A PARSEVAL-GOLDSTEIN TYPE THEOREM ON THE WIDDER POTENTIAL TRANSFORM AND ITS APPLICATIONS

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(Received March 19, 1990 and in revised form November 20, 1990)

ABSTRACT. In this paper a Parseval-Goldstein type theorem involving the Widder potential transform and a Laplace type integral transform is given. The theorem is then shown to yield a relationship between the $K$-transform and the Laplace type integral transform. The theorem yields some simple algorithms for evaluating infinite integrals. Using the theorem and its results, a number of new infinite integrals of elementary and special functions are presented. Some illustrative examples are also given.

KEY WORDS AND PHRASES. The Widder potential transform, the Laplace transform, the $L_2$-transform, the $K$-transform the modified Bessel function of the third kind or MacDonald’s function.

1980 AMS SUBJECT CLASSIFICATION CODES: Primary 44A10, 44A15; Secondary 33A40, 44A35.

1. INTRODUCTION

Widder [1,2] presented a systematic account of the potential transform

$$\mathcal{P}[f(x);y] = \int_{0}^{\infty} \frac{xf(x)}{x^2 + y^2} \, dx.$$  \hspace{1cm} (1.1)

Widder pointed out that the potential transform is related to the Poisson integral representation of a function which is harmonic in a half plane and gave several inversion formulae for the transform and applied his results to harmonic functions. Srivastava and Singh [3] gave the following Parseval-Goldstein type formula:

$$\int_{0}^{\infty} x \mathcal{P}[f(u);x] \, g(x) \, dx = \int_{0}^{\infty} x f(x) \mathcal{P}[g(u);x] \, dx$$  \hspace{1cm} (1.2)

for the Widder potential transform. Srivastava and Yürekli [4] gave the following Parseval-Goldstein type relation involving the Laplace transform, the Fourier sine transform and the
Widder potential transform:

\[
\int_0^\infty \mathcal{L}[f(u), x] \mathcal{F}_x[g(u), x] \, dx = \int_0^\infty f(x) \mathcal{P}[g(u), x] \, dx
\]

(1.3)

where \( \mathcal{L} \) is the Laplace transform and \( \mathcal{F}_x \) is the Fourier sine transform. There are numerous analogous results in the literature on integral transforms. (See, for instance, Goldstein [5], Srivastava [6], Srivastava and Panda [7] and Yürekli [8].)

The objective of this paper is to establish a Parseval-Goldstein type relation between the potential transform and a Laplace type integral transform called \( \mathcal{L}_2 \)-transform where the \( \mathcal{L}_2 \)-transform is defined as

\[
\mathcal{L}_2[f(x); y] = \int_0^\infty x e^{-x^2 y^2} f(x) \, dx.
\]

(1.4)

If we make a change of variable in the integral on the right-hand side of (1.4), one obtains

\[
\mathcal{L}_2[f(x); y] = \frac{1}{2} \int_0^\infty e^{-x^2 y^2} f(\sqrt{x}) \, dx.
\]

(1.5)

Comparing (1.5) with the definition of Laplace transform we obtain the following relationship between the Laplace transform and the \( \mathcal{L}_2 \)-transform

\[
\mathcal{L}_2[f(x); y] = \frac{1}{2} \int \mathcal{L}[f(\sqrt{x}); y^2].
\]

(1.6)

We also obtain identities relating the \( \mathcal{K} \)-transform

\[
\mathcal{K}_\nu [f(x); y] = \int_0^\infty x^{\nu} K_\nu(xy) f(x) \, dx
\]

(1.7)

to the \( \mathcal{L}_2 \)-transform, where \( K_\nu(x) \) is the Bessel function of the third kind (it is also known as the Macdonald function), and the Laplace transform to the \( \mathcal{L}_2 \)-transform. Using these results we show how one can extend tables of Laplace and Hankel transforms. (See Erdélyi et al. [9, 10], Oberhettinger [11], Oberhettinger and Badii [12].) For definitions of special functions that are used in the paper, the reader is referred to Oldham and Spanier [13], and Erdélyi et al. [14].

