AN INEQUALITY OF W.L. WANG AND P.F. WANG

HORST ALZER

Department of Mathematics
University of the Witwatersrand
Johannesburg, South Africa

(Received June 7, 1988 and revised form May 1, 1989)

ABSTRACT. In this note we present a proof of the inequality \( \frac{H_n}{H'_n} \leq \frac{G_n}{G'_n} \) where \( H_n \) and \( G_n \) (resp. \( H'_n \) and \( G'_n \)) denote the weighted harmonic and geometric means of \( x_1, \ldots, x_n \) (resp. \( 1-x_1, \ldots, 1-x_n \)) with \( x_i \in (0, 1/2] \), \( i = 1, \ldots, n \).

KEY WORDS AND PHRASES. Geometric and harmonic means, inequalities.

1980 AMS SUBJECT CLASSIFICATION CODE. 26D15.

1. INTRODUCTION.

Let \( p_1, \ldots, p_n \) and \( x_1, \ldots, x_n \) be two sequences of positive real numbers with
\[
\sum_{i=1}^{n} p_i = 1 \quad \text{and} \quad x_i \in (0, 1/2], \quad i = 1, \ldots, n.
\]
In what follows we denote by \( A_n, G_n \) and \( H_n \) (resp. \( A'_n, G'_n \) and \( H'_n \)), the weighted arithmetic, geometric and harmonic means of \( x_1, \ldots, x_n \) (resp. \( 1-x_1, \ldots, 1-x_n \)), i.e.
\[
A_n = \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}, \quad G_n = \prod_{i=1}^{n} x_i^{p_i} \quad \text{and} \quad H_n = \frac{1}{\sum_{i=1}^{n} p_i x_i}, \quad \text{(1.1)}
\]
resp. \( A'_n = \frac{\sum_{i=1}^{n} p_i (1-x_i)}{\sum_{i=1}^{n} p_i}, \quad G'_n = \prod_{i=1}^{n} (1-x_i)^{p_i} \quad \text{and} \quad H'_n = \frac{1}{\sum_{i=1}^{n} p_i (1-x_i)}, \quad \text{(1.2)}
\]
Setting \( p_1 = \ldots = p_n = 1/n \) in (1.1) and (1.2) we obtain the unweighted arithmetic, geometric and harmonic means of \( x_1, \ldots, x_n \) (resp. \( 1-x_1, \ldots, 1-x_n \)), designated by \( a_n, g_n \) and \( h_n \) (resp. \( a'_n, g'_n \) and \( h'_n \)).

In 1961 E.F. Beckenbach and R. Bellman [1] published a remarkable counterpart of the classical arithmetic mean-geometric mean inequality which is due to Ky Fan, namely
\[
g_n / g'_n < a_n / a'_n \quad \text{(1.3)}
\]
with equality holding in (1.3) if and only if \( x_1 = \ldots = x_n \). Since then Fan's inequality has been subjected to considerable investigations resulting in many proofs, sharpenings and refinements (see Alzer [2] and the references therein). It is natural to ask whether there exists a corresponding inequality for geometric and harmonic means. In 1984 W.L. Wang and P.F. Wang [3] have answered this question. They
established the inequality
\[ \frac{h_n}{h'_n} < \frac{g_n}{g'_n} \] (1.4)
where the sign of equality is valid if and only if \( x_1 = \ldots = x_n \). It is worth mentioning that not only (1.3) but also (1.4) has been originally proved by using Cauchy's method of forward and backward induction.

In the last year, different authors have verified that Fan's inequality holds for weighted mean values, i.e.
\[ \frac{G_n}{G'_n} < \frac{A_n}{A'_n} \] (1.5)
with equality if and only if \( x_1 = \ldots = x_n \) as in Flanders [4], Levinson [5] and Wang [6-8]. The aim of this note is to show that inequality (1.4) can also be extended to weighted means.

2. AN INEQUALITY FOR WEIGHTED GEOMETRIC AND HARMONIC MEANS.

We establish the following counterpart of (1.5):

**THEOREM 2.1.** If \( x_i \in (0, 1/2] \), \( i = 1, \ldots, n \), then
\[ \frac{H_n}{H'_n} < \frac{G_n}{G'_n} \] (2.1)
with equality holding in (2.1) if and only if \( x_1 = \ldots = x_n \).

