

BOUNDARY VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT DELAY

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(Received February 21, 1990)

ABSTRACT. The influence of certain discontinuous delays on the behavior of solutions to some typical equations of mathematical physics is studied.

KEY WORDS AND PHRASES. Partial Differential Equation, Piecewise Constant Delay, Boundary Value Problem, Fourier Method, Positive Operator, Weak Solution.

1980 AMS Subject Classification Codes. 35A05, 35B25, 35L10, 34K25.

1. INTRODUCTION.

Functional differential equations(FDE) with delay provide a mathematical model for a physical or biological system in which the rate of change of the system depends upon its past history. The theory of FDE with continuous argument is well developed and has numerous applications in natural and engineering sciences. This note continues our earlier work [1-5] in an attempt to extend this theory to differential equations with discontinuous argument deviations. In these papers, ordinary differential equations with arguments having intervals of constancy have been studied. Such equations represent a hybrid of continuous and discrete dynamical systems and combine properties of both differential and difference equations. They include as particular cases loaded and impulse equations, hence their importance in control theory and in certain biomedical models. Continuity of a solution at a point joining any two consecutive intervals implies recursion relations for the values of the solution at such points. Therefore, differential equations with piecewise constant argument (EPCA) are intrinsically closer to difference rather than differential

equations. Here boundary value problems for some linear EPCA in partial derivatives are considered and the behavior of their solutions studied. The results are also extended to equations with positive definite operators in Hilbert spaces.

2. BOUNDARY VALUE PROBLEMS.

The equation

$$u_t = a^2 u_{xx} - b(u - u_0)$$

describes heat flow in a rod with both diffusion $a^2 u_{xx}$ along the rod and heat loss (or gain) across the lateral sides of the rod. Heat loss ($b > 0$) or gain ($b < 0$) is proportional to the difference between the temperature $u(x,t)$ of the rod and u_0 of the surrounding medium.

In chemistry where u may stand for concentration, the above equation says that the rate of change u_t of the substance is due both to the diffusion $a^2 u_{xx}$ (in the x -direction) and to the fact that the substance is being created ($b < 0$) or destroyed ($b > 0$) by a chemical reaction proportional to the difference between two concentrations u and u_0 [6]. We may change $u - u_0$ to u and consider the equation

$$u_t = a^2 u_{xx} - bu.$$

Measuring the lateral heat change (or substance change due to a chemical reaction) at discrete moments of time leads to an equation with piecewise constant argument

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x,nh),$$

$$t \in [nh, (n+1)h], \quad n = 0, 1, \dots$$

where $h > 0$ is some constant. This equation can be written in the form

$$u_t(x,t) = a^2 u_{xx}(x,t) - bu(x, [t/h]h), \quad (2.1)$$

where $[\cdot]$ designates the greatest integer function. Ordinary differential equations with arguments $[t]$, $[t-n]$, $[t+n]$ have been investigated in [1-4], with $[t+1/2]$ in [5], and with $[t/h]h$ in [7,8].

Furthermore, EPCA have been used recently in [8] to approximate solu-

tions of equations with continuous delay.

The diffusion-convection equation

$$u_t = a^2 u_{xx} - ru_x$$

describes, for instance, the concentration $u(x,t)$ of a pollutant carried along in a stream moving with velocity r . The term $a^2 u_{xx}$ is the diffusion contribution and $-ru_x$ is the convection component. If the convection part is measured at discrete times nh , the process results in the equation

$$u_t(x,t) = a^2 u_{xx}(x,t) - ru_x(x, [t/h]h). \tag{2.2}$$

We consider the boundary value problem (BVP) consisting of the equation

$$\frac{\partial u(x,t)}{\partial t} + P\left[\frac{\partial}{\partial x}\right]u(x,t) = Q\left[\frac{\partial}{\partial x}\right]u(x, [t/h]h), \tag{2.3}$$

where P and Q are polynomials of the highest degree m with coefficients that may depend only on x , the boundary conditions

$$L_j u = \sum_{k=1}^m \left[M_{jk} u^{(k-1)}(0) + N_{jk} u^{(k-1)}(1) \right] = 0, \tag{2.4}$$

(M_{jk} and N_{jk} are constants, $j = 1, \dots, m$)

and the initial condition

$$u(x,0) = u_0(x). \tag{2.5}$$

Here $[\cdot]$ designates the greatest integer function, $(x,t) \in [0,1] \times [0,\infty)$, and $h = \text{const} > 0$. Equations (2.4) will be written briefly as

$$Lu = 0. \tag{2.4'}$$

DEFINITION 2.1. A function $u(x,t)$ is called a solution of the above BVP if it satisfies the conditions: (i) $u(x,t)$ is continuous in $G = [0,1] \times [0,\infty)$; (ii) $\partial u / \partial t$ and $\partial^k u / \partial x^k$ ($k = 0, 1, \dots, m$) exist and are continuous in G , with the possible exception of the points (x, nh) , where one-sided derivatives exist ($n = 0, 1, 2, \dots$); (iii) $u(x,t)$ satisfies equation (2.3) in G , with the possible exception of the points (x, nh) , and conditions (2.4)-(2.5).