We note that if we write \( \mathcal{M}[f(x); y] = F(y) \) where \( \mathcal{M} \) represents any integral transform, we mean the \( \mathcal{M} \)-transform of \( f(x) \) exists and it is \( F(y) \).

2. A PARSEVAL-GOLDSTEIN THEOREM AND ITS COROLLARIES

LEMMA 2.1. We have

\[
\mathcal{L}_2[\mathcal{L}_2[f(x); z]; y] = \frac{1}{2} \mathcal{P}[f(x); y],
\]

(2.1)

provided that the integrals involved converge absolutely.

PROOF: Using the definition of the \( \mathcal{L}_2 \)-transform we obtain

\[
\mathcal{L}_2[\mathcal{L}_2[f(x); z]; y] = \int_0^\infty ze^{-y^2 x^2} \left( \int_0^\infty x e^{-x^2 z^2} f(x) \, dx \right) \, dz.
\]

(2.2)

Changing the order of integration on the right side of (2.2), which is permissible by the absolute convergence of integrals, we have from (2.2) that

\[
\mathcal{L}_2[\mathcal{L}_2[f(x); z]; y] = \int_0^\infty zf(x) \left( \int_0^\infty ze^{-(x^2+y^2)z^2} \, dz \right) \, dx
\]

\[
= \frac{1}{2} \int_0^\infty \frac{zf(x)}{x^2+y^2} \, dx.
\]

(2.3)
Now the result follows from (1.1) and (2.3).

THEOREM 2.1. We have

\[ \int_0^{\infty} x \mathcal{L}_2[f(y); z] \mathcal{L}_2[g(z); x] \, dx = \frac{1}{2} \int_0^{\infty} y f(y) \mathcal{P}[g(z); y] \, dy. \]  

(2.4)

provided that the integrals involved converge absolutely.

PROOF: Using the definition of the \( \mathcal{L}_2 \)-transform we obtain

\[ \int_0^{\infty} x \mathcal{L}_2[f(y); z] \mathcal{L}_2[g(z); x] \, dx = \int_0^{\infty} x \mathcal{L}_2[g(z); x] \left( \int_0^{\infty} y e^{-x^2 y^2} \, dy \right) \, dx. \]  

(2.5)

Changing the order of integration, which is permissible by the hypothesis, and then using the definition of the \( \mathcal{L}_2 \)-transform we find that

\[ \int_0^{\infty} x \mathcal{L}_2[f(y); z] \mathcal{L}_2[g(z); x] \, dx = \int_0^{\infty} y f(y) \mathcal{L}_2[g(z); x] \, dy. \]  

(2.6)

Now the result follows from Lemma 2.1.

REMARK 2.1. We have

\[ \int_0^{\infty} x \mathcal{L}_2[g(y); z] \mathcal{L}_2[f(z); x] \, dx = \frac{1}{2} \int_0^{\infty} y g(y) \mathcal{P}[f(z); y] \, dy, \]  

(2.7)

since the relation (2.4) is symmetrical with respect to \( f \) and \( g \). Using the relations (2.4) and (2.7) we obtain the Parseval-Goldstein type formula (1.2). Thus, Theorem 2.1 generalizes relation (1.2).

COROLLARY 2.1. We have

\[ \int_0^{\infty} x h(x) \mathcal{L}_2[f(z); x] \, dx = \int_0^{\infty} y f(y) \mathcal{L}_2[h(z); y] \, dy. \]  

(2.8)

provided that the integrals involved converge absolutely.

PROOF: The identity (2.8) follows immediately after letting \( h(x) = \mathcal{L}_2[g(z); x] \) in the relation (2.4).

COROLLARY 2.2. We have

\[ \mathcal{P}[\mathcal{L}_2[g(u); z]; z] = \mathcal{L}_2[\mathcal{P}[g(u); z]; z] \]  

(2.9)

provided that the integrals involved converge absolutely.