**PROOF.** If we set
\[ z_i = x_i/(1-x_i), \quad 0 < z_i < 1, \quad i = 1, \ldots, n, \]
then (2.1) can be rewritten as
\[ \sum_{i=1}^{n} \frac{p_i(1+z_i)}{p_i(1+1/z_i)} < \prod_{i=1}^{n} z_i. \] (2.2)

Since equality holds in (2.2) if \( z_1 = \ldots = z_n \), it remains to show that (2.2) is strict if the numbers \( z_1, \ldots, z_n \) are not all equal. We use induction on \( n \). Let \( n = 2 \); then we have to prove that the function
\[ f(z_2) = (p_1z_1 + z_1)z_2 + p_2z_1z_2 - p_2z_2 - p_1z_1 - 1 \]
is positive for \( 0 < z_1 < z_2 < 1 \).

A simple calculation yields
\[ f''(z_2) = p_1p_2z_1^2z_2 - p_2p_1z_1^3 + (p_1+z_1)(z_0-z_2) \] with
\[ z_0 = (2-p_2)z_1/(p_1+z_1) \in (z_1,1), \]
so
\[ f''(z_2) > 0 \] for \( z_1 < z_2 < z_0 \)
and
\[ f''(z_2) < 0 \] for \( z_0 < z_2 < 1. \]
Since $f(z_1) = f'(z_1) = 0$ and $f'(1) > 0$ we obtain

$$f(z_2) > 0 \quad \text{for } z_1 < z_2 < 1.$$ 

Next we assume that (2.2) is true for $n > 2$. Let us put $z = z_{n+1}$ and $p = p_{n+1}$. Without loss of generality we set $0 < z_1 < \ldots < z_n < 1$, $z_1 < z$. (2.3)

Since

$$\frac{1}{1-p} \sum_{i=1}^{n} p_i = 1$$

we get from the induction hypothesis

$$z^p \prod_{i=1}^{n} z_i^{p_i} > z^p \left[ \frac{\sum_{i=1}^{n} p_i(1+z_i)}{\sum_{i=1}^{n} p_i(1+1/z_i)} \right]^{1-p}$$

and it remains to prove

$$z^p \left[ \frac{\sum_{i=1}^{n} p_i(1+z_i)}{\sum_{i=1}^{n} p_i(1+1/z_i)} \right]^{1-p} > \frac{\sum_{i=1}^{n} p_i(1+z_i) + p(1+z)}{\sum_{i=1}^{n} p_i(1+1/z_i) + p(1+1/z)}.$$ (2.4)

We set

$$a = \sum_{i=1}^{n} p_i(1+z_i) \quad \text{and} \quad b = \sum_{i=1}^{n} p_i(1+1/z_i).$$

Then (2.4) can be written as

$$z^p \frac{(a/b)^{1-p}}{a + p(1+z)} > \frac{a + p(1+z)}{b + p(1+1/z)}$$

and this is equivalent to

$$g(a, b, z) = p \ln(z) + (1-p) \ln(a) - (1-p) \ln(b) - \ln(a+p(1+z)) + \ln(b+p(1+1/z)) > 0.$$ 

Partial differentiation reveals

$$\frac{\partial}{\partial a} g(a, b, z) = \frac{p}{a} \left[ (1-p)(1+z) - a \right] / [a+p(1+z)]$$

and

$$\frac{\partial}{\partial b} g(a, b, z) = \frac{p}{b} \left[ (p-1)(1+1/z)+b \right] / [b+p(1+1/z)].$$

From (2.3) we conclude

$$a < (1-p)(1+z) \quad \text{and} \quad b > (1-p)(1+1/z)$$

hence we obtain
\[
\frac{\partial^2}{\partial a^2} g(a,b,z) > 0 \quad \text{and} \quad \frac{\partial^2}{\partial b^2} g(a,b,z) > 0.
\]

Since \(l - p < a < b\) we get
\[
g(a,b,z) > g(l-p, l-p,z).
\]

We define
\[
h(p) = g(l-p, l-p,z)
\]
then we get
\[
h''(p) = (z/(l+pz))^2 - (1/(p+z))^2 < 0
\]
and because of
\[
h(0) = h(1) = 0
\]
we have
\[
h(p) > 0 \text{ for } 0 < p < 1,
\]
which completes the proof of inequality (2.4).

REMARK 2.1. We notice that the method used to establish inequality (2.2) for \(n = 2\), can also be used to prove (2.4). And the technique applied to establish inequality (2.4) can be used to prove (2.2) for \(n = 2\) as well.

ACKNOWLEDGEMENT. I want to thank the referee for helpful remarks.

REFERENCES

Special Issue on
Singular Boundary Value Problems for Ordinary
Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Deadline</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie

Hindawi Publishing Corporation
http://www.hindawi.com