Let $u_n(x,t)$ be the solution of the given problem on the interval $nh \leq t < (n+1)h$, then

$$\partial u_n(x,t) / \partial t + Pu_n(x,t) = Qu_n(x), \tag{2.6}$$

where

$$u_n(x) = u_n(x, nh).$$

Write

$$u_n(x, t) = w_n(x, t) + v_n(x),$$

which gives the equation

$$\partial w_n / \partial t + Pw_n + Pv_n(x) = Qu_n(x),$$

and require that

$$\partial w_n / \partial t + Pw_n = 0, \quad (2.7)$$

$$Pv_n(x) = Qu_n(x). \quad (2.8)$$

Assuming both w_n and v_n satisfy (2.4') leads to an ordinary BVP (2.8)-(2.4'), whose solution is denoted by

$$v_n(x) = P^{-1}Qu_n(x),$$

and to BVP (2.7)-(2.4'), whose solution is sought in the form

$$w_n(x, t) = X(x)T_n(t). \quad (2.9)$$

Separation of variables produces the ODE

$$T_n' + \lambda T_n = 0$$

with a solution

$$T_n(t) = e^{-\lambda(t-nh)},$$

and the BVP

$$P(d/dx)X - \lambda X = 0, \quad LX = 0 \quad (2.10)$$

where L is defined in (2.4) and (2.4'). If BVP (2.10) has an infinite countable set of eigenvalues λ_j and corresponding eigenfunctions

$X_j(x) \in C^m[0, 1]$, then the series

$$w_n(x, t) = \sum_{j=1}^{\infty} c_{nj} e^{-\lambda_j(t-nh)} X_j(x), \quad c_{nj} = \text{const}$$

represents a formal solution of problem (2.7)-(2.4') and

$$u_n(x, t) = \sum_{j=1}^{\infty} c_{nj} e^{-\lambda_j(t-nh)} X_j(x) + P^{-1}Qu_n(x) \quad (2.11)$$

is a formal solution of (2.3)-(2.4). At $t = nh$ we have

$$u_n(x) = \sum_{j=1}^{\infty} c_{nj} X_j(x) + P^{-1}Qu_n(x). \quad (2.12)$$

Therefore, assuming the sequence $\{X_j\}$ is complete and orthonormal in

$C^m[0,1]$ yields for the coefficients C_{nj} the formula

$$C_{nj} = \int_0^1 X_j(x) (I - P^{-1}Q) u_n(x) dx, \tag{2.13}$$

($n = 0, 1, 2, \dots$).

Substituting the initial function $u_0(x) \in C^m[0,1]$ in (2.13) produces the coefficients C_{0j} , and putting them together with $u_0(x)$ in (2.11) as $n=0$ gives the solution $u_0(x,t)$ of BVP (2.3), (2.4), (2.5) on the interval $0 \leq t \leq h$. Since $u_0(x,h) = u_1(x,h) = u_1(x)$, we can find from (2.13) the numbers C_{1j} and then substitute them along with $u_1(x)$ in (2.11), to obtain the solution $u_1(x,t)$ on $h \leq t \leq 2h$. This method of steps allows to extend the solution to any interval $nh \leq t \leq (n+1)h$. Furthermore, continuity of the solution $u(x,t)$ implies

$$u_n(x, (n + 1)h) = u_{n+1}(x, (n + 1)h) = u_{n+1}(x),$$

hence, at $t = (n + 1)h$ we get from (2.11) the recursion relations

$$u_{n+1}(x) = \sum_{j=1}^{\infty} C_{nj} e^{-\lambda_j h} X_j(x) + P^{-1}Q u_n(x). \tag{2.14}$$

Therefore,

$$u_{n+1}(x) = u_n(x) - \sum_{j=1}^{\infty} C_{nj} (1 - e^{-\lambda_j h}) X_j(x)$$

and

$$(I - P^{-1}Q) u_{n+1}(x) = (I - P^{-1}Q) u_n(x) - \sum_{j=1}^{\infty} C_{nj} (1 - e^{-\lambda_j h}) (I - P^{-1}Q) X_j(x).$$