PROOF: We set \( f(y) = e^{-x^2 y^2} \) in Theorem 2.1. Then

\[ \mathcal{L}_2[f(y); z] = \int_0^{\infty} y e^{-(z^2 + x^2) y^2} \, dy = \frac{1}{2(z^2 + x^2)}. \]  

(2.10)

Now the assertion (2.9) follows from (2.4) and (2.10).

THEOREM 2.2. If \( \text{Re} \nu \geq -1 \)

\[ \mathcal{K}_\nu[y^{\nu+\frac{1}{2}} f(y); z] = 2^\nu z^{\nu+\frac{1}{2}} \mathcal{L}_2 \left[ z^{2\nu-2} \mathcal{L}_2 \left[ f(y); \frac{1}{2x} \right]; z \right]. \]  

(2.11)

provided that the integrals involved are absolutely convergent.
PROOF: We set \( g(u) = u^{\nu} J_{\nu}(zu) \) in Theorem 2.1, where \( J_{\nu} \) is the Bessel function of the first kind of order \( \nu \). Using relation (1.6) and then making use of the Laplace transform table (see [9, formula (30), p. 185]) we have

\[
L_2[u^{\nu} J_{\nu}(zu); z] = \frac{1}{2} L \left[ u^{\frac{\nu}{2}} J_{\nu} \left( z u^{\frac{1}{2}} \right); z^2 \right] = \frac{1}{2} \left( \frac{z}{2} \right)^{\nu} z^{-2\nu - 2} \exp \left( -\frac{z^2}{4z^2} \right). \tag{2.12}
\]

Now in order to evaluate the potential transform of the function \( g(u) \) we use Lemma 2.1 and obtain

\[
P[g(u); y] = \left( \frac{z}{2} \right)^{\nu} L_2 \left[ z^{-2\nu - 2} \exp \left( -\frac{z^2}{4z^2} \right); y \right]. \tag{2.13}
\]

The \( L_2 \)-transform on the right-hand side of (2.12) may be evaluated by using the relation (1.6) and then the Laplace transform table (see [9, formula (20), p. 146]). Thus

\[
P[g(u); y] = y^{\nu} K_{\nu}(zy). \tag{2.14}
\]

Substituting the results (2.12) and (2.14) into (2.4) of Theorem 2.1 gives

\[
\int_0^{\infty} y^{\nu+1} K_{\nu}(zy) f(y) \, dy = \left( \frac{z}{2} \right)^{\nu} \int_0^{\infty} z^{-2\nu - 1} \exp \left( -\frac{z^2}{4z^2} \right) L_2[f(y); z] \, dx. \tag{2.15}
\]

Now the assertion follows by making the change of variable \( x = t/2 \) and then by using the definitions of the \( K \)-transform and the \( L_2 \)-transform.

It is well known that

\[
K_{\frac{1}{4}}(x) = K_{-\frac{3}{4}}(x) = \left( \frac{\pi}{2x} \right)^{\frac{1}{4}} e^{-x}, \tag{2.16}
\]

(see [13, p. 306]). Using (2.11) and (2.16) we obtain the identities in the following corollary:

**COROLLARY 2.3.** We have

\[
L[yf(y); z] = \frac{2}{\sqrt{\pi}} z L_2 \left[ \frac{1}{x} L_2 \left[ f(y); \frac{1}{2x} \right]; z \right], \tag{2.17}
\]

\[
L[yf(y); z] = \frac{1}{\sqrt{\pi}} L_2 \left[ \frac{1}{x^3} L_2 \left[ f(y); \frac{1}{2x} \right]; z \right], \tag{2.18}
\]

provided that the integrals involved are absolutely convergent.

3. EXAMPLES

We shall illustrate the above results by several examples. In the following example we present a known result (see [10, formula (1), p. 127]) in order to verify relation (2.11).

