A COMMON FIXED POINT THEOREM FOR TWO SEQUENCES OF SELF-MAPPINGS

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ABSTRACT. In this paper a common fixed point theorem for two sequences of self-mappings from a complete metric space M to M is proved. Our theorem is a generalization of Hadzic's fixed point theorem[1].

KEY WORDS AND PHRASES. A common fixed point, self-mappings and complete metric spaces.

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1. INTRODUCTION.

Banach's fixed point theorem has been generalized by many authors. Among such investigations there are several, interesting and important studies[2]. Particularly, K. Iseki[3] proved a fixed point theorem of a sequence of self-mappings from a complete metric space M to M. We are interested in fixed point theorems of a sequence of self-mappings since they pertain to the problem of finding an equilibrium point of a difference equation \( x_{n+1} = f(n, x_n) \) \( (n = 1, 2, \ldots) \).

Recently O. Hadzic proved the existence of a common fixed point for the sequence of self-mappings \( \{A_j\}(j = 1, 2, \ldots) \), S and T where \( A_j \) commutes with S and T. His result is as follows:

THEOREM 1. Let \((M, d)\) be a complete metric space, \( S, T: M \rightarrow M \) be continuous, \( A_j: M \rightarrow SM \cap TM(j = 1, 2, \ldots) \) so that \( A_j \) commutes with S and T and for every \( i, j \) \((i \neq j, i, j = 1, 2, \ldots)\) and every \( x, y \in M \):

\[
d(A_i x, A_j y) \leq qd(Sx, Ty), \quad 0 < q < 1 
\] (1.1)

Using Theorem 1, he gave a generalization of Gohde's fixed point theorem and extended Krasnoseliski's fixed point theorem.

In this paper we shall present a generalization of Hadzic's fixed point theorem.
2. MAIN THEOREMS.
Let \( N \) denote the set of all positive integers. In this section we shall prove the following theorem.

**THEOREM A.** Let \((M,d)\) be a complete metric space and let \( \{A_p\}, \{B_q\} (p, q = 1, 2, \ldots) \) be two sequences of mappings from \( M \) to \( M \).

Suppose that the following conditions are satisfied: for all \( m, n \in N \) and all \( x, y \in M \),

(a) there exists a constant \( k \) \((0 < k < 1)\) such that

\[
d(A_{2n-1}x, A_{2n}y) \leq kd(B_{2n-1}x, B_{2n}y),
\]

\[
d(A_{2n-1}x, A_{2n+1}y) \leq kd(B_{2n+1}x, B_{2n+1}y), \text{ for all } m \leq n \geq 1,
\]

(b) \( A_{2n}B_{2m} = A_{2m}A_{2n} \) and \( A_{2n-1}B_{2m-1} = B_{2m-1}A_{2n-1} \),

(c) \( B_{2n}B_{2m} = B_{2m}B_{2n} \) and \( B_{2m-1}B_{2n-1} = B_{2n-1}B_{2m-1} \),

(d) \( A_{2n-1}(M) \subseteq B_{2n}(M) \) and \( A_{2n}(M) \subseteq B_{2n+1}(M) \).

If each \( B_q(q = 1, 2, \ldots) \) is continuous, then there exists a unique fixed point for two sequences \( \{A_p\} \) and \( \{B_q\} \) \((p, q = 1, 2, \ldots)\).

**PROOF.** Let \( x_0 \) be an arbitrary point in \( M \). By condition (d) there exists a point \( x_1 \in M \) such that \( A_1x_0 = B_2x_1 \). Next we choose a point \( x_2 \in M \) such that \( A_2x_1 = B_3x_2 \). Inductively, we can define by condition (d), the sequence \( \{x_n\} \) such that

\[
A_{2n-1}x_{2n-2} = B_{2n}x_{2n-1} \quad \text{and} \quad A_{2n}x_{2n} - 1 = B_{2n+1}x_{2n}, \quad n \in N. \tag{2.1}
\]

First of all we shall show that \( \{B_nx_{n-1}\} \) is a Cauchy sequence. By (2.1) and condition (a), we obtain that for all \( n \in N \)

\[
d(B_{2n-1}x_{2n-2}, B_{2n}x_{2n} - 1) \leq kd(B_{2n}x_{2n-3}, A_{2n}x_{2n-2})
\]

\[
\leq kd(B_{2n}x_{2n-3}, B_{2n-1}x_{2n-2}) = kd(A_{2n-1}x_{2n-3}, A_{2n}x_{2n-2})
\]

\[
\leq kd(B_{2n}x_{2n-4}, B_{2n-2}x_{2n-3}) \leq \cdots \leq k^{2n-3}d(B_1x_0, B_2x_1)
\]

and similarly that

\[
d(B_{2n+1}x_{2n+1}, B_{2n+2}x_{2n}) = d(A_{2n-1}x_{2n-2}, A_{2n}x_{2n} - 1)
\]

\[
\leq kd(B_{2n-1}x_{2n-2}, B_{2n}x_{2n} - 1) \leq \cdots \leq k^{2n-1}d(B_1x_0, B_2x_1).
\]
Since \( 0 < k < 1 \), this implies that the sequence \( \{B_n x_n - 1\} \) is a Cauchy sequence. Thus \( \{B_n x_n - 1\} \) converges to some point \( v \) in \( M \) because \( M \) is complete. Now since each \( B_q(q \in N) \) is continuous, we obtain that

\[
B_{2m} = B_{2m} \left( \lim_{n \to \infty} B_{2n} + 1 x_{2n} \right) = \lim_{n \to \infty} \left( B_{2m} B_{2n} + 1 x_{2n} \right)
\]

\[
= \lim_{n \to \infty} \left( B_{2m} A_{2n} x_{2n} - 1 \right) = \lim_{n \to \infty} \left( A_{2n} B_{2m} x_{2n} - 1 \right)
\]

and similarly that \( B_{2m_1} + 1 v = \lim_{n \to \infty} \left( A_{2n_1} + 1 B_{2m} + 1 x_{2n} \right) \) and \( B_{2m_1} - 1 v = \lim_{n \to \infty} \left( A_{2n_1} - 1 B_{2m} - 1 x_{2n_1} - 1 \right) \).