Multiplying by $X_k(x)$ and integrating between 0 and 1 yields the recursion formulas

$$C_{n+1,k} = C_{nk} - \sum_{j=1}^{\infty} C_{nj} (1 - e^{-\lambda_j h}) X_{jk},$$

where

$$X_{jk} = \int_0^1 X_k(x) (I - P^{-1}Q) X_j(x) dx.$$

THEOREM 2.1. Formula (2.11), with coefficients C_{nj} and functions $u_n(x)$ defined by recursion relations (2.13) and (2.14), represents a formal solution of BVP (2.3), (2.4), (2.5) in $[0,1] \times [nh, (n + 1)h]$, for $n = 0, 1, \dots$, if BVP (2.10) has a countable number of eigenvalues λ_j and a complete orthonormal set of eigenfunctions $X_j(x) \in C^m[0,1]$ and the

initial function $u_0(x) \in C^m[0,1]$ satisfies (2.4).

A different method can be used if we look for a solution with continuous derivatives $\partial^2 u / \partial t^2$ and $\partial^{k+1} u / \partial t \partial x^k$ ($k = 0, 1, \dots, m$) for $t \in (nh, (n+1)h)$. In this case we differentiate (2.6) with respect to t and obtain the equation

$$\partial y_n / \partial t + P(\partial / \partial x) y_n = 0, \quad y_n = \partial u_n / \partial t,$$

whose solution is sought in form (2.9). Again, separation of variables produces $T_n(t)$ and BVP (2.10). Integrating the solution

$$y_n(x, t) = \sum_{j=1}^{\infty} B_{nj} e^{-\lambda_j(t-nh)} X_j(x) \quad (2.15)$$

between nh and t gives

$$u_n(x, t) = u_n(x) + \sum_{j=1}^{\infty} B_{nj} (1 - e^{-\lambda_j(t-nh)}) X_j(x) / \lambda_j. \quad (2.16)$$

Continuity of the solution at $t = (n+1)h$ implies

$$u_{n+1}(x) = u_n(x) + \sum_{j=1}^{\infty} B_{nj} (1 - e^{-\lambda_j h}) X_j(x) / \lambda_j. \quad (2.17)$$

From (2.6) and (2.15) at $t = nh$ we have

$$y_n(x, nh) = (Q - P) u_n(x),$$

$$y_n(x, nh) = \sum_{j=1}^{\infty} B_{nj} X_j(x),$$

and consequently,

$$B_{nj} = \int_0^1 X_j(x) (Q - P) u_n(x) dx. \quad (2.18)$$

THEOREM 2.2. Series (2.16), with coefficients B_{nj} and functions $u_n(x)$ defined by (2.17) and (2.18), formally represents a solution of BVP (2.3), (2.4), (2.5) whose derivatives $\partial u_n / \partial t$, $\partial^k u_n / \partial x^k$ ($k = 0, 1, \dots, m$) are continuous in $[0, 1] \times [nh, (n+1)h]$ and $\partial^2 u_n / \partial t^2$, $\partial^{k+1} u_n / \partial t \partial x^k$ are continuous in $[0, 1] \times (nh, (n+1)h)$ if, in addition to the other conditions of Theorem 2.1, the initial function $u_0(x)$ and $(Q - P)u_0(x)$ satisfy (2.4).

The solution $u_n(x, t)$ of the nonhomogeneous equation

$$\frac{\partial u(x,t)}{\partial t} + P\left[\frac{\partial}{\partial x}\right]u(x,t) = Q\left[\frac{\partial}{\partial x}\right]u(x, [t/h]h) + f(x,t) \quad (2.19)$$

on $nh \leq t < (n + 1)h$ is also sought in the form

$$u_n(x,t) = \sum_{j=1}^{\infty} X_j(x) T_{nj}(t), \quad (2.20)$$

where $X_j(x)$ are the eigenfunctions of the operator P . Upon multiplying (2.19) by $X_k(x)$, then integrating between 0 and 1 and changing k to j , we obtain

$$\begin{aligned} T'_{nj}(t) + \lambda_j T_{nj}(t) &= q_{nj} + f_j(t), \\ q_{nj} &= \int_0^1 X_j(x) Q(d/dx) u_n(x) dx, \\ f_j(t) &= \int_0^1 X_j(x) f(x,t) dx, \end{aligned}$$