**EXAMPLE 3.1.** We show that

\[
K_{\nu}[y^{\rho - \frac{1}{2}}; z] = 2^{\rho - \frac{1}{2}} z^{-\rho} \Gamma \left( \frac{\rho}{2} - \frac{\nu}{2} + \frac{1}{4} \right) \Gamma \left( \frac{\rho}{2} + \frac{\nu}{2} + \frac{1}{4} \right) \tag{3.1}
\]

provided that \( \text{Re} \rho > |\text{Re} \nu| - \frac{3}{2} \).

We set \( f(y) = y^{\rho - \nu - \frac{3}{4}} \) in Theorem 3.2. Making use of identity (1.6) we obtain

\[
L_2 \left[ y^{\rho - \nu - \frac{3}{4}}; \frac{1}{2x} \right] = \frac{1}{2} L \left[ y^{\frac{\rho}{2} - \frac{\nu}{2} - \frac{3}{4}}; \frac{1}{4x^2} \right] = 2^{\rho - \nu + \frac{1}{4}} \Gamma \left( \frac{\rho}{2} - \frac{\nu}{2} + \frac{1}{4} \right) x^{\rho - \nu - \frac{1}{4}}. \tag{3.2}
\]
provided that \( \Re(\rho - \nu) > \frac{1}{2} \). Substituting (3.2) into (2.11) we find

\[
\mathcal{K}_{\nu} [y^{\rho-1};z] = 2^\rho - \frac{1}{2} \Gamma\left(\frac{\rho}{2} - \frac{\nu}{2} + \frac{1}{4}\right) z^{\nu+\frac{1}{4}} L_2 \left[x^{\rho+\nu-\frac{1}{4}};z\right]
\]  

(3.3)

Now formula (3.1) follows after evaluating the \( L_2 \)-transform on the right side of equation (3.3).

Integral transforms evaluated in Examples 3.2, 3.4 and 3.5, and in the appendix, to the best of the author's knowledge, are all new.

**EXAMPLE 3.2.** We show that

\[
g + i\gamma = \frac{\Gamma(\nu + 2) z^{\nu+\frac{1}{4}}}{4a^{\nu+\frac{1}{4}}} \sin\left(\frac{z^2}{4a}\right)
\]

where \( \Re \nu > -2 \) and \( S_{\mu,\nu} \) is the Lommel function.

We set \( f(y) = \sin ay^2 \) in Theorem 3.2. Making use of (1.6) and then tables of Laplace transforms (see [12, formula 7.1, p. 4]), we obtain

\[
\mathcal{L}_2 \left[y^{\nu+1/4}\sin ay^2; \frac{1}{4a^2}\right] = \frac{1}{2} \mathcal{L} \left[\sin ay; \frac{1}{4a^2}\right]
\]

\[
= \frac{2\alpha z^4}{z^4 + \alpha^2}
\]

(3.5)

where \( \alpha = 1/4a \). Substituting the function \( f \) into (2.11), and using (3.5) and (1.6), we find

\[
\mathcal{K}_{\nu} \left[y^{\nu+1/4}\sin ay^2; z\right] = 2^\rho \alpha z^{\nu+1/4} \mathcal{L} \left[\frac{z^{\nu+1}}{z^2 + \alpha^2};z^2\right]
\]

(3.6)

Now (3.4) follows from tables of Laplace transforms (see [12, formula 3.11, p. 22]).

Using the technique of Example 3.2, we provide additional results in the appendix.

In the following example we obtain a well known result (Erdélyi [9, formula (30) p. 153]) as a special case of Example 3.2.

**EXAMPLE 3.3.** We show that

\[
\mathcal{L} \left[\sin (ay^2); z\right] = \sqrt{\frac{\pi}{2a}} \left\{ \left(\frac{1}{2} - C(t)\right) \cos t + \left(\frac{1}{2} - S(t)\right) \sin t \right\}
\]

(3.7)

where \( t = z^2/(4a) \), and \( C(t) \) and \( S(t) \) are the Fresnel integrals.

