In support of the above, we present an example of an isometric embedding of the type of space-time described in this paper viz., 4-dimensional space-time of general relativity with an isotropic matter of type I [3]. For details, we refer [7].

A 4-dimensional space-time is of embedding class 1 (i.e., a hypersurface) if there exists a symmetric tensor  $\Omega_{ab}$  satisfying:

$$R_{abcd} = e(\Omega_{ac}\Omega_{bd} - \Omega_{ad}\Omega_{bc}), \quad e = \pm 1 \quad (\text{Gauss})$$

$$\Omega_{ab;c} = \Omega_{a;c;b} \quad (\text{Codazzi})$$

The field equations then yield (for  $\Lambda = 0$ ).

$$R_{ab} = T_{ab} + \frac{1}{2}r g_{ab} = e(\Omega_{ab}\Omega_c^c - \Omega_{ac}\Omega_b^c)$$

All possible tensors  $\Omega$  corresponding to the isotropic matter or Maxwell type are known. Precisely, there are four different cases (for details, see [7], pages 360-367). Here we mention the following one case related to this paper:

ISOTROPIC PETROV TYPE 0 SOLUTION.

$$\begin{aligned} T &= (\mu + p)u \otimes u + pg, & \Omega &= Au \otimes u + Bg, \\ \mu + p &= 2AB > 0, & \mu &= 3B^2 > 0, & e &= 1 \end{aligned}$$

Relating this case to the equations (3.2) and (3.3) we get

$$\eta = 2AB, \quad \chi = 2AB - 3B^2 + \frac{1}{2}r \text{ and } \Lambda = 0$$

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