whence

$$\begin{aligned} T_{nj}(t) &= (T_{nj}(nh) - \lambda_j^{-1} q_{nj}) e^{-\lambda_j(t-nh)} \\ &+ \lambda_j^{-1} q_{nj} + \int_{nh}^t e^{-\lambda_j(t-s)} f_j(s) ds, \\ &\quad (nh \leq t \leq (n + 1)h) \\ T_{nj}(nh) &= \int_0^1 u_n(x) X_j(x) dx, \end{aligned}$$

that is,

$$\begin{aligned} T_{nj}(t) &= \left[\int_0^1 X_j(x) (I - \lambda_j^{-1} Q) u_n(x) dx \right] e^{-\lambda_j(t-nh)} + \lambda_j^{-1} \int_0^1 X_j(x) Q u_n(x) dx \\ &+ \int_{nh}^t e^{-\lambda_j(t-s)} f_j(s) ds. \end{aligned} \quad (2.21)$$

The principal role of the operator P emerges from the above three methods of constructing the solution. Let

$$Py = \sum_{j=0}^m p_j y^{(m-j)},$$

where p_j are real-valued functions of classes C^{m-j} on $0 \leq x \leq 1$ and

$p_0(x) \neq 0$ on $[0,1]$. Assuming $C^m[0,1]$ is embedded in $L^2[0,1]$ with the inner product

$$(y,z) = \int_0^1 y(x) z(x) dx,$$

BVP (2.10) is called self-adjoint if

$$(Py,z) = (y,Pz),$$

for all $y, z \in C^m[0,1]$ that satisfy the boundary conditions

$$Ly = Lz = 0.$$

If BVP (2.10) is self-adjoint, then all its eigenvalues are real and form at most a countable set without finite limit points. The eigenfunctions corresponding to different eigenvalues are orthogonal.

THEOREM 2.3. BVP (2.3), (2.4), (2.5) has a solution in

$[0,1] \times [nh, (n+1)h]$, for each $n = 0, 1, \dots$, given by formula (2.11) if the following hypotheses hold true.

(i) BVP (2.10) is self-adjoint, all its eigenvalues λ_j are positive.

(ii) For each λ_j , the roots of the equation $P(z) - \lambda_j = 0$ have non-positive real parts.

(iii) The initial function $u_0(x) \in C^m[0,1]$ satisfies (2.4).

PROOF. According to (2.8), we find the solution $v_0(x) = P^{-1}Qu_0(x)$ of the equation $Pv_0(x) = Qu_0(x)$ satisfying the boundary conditions $Lv_0 = 0$. Then the difference $u_0(x) - P^{-1}Qu_0(x) \in C^m[0,1]$ satisfies (2.4'), and therefore we conclude from (2.12) that the Fourier series $\sum c_{0j} X_j(x)$ converges to it absolutely and uniformly on $[0,1]$, where $\{X_j(x)\}$ is the set of the orthonormal eigenfunctions of (2.10). Since $\lambda_j > 0$, the series in (2.11) also converges absolutely and uniformly on $[0,1] \times [0,h]$. Furthermore, the same is true on $[0,1]$ for the series in (2.14) at $n=0$, and $u_1(x)$ satisfies (2.4). Hence, $u_1(x)$ should be used now to find the solution $v_1(x) = P^{-1}Qu_1(x)$ of the equation $Pv_1(x) = Qu_1(x)$ satisfying $Lv_1 = 0$, then to calculate the coefficients c_{1j} by (2.13) and the solution $u_1(x,t)$ of the given BVP on $[0,1] \times [h,2h]$, according to (2.11). This procedure can be continued successively to construct the solution $u_n(x,t)$ for any $n \geq 0$. From (2.12) we conclude that all $u_n(x)$ satisfy (2.4'). Differentiating (2.11) term by term

with respect to t produces a series which converges to $\partial u_n / \partial t$ uniformly on $[0,1] \times [nh + \delta, (n + 1)h]$, for sufficiently small $\delta > 0$, since

$\lambda_j > 0$. Furthermore, it follows from (2.13) that

$$C_{nj} = \lambda_j^{-1} \int_0^1 (PX_j)(I - P^{-1}Q)u_n(x) dx$$

and since $X_j(x)$, $u_n(x)$, and $P^{-1}Qu_n(x)$ satisfy (2.4'), then

$$C_{nj} = \lambda_j^{-1} \int_0^1 X_j(x)(P - Q)u_n(x) dx.$$

Hence,

$$|C_{nj}| \leq \lambda_j^{-1} \left[\int_0^1 X_j^2 dx \right]^{1/2} \left[\int_0^1 (Pu_n - Qu_n)^2 dx \right]^{1/2} \leq c_n \lambda_j^{-1}. \quad (2.22)$$