We set \( \nu = -1/2 \) in (3.4). Using (2.16) and the definition of the \( K \)-transform we obtain

\[
\mathcal{L} \left[\sin ay^2; z\right] = \frac{z}{4a} S_{-1,4} \left(\frac{z^2}{4a}\right)
\]

(3.8)

It follows from a formula on the Lommel function (see [12, p. 416]) that

\[
S_{-1,4}(t) = \pi \left[ J_{\frac{1}{4}}(t) + J_{-\frac{1}{4}}(t) - J_{\frac{1}{4}}(t) - J_{-\frac{1}{4}}(t)\right]
\]

(3.9)

where \( J_{\nu}(t) \) is the Bessel function of order \( \nu \) and \( J_{\nu}(t) \) is the Anger-Weber function of order \( \nu \). However, we have

\[
J_{\frac{1}{4}}(t) = \sqrt{\frac{2}{\pi t}} \sin t \quad \text{and} \quad J_{-\frac{1}{4}}(t) = \sqrt{\frac{2}{\pi t}} \cos t,
\]

(3.10)
see [13, p. 306] and

\[
J_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}} \left\{ \left[ C(t) - S(t) \right] \cos t + \left[ C(t) + S(t) \right] \sin t \right\}
\]

\[
J_{-\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}} \left\{ \left[ C(t) + S(t) \right] \cos t - \left[ C(t) - S(t) \right] \sin t \right\},
\]

see Oberhettinger and Badii [12, p. 415].

Now, substituting (3.10), (3.11) and (3.12) into (3.9) and then using (3.8) we obtain formula (3.7).

EXAMPLE 3.4. We show that

\[
\mathcal{L} \left[ (x + 4ax^3)^{\frac{-1}{2}}; z \right] = 2^{\nu} (\pi a)^{-\frac{1}{2}} \Gamma \left( \frac{\nu}{2} + \frac{1}{2} \right) z^{1/2} \exp \left( \frac{z}{8a} \right) K_{\frac{1}{2}} \left( \frac{z}{8a} \right),
\]

providing that \(-1 < \text{Re} \nu < 1, \text{Re} a > 0\).

We set \(f(y) = y^{-\nu-1} \exp(-ay^2)\) in Theorem 2.2. Making use of (1.6) and then using tables of Laplace transforms (see [12, formula 5.3, p. 37]), we obtain

\[
\mathcal{L}_2 \left[ y^{-\nu-1} e^{-ay^2}; \frac{1}{2ax} \right] = \frac{1}{2} \mathcal{L} \left[ y^{\frac{-1}{2}} e^{-ay^2}; \frac{1}{4ax^2} \right] = 2^{-\nu} \Gamma \left( \frac{1}{2} - \nu \right) x^{-\nu} (4ax^2 + 1)^{-\frac{1}{2}},
\]

providing that \(\text{Re} \nu < 1\). Using tables of Hankel transforms (see [10, formula (24), p. 132]) we obtain

\[
\mathcal{K}_\nu \left[ y^{-\frac{1}{2}} e^{-ay^2}; z \right] = \frac{1}{4} \sec \left( \frac{\nu \pi}{2} \right) \left( \frac{\nu \pi}{a} \right)^{\frac{1}{2}} K_{\frac{1}{2}} \left( \frac{z^2}{8a} \right)
\]

providing that \(\text{Re} a > 0\) and \(-1 < \text{Re} \nu < 1\). Now formula (3.13) follows from substituting (3.14) and (3.15) into (2.10) and then using (1.6).

EXAMPLE 3.5. We show that

\[
\mathcal{L} \left[ x^{\frac{1}{2}} \text{Erf} \left( ax^{\frac{1}{2}} \right); z \right] = \frac{2}{\sqrt{\pi}} z^{-\frac{3}{4}} \arctan \left( az^{-\frac{1}{4}} \right)
\]

\[
\mathcal{L} \left[ x^{\frac{3}{2}} \text{Erf} \left( ax^{\frac{3}{2}} \right); z \right] = \frac{4}{\sqrt{\pi}} \arctan \left( az^{-\frac{1}{4}} \right),
\]

where \(\text{Erf}(z)\) is the error function.