Hence by condition (c), we have

\[
d(B_{2m} v, B_{2m_1} + 1 v) = \lim_{n \to \infty} \left( A_{2n} B_{2m} x_{2n} - 1, A_{2n} + 1 B_{2m} + 1 x_{2n} \right)
\]

\[
\leq \lim_{n \to \infty} \left( B_{2n} B_{2m} x_{2n} - 1, B_{2n} + 1 B_{2m} + 1 x_{2n} \right)
\]

\[
= kd(B_{2m} v, B_{2m_1} + 1 v)
\]

and \( d(B_{2m} v, B_{2m_1} - 1 v) \leq kd(B_{2m} v, B_{2m_1} - 1 v) \) \((m \in N)\) in like manner, which implies that \( B_m v = B_m + 1 v \) for all \( m \geq 1 \). Next we shall show that \( A_n v = A_n v \) for all \( n \leq 1 \).

By (2.1), conditions (b) and (c), we have

\[
d(B_{2n_1} + 1 B_{2m} + 2 x_{2m_1} + 1, A_{2n} v) = d(A_{2m} + 1 B_{2n} + 1 x_{2m_1} A_{2n} v)
\]

\[
\leq kd(B_{2m} + 1 B_{2n} + 1 x_{2m_1} B_{2n} v)
\]

\[
= kd(B_{2n_1} + 1 B_{2m} + 1 x_{2m_1} B_{2n} v)
\]

Thus letting \( m \to \infty \), we obtain that \( d(B_{2n_1} + 1 v, A_{2n} v) \leq kd(B_{2n_1} + 1 v, B_{2n} v) \) from which it follows that \( A_{2n} v = B_{2n} + 1 v \) for all \( n \geq 1 \). And since

\[
d(A_{2n - 1} v, A_{2n} v) \leq kd(B_{2n - 1} v, B_{2n} v) \) and \( d(A_{2n} + 1 v, A_{2n} v) \leq kd(B_{2n} + 1 v, B_{2n} v) \),

we obtain that \( A_n v = A_n + 1 v = B_n + 1 v = B_n v \) for all \( n \in N \). Furthermore, for all \( n \in N \), we obtain

\[
d(A_{2n} v, A_{2n - 1} A_{2n} + 1 v) \leq kd(B_{2n} v, B_{2n - 1} A_{2n} + 1 v) = kd(A_{2n} v, A_{2n - 1} A_{2n} + 1 v)
\]

and \( d(A_{2n} v, A_{2n} A_{2n} + 1 v) \leq kd(B_{2n} v, B_{2n} A_{2n} + 1 v) = kd(A_{2n} v, A_{2n} A_{2n} + 1 v) \).

Therefore we obtain \( u = A_p(u) = B_p(u) \) for all \( p \geq 1 \) setting \( u = A_n v \) because \( 0 < k < 1 \).
Now we shall prove that $u$ is a unique common fixed point of $(A_p)$ and $(B_p)$. If there exists another point $w$ such that $w = A_p w = B_p w$ for all $p > 1$, then

$$d(u, w) = d(A_{2m-1} u, A_{2m} w) \leq kd(B_{2m-1} u, B_{2m} w)$$

$$\leq kd(u, w),$$

which is a contradiction since $0 < k < 1$. Therefore $u$ is a unique common fixed point of two sequences of self-mappings $(A_p)$ and $(B_p)$. This completes the proof.

If $S = B_{2n-1}$ and $T = B_{2n}(n = 1, 2, \ldots)$, we obtain Theorem 1 as the corollary of Theorem A. Next we obtain the following theorem which is a generalization of Theorem 1 in [4].

**THEOREM B.** Let $(M, d)$ be a complete metric space and let $(T_p)$ $(p = 1, 2, \ldots)$ be a sequence of mappings from $M$ to $M$. Suppose that the following conditions as satisfied for all $m > n > 0$ and $x, y \in M$

(e) there exists a constant $h$ ($h > 1$) such that

$$d(T_{2n-1} x, T_{2n} y) \geq hd(x, y)$$

$$d(T_{2n} x, T_{2n+1} y) \geq hd(x, y),$$

(f) $T_p T_q = T_q T_p$ (p, q are even or odd respectively).

If every $T_n$ is continuous on $M$ and $T_n(M) = M(n = 1, 2, \ldots)$, then there exists a unique fixed point for $T_n$.

**PROOF.** Set $A_n = I$ (I is the identity map from $M$ to $M$) in Theorem A. The proof is complete.

**REMARK 1.** We remark that the mapping $f: X \to X$ in Theorem 1 of [4] is continuous from the condition of the theorem.

**REFERENCES**

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