Let $p_0(x) = 1$ and $\lambda = \rho^m$, then in any domain T of the complex ρ -plane the equation

$$P(d/dx)y - \lambda y = 0$$

has m linearly independent solutions y_1, \dots, y_m which are regular with respect to $\rho \in T$, for sufficiently large $|\rho|$, and satisfy the relations

$$y_k^{(r-1)}(x) = \rho^{r-1} e^{\rho \omega_k x} \left[\omega_k^{r-1} + o(\rho^{-1}) \right],$$

$$(k, r = 1, \dots, m)$$

where $\omega_1, \dots, \omega_m$ are the different m -order roots of unity [9]. Therefore, by virtue of condition (ii) and estimates (2.22), differentiating series (2.11) term by term r times ($r = 1, \dots, m$) with respect to x produces series that converge iniformly on $[0,1] \times [nh + \delta, (n + 1)h]$, for sufficiently small δ and large λ_j . Letting $t = (n+1)h$ in each of these series and taking into account (2.14) shows that $u_{n+1}(x) \in C^m[0,1]$

if $u_n(x) \in C^m[0,1]$. By virtue of (iii), the proof is complete.

REMARK 1. We assumed in this theorem that $p_0(x) = 1$, where $p_0(x)$ is the leading coefficient of the operator $P(d/dx)$. If $p_0 = \text{const} \neq 1$, then dividing the equation $Py - \lambda y = 0$ by p_0 produces an equation whose leading coefficient is 1. If $p_0(x) \neq \text{const}$ on $[0,1]$ and retains

its sign, then we may assume $p_0(x) > 0$ and use the substitution [9]

$$x_1 = \int_0^x P_0^{-1/m}(s) ds / \int_0^1 P_0^{-1/m}(s) ds,$$

to reduce the above equation to a new one in the interval $0 \leq x_1 \leq 1$, with a constant leading coefficient.

REMARK 2. The Fourier coefficients used in the above three methods of solving BVP (2.3), (2.4), (2.5) are closely interrelated. Indeed, differentiating (2.11) with respect to t and comparing with (2.15) shows the $B_{nj} = -\lambda_j C_{nj}$. Furthermore, comparing (2.11) with (2.20) and (2.13) with (2.21), we have to prove that

$$\int_0^1 \lambda_j^{-1} X_j(x) Q u_n(x) dx = \int_0^1 X_j(x) P^{-1} Q u_n(x) dx.$$

Since $P^{-1} Q u_n(x)$ satisfies (2.4), then

$$P^{-1} Q u_n(x) = \sum_{k=1}^{\infty} X_k(x) \int_0^1 X_k(x) P^{-1} Q u_n(x) dx,$$

and applying the operator P to this equation yields

$$Q u_n(x) = \sum_{k=1}^{\infty} \lambda_k X_k(x) \int_0^1 X_k(x) P^{-1} Q u_n(x) dx.$$

It remains to multiply this expansion by $X_j(x)$ and to integrate between 0 and 1.

EXAMPLE 2.1. The solution $u_n(x, t)$ of equation (2.1) in

$[0, 1] \times [nh, (n+1)h]$, with the boundary conditions $u_n(0, t) = u_n(1, t) = 0$ and initial condition $u_n(x, nh) = u_n(x)$ is sought in form (2.20). Separation of variables produces

$$X_j(x) = \sqrt{2} \sin(\pi j x), \quad T'_{nj}(t) + a^2 \pi^2 j^2 T_{nj}(t) = -b T_{nj}(nh),$$

whence

$$T_{nj}(t) = c_{nj} e^{-a^2 \pi^2 j^2 (t-nh)} - \frac{b}{a^2 \pi^2 j^2} T_{nj}(nh).$$

We put $t = nh$ in this equation and get

$$c_{nj} = \left[1 + \frac{b}{a^2 \pi^2 j^2} \right] T_{nj}(nh),$$

that is,

$$T_{nj}(t) = E_j(t - nh) T_{nj}(nh),$$

where

$$E_j(t) = e^{-a^2 \pi^2 j^2 t} - \left[1 - e^{-a^2 \pi^2 j^2 t} \right] \frac{b}{a^2 \pi^2 j^2} . \tag{2.23}$$