We set \(f(y) = y^{-2} \sin ay\) in Corollary 2.3. Making use of (1.6) and then tables of integral transforms (see [12, formula 7.76, p. 66]) we obtain

\[
\mathcal{L}_2 \left[ \frac{1}{y^2} \sin ay; \frac{1}{2ax} \right] = \frac{1}{2} \mathcal{L} \left[ \frac{1}{y} \sin ay^{\frac{1}{2}}; \frac{1}{4ax^2} \right] = \frac{\pi}{2} \text{Erf}(ax).
\]

It follows from tables of Laplace transforms (see [12, formula 7.5, p. 54])

\[
\mathcal{L} \left[ \frac{1}{y} \sin ay; z \right] = \arctan \left( \frac{a}{z} \right)
\]

Now formula (3.16) follows from substituting (3.18) and (3.19) into (2.17). Similarly, formula (3.17) follows from substituting (3.18) and (3.19) into (2.18) of Corollary 2.3.
ACKNOWLEDGEMENT We would like to thank Thomas K. Boehme for his interest and valuable suggestions. We are also indebted to John Maceli for a careful reading of the preprint. Referee’s remarks are also acknowledged.

REFERENCES


APPENDIX

A. SOME $\mathcal{K}$-TRANSFORM PAIRS

The following formulae (A.1) through (A.5) are consequences of Theorem 2.2. The techniques of Example 3.1 and 3.2 are used to obtain these results.

\[ \mathcal{K}_\nu \left[ y^{\nu+\frac{1}{2}} \sin ay^2; z \right] = 2(2a)^{-\nu+\frac{1}{2}} \Gamma(\nu + 2) z^{\nu+\frac{3}{2}} S_{-\nu-\frac{1}{2}, \frac{1}{2}} \left( \frac{z^2}{4a} \right), \]  \hspace{1cm} (A.1)

where \( \Re \nu > -2. \)

\[ \mathcal{K}_\nu \left[ y^{\nu+\frac{1}{2}} \cos ay^2; z \right] = 2^{-\frac{1}{2}} (2a)^{-\nu-\frac{1}{2}} \Gamma(\nu + 1) z^{\nu+\frac{3}{2}} S_{-\nu-\frac{1}{2}, \frac{1}{2}} \left( \frac{z^2}{4a} \right), \]  \hspace{1cm} (A.2)

where \( \Re \nu > -1. \)

\[ \mathcal{K}_1 \left[ y^{-\frac{1}{2}} \sin ay^2; z \right] = \frac{1}{2} z^{-\frac{1}{2}} \left[ \text{Ci} \left( \frac{z^2}{4a} \right) \sin \left( \frac{z^2}{4a} \right) - \text{si} \left( \frac{z^2}{4a} \right) \cos \left( \frac{z^2}{4a} \right) \right], \]  \hspace{1cm} (A.3)
where $Ci(x)$ and $si(x)$ are cosine and sine integrals, respectively.

$$K_2 \left[ y^{-\frac{1}{2}} \sin ay^2 ; z \right] = z^{-\frac{1}{2}} \left[ -Ci \left( \frac{z^2}{4a} \right) \sin \left( \frac{z^2}{4a} \right) - si \left( \frac{z^2}{4a} \right) \cos \left( \frac{z^2}{4a} \right) \right] + \frac{1}{4a} z^{-\frac{1}{2}} \left[ -Ci \left( \frac{z^2}{4a} \right) \cos \left( \frac{z^2}{4a} \right) - si \left( \frac{z^2}{4a} \right) \sin \left( \frac{z^2}{4a} \right) \right]. \quad (A.4)$$

$$K_\nu \left[ y^{\nu+\frac{1}{2}} \sin^2 ay^2 ; z \right] = 2^{\nu-4}(2a)^{-\nu-\frac{3}{2}} \Gamma(\nu + 3) z^{\nu+\frac{3}{2}} S_{-\nu-\frac{1}{4}} \left( \frac{z^2}{8a} \right), \quad (A.5)$$

where $\Re \nu > -3$.