At $t = (n + 1)h$ we have

$$T_{nj}((n + 1)h) = E_j(h)T_{nj}(nh)$$

and since

$$T_{nj}((n + 1)h) = T_{n+1,j}((n + 1)h) ,$$

then

$$T_{n+1,j}((n + 1)h) = E_j(h)T_{nj}(nh)$$

and

$$T_{nj}(nh) = E_j^n(h)T_{0j}(0) .$$

Therefore,

$$T_{nj}(t) = E_j(t - nh)E_j^n(h)T_{0j}(0)$$

and

$$u_n(x,t) = \sum_{j=1}^{\infty} \sqrt{2} E_j^n(h)T_{0j}(0)E_j(t - nh)\sin(\pi jx) . \tag{2.24}$$

Putting $t = 0, n = 0$ gives

$$u_0(x) = \sum_{j=1}^{\infty} T_{0j}(0)\sqrt{2}\sin(\pi jx)dx$$

and

$$T_{0j}(0) = \sqrt{2} \int_0^1 u_0(x)\sin(\pi jx)dx .$$

If $|E_j(h)| < 1$, then solution (2.24) decays exponentially as $t \rightarrow \infty$,

uniformly with respect to x . From (2.23) it follows that this is true if

$$-a^2 \pi^2 < b < a^2 \pi^2 \left[e^{a^2 \pi^2 h} + 1 \right] / \left[e^{a^2 \pi^2 h} - 1 \right] .$$

Furthermore, from the equations

$$T_{nj}(nh) = E_j^n(h)T_{0j}(0) , T_{nj}((n + 1)h) = E_j^{n+1}(h)T_{0j}(0)$$

we see that $T_{nj}(nh)T_{nj}((n + 1)h) < 0$ if $E_j(h) < 0$. The latter inequality holds true if

$$b > a^2 \pi^2 / \left[e^{a^2 \pi^2 h} - 1 \right] . \tag{2.25}$$

Hence, under condition (2.25), each function $T_{nj}(t)$ ($j = 1, 2, \dots$) has a zero in the interval $[nh, (n+1)h]$, in sharp contrast to the functions $T_j(t)$ in the Fourier expansion for the solution of the equation $u_t = a^2 u_{xx} - bu$ without time delay. Moreover, the inequality $E_j(h) < 0$ takes place for sufficiently large j and any $b > 0$. Therefore, for $b > 0$ and sufficiently large j , the functions $T_{nj}(t)$ are oscillatory.

EXAMPLE 2.2. Equation (2.2) on $nh \leq t < (n+1)h$ becomes

$$\partial u_n(x, t) / \partial t = a^2 \partial^2 u_n(x, t) / \partial x^2 - r u_n'(x),$$

and we differentiate the latter with respect to t to obtain the equation

$$\partial y_n / \partial t = a^2 \partial^2 y_n / \partial x^2, \quad y_n = \partial u_n / \partial t,$$

whose solution is sought in form (2.9). Separation of variables leads to the equations

$$X''(x) + \lambda X(x) = 0, \quad T_n'(t) + a^2 \lambda T_n(t) = 0,$$

and posing the boundary conditions $u_n(0, t) = u_n(1, t) = 0$ gives $\lambda_j = j^2 \pi^2$

and

$$y_n(x, t) = \sum_{j=1}^{\infty} \sqrt{2} T_{nj}(nh) e^{-a^2 \pi^2 j^2 (t-nh)} \sin(\pi j x).$$

Since

$$y_n(x, nh) = a^2 u_n''(x) - r u_n'(x), \quad u_n(x) = u_n(x, nh),$$

then

$$a^2 u_n''(x) - r u_n'(x) = \sum_{j=1}^{\infty} \sqrt{2} T_{nj}(nh) \sin(\pi j x)$$

and

$$\begin{aligned} T_{nj}(nh) &= -a^2 \pi^2 j^2 \sqrt{2} \int_0^1 u_n(x) \sin(\pi j x) dx \\ &\quad + r \pi j \sqrt{2} \int_0^1 u_n(x) \cos(\pi j x) dx. \end{aligned}$$

Finally,

$$u_n(x, t) = u_n(x) + \sum_{j=1}^{\infty} \sqrt{2} T_{nj}(nh) \left[1 - e^{-a^2 \pi^2 j^2 (t-nh)} \right] \sin(\pi j x) / a^2 \pi^2 j^2.$$

Given the initial function $u(x, 0) = u_0(x)$, we can find the coefficients $T_{0j}(0)$ and the solution $u_0(x, t)$ on $0 \leq t \leq h$. Since $u_0(x, h) = u_1(x)$, we can calculate the coefficients $T_{1j}(h)$ and the solution

$u_1(x, t)$ on $h \leq t \leq 2h$. By the method of steps the solution can be extended to any interval $[nh, (n + 1)h]$.

EXAMPLE 2.3. Separation of variables for the equation

$$\partial u(x, t) / \partial t = a^2 \partial^2 u(x, t) / \partial x^2 - b \partial^2 u(x, [t]) / \partial x^2$$

with the boundary conditions $u(0, t) = u(1, t) = 0$ produces the eigenfunctions $X_j(x) = \sqrt{2} \sin(\pi j x)$ and the equation

$$T'(t) + a^2 \pi^2 j^2 T(t) = b \pi^2 j^2 T([t]),$$

which on the interval $n \leq t < n + 1$ becomes

$$T'_{nj}(t) + a^2 \pi^2 j^2 T_{nj}(t) = b \pi^2 j^2 T_{nj}(n).$$

From here,

$$T_{nj}(t) = F_j(t - n) T_{nj}(n),$$

where

$$F_j(t) = e^{-a^2 \pi^2 j^2 t} + \left[1 - e^{-a^2 \pi^2 j^2 t} \right] b / a^2.$$

At $t = n + 1$ we have

$$T_{nj}(n + 1) = F_j(1) T_{nj}(n)$$

and since $T_{nj}(n + 1) = T_{n+1, j}(n + 1)$, then

$$T_{n+1, j}(n + 1) = F_j(1) T_{nj}(n)$$

and

$$T_{nj}(n) = F_j^n(1) T_{0j}(0).$$

Hence,

$$u_n(x, t) = \sum_{j=1}^{\infty} \sqrt{2} F_j^n(1) T_{0j}(0) F_j(t - n) \sin(\pi j x),$$

where

$$T_{0j}(0) = \sqrt{2} \int_0^1 u_0(x) \sin(\pi j x) dx.$$

The inequalities

$$-a^2 \left[e^{a^2 \pi^2} + 1 \right] / \left[e^{a^2 \pi^2} - 1 \right] < b < a^2$$

are equivalent to $|F_j(1)| < 1$ and ensure the exponential decay of

$u_n(x, t)$ as $t \rightarrow \infty$, uniformly with respect to x . For

$$b < -a^2 / (e^{a^2\pi^2} - 1),$$

each function $T_{nj}(t)$ has a zero in the interval $[n, n + 1]$, which is impossible for the equation $u_t = (a^2 - b)u_{xx}$.

EXAMPLE 2.4. The equation

$$iq \frac{\partial u(x, t)}{\partial t} = -\frac{q^2}{2m_0} \frac{\partial^2 u(x, t)}{\partial x^2} + V(x)u(x, [t/h]h)$$

is a piecewise constant analogue of the one-dimensional Schrödinger equation

$$iq \psi_t(x, t) = -q^2 \psi_{xx}(x, t)/2m_0 + V(x)\psi(x, t).$$

If $u(x, t)$ satisfies conditions (2.4) and (2.5), with $m = 2$, then separation of variables produces a formal solution

$$u_n(x, t) = \sum_{j=1}^{\infty} c_{nj} e^{-i\lambda_j(t-nh)/q} x_j(x) + P^{-1}Qu_n(x),$$

for $nh \leq t \leq (n + 1)h$. Here $x_j(x)$ are the eigenfunctions of the operator $q^2(d^2/dx^2)/2m_0$, and $P^{-1}Qu_n(x)$ is the solution $v_n(x)$ of the equation

$$q^2 v_n''(x) = 2m_0 V(x)u_n(x)$$

that satisfies (2.4) and c_{nj} are given by (2.13).

The Fourier method can be also used to find weak solutions of BVP (2.3), (2.4), (2.5) and it is easily generalized to similar problems in Hilbert space. First, we remind a few well known definitions. Let H be a Hilbert space and let P be a linear operator in H (additive and homogeneous but, possibly, unbounded) whose domain $D(P)$ is dense in H , that is $\overline{D(P)} = H$. The operator P is called symmetric if $(Pu, v) = (u, Pv)$, for any $u, v \in D(P)$. If P is symmetric, then (Pu, v) is a symmetric bilinear functional and (Pu, u) is a quadratic form. A symmetric operator P is called positive if $(Pu, u) \geq 0$ and $(Pu, u) = 0$ if and only if $u = 0$. A symmetric operator P is called positive definite if there exists a constant $\gamma^2 > 0$ such that $(Pu, u) \geq \gamma^2 \|u\|^2$. With every positive operator P a certain Hilbert space H_P can be associated, which is called the energy space of P . It is the completion of $D(P)$, with the inner product $(u, v)_P = (Pu, v)$; $u, v \in D(P)$. This pro-

duct induces a new norm $\|u\|_P = (Pu, u)^{1/2}$, $u \in D(P)$, and if P is positive definite, then $\|u\| \leq \gamma^{-1} \|u\|_P$. Since $D(P)$ is dense in H , it follows by using the latter inequality that the energy space H_P of a positive definite operator P is dense in the original space H .