**B. SOME LAPLACE TRANSFORM PAIRs**

The following formulae (B.1) through (B.5) result from Theorem 2.2 and Corollary 2.3. The techniques of Example 3.4 and 3.5 are used to obtain these results.

$$L \left[ z^{-2\mu} (4az^2 - z)^{\frac{\nu+1}{2} + \mu} ; z \right] = \frac{(4a)^\mu}{z^{\frac{\nu+1}{2}}} \Gamma \left( \frac{\nu + 1}{2} - \mu \right) \exp \left( \frac{z}{8a} \right) W_{\mu, \xi} \left( \frac{z}{4a} \right) \quad (B.1)$$

where $2\Re \mu < 1 - |\Re \nu|$, $\Re a > 0$ and $W_{\mu, \nu}$ is the Whittaker function.

$$L \left[ z^{\frac{\nu}{2} + \frac{3}{4}} \exp \left( \frac{a^2 z}{2} \right) D_{\nu - \mu - \frac{1}{2}} \left( \frac{a\sqrt{2z}}{2} \right) ; z \right] = \frac{\pi^{\frac{1}{2}} 2^{\nu+\frac{3}{4}} \Gamma(\mu + \nu + \frac{1}{2})}{\Gamma(\mu + 1)} \left( a + z^{\frac{1}{2}} \right)^{\mu + \nu + \frac{3}{4}} _2F_1 \left( \mu + \nu + \frac{1}{2}, \nu + \frac{1}{2}, \mu + 1, \frac{a - z^{\frac{1}{2}}}{a + z^{\frac{1}{2}}} \right) \quad (B.2)$$

where $\Re \mu > |\Re \nu| - \frac{1}{2}$, $D_\nu$ is the parabolic cylinder function and $mF_n$ is the hypergeometric function.

$$L \left[ \sin \left( \mu \arctan \left( \frac{x}{2} \right) \right) ; z \right] = \frac{\pi^{\frac{1}{2}} \Gamma(1 - \mu)}{4x^{\frac{\nu+1}{2}}} \times \left\{ J_{\nu - \frac{1}{2}} \left( \frac{\alpha x}{2} \right) \sin \left( \frac{\alpha x}{2} - \frac{\pi}{2} + \frac{\pi}{4} \mu \right) - Y_{\nu - \frac{1}{2}} \left( \frac{\alpha x}{2} \right) \cos \left( \frac{\alpha x}{2} - \frac{\pi}{2} + \frac{\pi}{4} \mu \right) \right\}, \quad (B.3)$$

where $-1 < \Re \mu < 2$ and $Y_\nu$ is the Bessel function of the second kind of order $\nu$.

$$L \left[ z^{-\frac{1}{4}} \exp \left( \frac{a^2}{4x} \right) \Gamma \left( -\nu, \frac{a^2}{4x} \right) ; z \right] = \pi 2^{2\nu+2} a^{\frac{3}{2}} z^{-\frac{3}{4}} \Gamma \left( \nu + \frac{3}{2} \right) S_{-2\nu - \frac{3}{4}} \left( a^{\frac{1}{2}} \right), \quad (B.4)$$

where $\Re \nu > -1$.

$$L \left[ z^{-\frac{3}{4}} \exp \left( \frac{a^2}{4x} \right) \Gamma \left( -\nu, \frac{a^2}{4x} \right) ; z \right] = \pi^{\frac{3}{4}} 2^{2\nu+3} a^{\frac{3}{2}} \Gamma \left( \nu + \frac{3}{2} \right) S_{-2\nu - \frac{3}{4}} \left( a^{\frac{1}{2}} \right), \quad (B.5)$$

where $\Re \nu > -1$. 
This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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