Assuming P is positive definite, we may consider the solution $u(x, t)$ of BVP (2.3), (2.4), (2.5) for a fixed t as an element of H_P . If $D(Q) \subset H$, then $Qu(x, [t/h]h)$ may be treated as an abstract function $Qu([t/h]h)$ with the values in H . Therefore, the given BVP is reduced to the abstract Cauchy problem

$$\frac{du}{dt} + Pu = Qu([t/h]h), \quad t > 0, \quad u|_{t=0} = u_0 \in H. \tag{2.26}$$

If (2.26) has a solution, we multiply each term by an arbitrary function $g(t) \in H_P$ in the sense of inner product in H and get on

$nh \leq t < (n+1)h$ the equation

$$\left[\frac{du}{dt}, g \right] + (u, g)_P = (Qu_n, g), \tag{2.27}$$

where $u_n = u(nh)$. Conversely, if $u \in C^1((nh, (n+1)h); D(P))$ for all integers $n \geq 0$ and satisfies (2.27), then it also satisfies equation (2.26). Indeed, if $u \in D(P)$, then $(u, g)_P = (Pu, g)$, and (2.27) can be written as

$$\left[\frac{du}{dt} + Pu - Qu_n, g \right] = 0, \quad nh \leq t < (n+1)h.$$

Since H_P is dense in H , then $u(t)$ is a solution of equation (2.26).

DEFINITION 2.2. An abstract function $u(t): [0, \infty) \rightarrow H$ is called a weak solution of problem (2.26) if it satisfies the conditions:

- (i) $u(t)$ is continuous for $t \geq 0$ and strongly continuously differentiable for $t > 0$, with the possible exception of the points $t = nh$ where one-sided derivatives exist;
- (ii) $u(t)$ is continuous for $t > 0$ as an abstract function with the values in H_P and satisfies equation (2.27) on each interval $nh \leq t < (n+1)h$, for any function $g(t): [0, \infty) \rightarrow H_P$;
- (iii) $u(t)$ satisfies initial condition (2.26), that is,

$$\lim_{t \rightarrow 0} \|u(t) - u_0\|_H = 0.$$

Clearly, a weak solution $u(t)$ is also an ordinary solution if $u(t) \in D(P)$, for any $t > 0$, and $u(x, t) \rightarrow u_0(x)$ as $t \rightarrow 0$ not only in

the norm of H but uniformly as well. It is said [10] that a symmetric operator P has a discrete spectrum if it has an infinite sequence $\{\lambda_j\}$ of eigenvalues with a single limit point at infinity and a sequence

$\{X_j\}$ of eigenfunctions which is complete in H . Suppose the operator P

in (2.27) is positive definite and has a discrete spectrum and assume existence of a solution $u(t) = u(x, t)$ to equation (2.27) with the condition $u(0) = u_0$. On the interval $nh \leq t < (n+1)h$ this solution can be expanded into series (2.20), where $T_j(t) = (u(t), X_j)$. To find the coefficients $T_j(t)$, we put $g(t) = X_k$ in (2.27) and since X_k does not depend on t , then

$$\left[\frac{du(t)}{dt}, X_k \right] = \frac{d}{dt} (u(t), X_k) = T_k'(t),$$

$$(u, X_k)_P = (Pu, X_k) = (u, PX_k) = \lambda_k (u, X_k) = \lambda_k T_k(t),$$

which again leads to the equation

$$T_{nj}'(t) + \lambda_j T_{nj}(t) = (Qu_n, X_j)$$

and to a generalization of (2.21). By selecting a proper space H , a weak solution corresponding to conditions (2.4) can be constructed. The proof of the following theorem is omitted.

THEOREM 2.4. If P and Q are linear operators in a Hilbert space and P is positive definite with a discrete spectrum, then there exists a unique weak solution of problem (2.26).

ACKNOWLEDGMENT: Research partially supported by U.S. Army Grant DAAL03-89-G-0107.

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