LOCATION OF APPROXIMATIONS OF A MARKOFF THEOREM

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ABSTRACT. Relative to the first two theorems of the well known Markoff Chain (J.W.S.
Cassels, "An introduction to diophantine approximation" approximations are well located.
Literature is silent on the question of location of approximations in reference to the
other theorems of the Chain. Here we settle it for the third theorem of the Chain.

KEY WORDS AND PHRASES. Continued fractions, rational approximation.

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1. INTRODUCTION

Suppose $\theta$ is an irrational number whose simple continued fraction expansion is

$[a_0, a_1, a_2, ..., a_n, ...]$. Let $a_n(\theta)$ denote

$[0, a_n, a_{n-1}, ..., a_1] + [a_{n+1}, a_{n+2}, ...]$

Markoff Chain (Cassels [1], Kokshma [2]) is the following chain of theorems about the
sequence $(a_n(\theta))_{n \geq 1}$:

$T_1$: For every irrational number $\theta$,

$a_j(\theta) > \sqrt{5}$

for infinity of $j$'s and $\sqrt{5}$ cannot be increased for $\theta \sim [0, (1)^*]$.

$T_2$: If $\theta \notin [0, (1)^*]$, then

$a_j(\theta) > \sqrt{8}$

for infinity of $j$'s and $\sqrt{8}$ cannot be increased for $\theta \sim [0, (2)^*]$. 


T3: If \( \theta \notin [0, (1) \} \) or \([0, (2) \} \) then
\[
a_j(\theta) > \left( \frac{\sqrt{21}}{5} \right)
\]  
for infinity of \( j \)'s and \( \left( \frac{\sqrt{21}}{5} \right) \) cannot be increased for \( \theta = [0, (2,2,1,1,1,1) \} \)

T4: If \( \theta \notin [0, (1) \} \) or \([0, (2) \} \) or \([0, (2,2,1,1) \} \) then
\[
a_j(\theta) > \left( \frac{\sqrt{1517}}{13} \right)
\]  
for infinity and \( j \)'s and \( \left( \frac{\sqrt{1517}}{13} \right) \) cannot be increased for
\( \theta = [0, (2,2,1,1,1,1) \} \) etc.

It is known that the sequence of constants \( \sqrt{5}, \sqrt{8}, \left( \frac{\sqrt{21}}{5} \right), \left( \frac{\sqrt{1517}}{13} \right), \ldots \), increases to 3. So the theorems say something non-trivial about \( \theta \)'s in which all quotients are eventually 1 or 2 only.

As regards (1.1) and (1.2) we have an ad-hoc idea of the \( j \)'s satisfying them. In reference to T1 we know that one \( j \) must occur in \( \{n, n+1, n+2\} \) and \( n \geq 1 \). Relative to T2, we have a similar result: if \( a_{n+2} = 2 \) then \( a_j \in \{n, n+1, n+2\} \). These may be found in Wright [3] or Prasad and Lari [4].

But the literature is surprisingly silent on such results in reference to T3, T4, etc. In this article we announce one such result in reference to T3 in the following theorem:

2. MAIN RESULTS

**Theorem.** \( a_{n+2} = 2 \) and \( a_{n+3} = 1 \)

then \( a_j(\theta) > \left( \frac{\sqrt{21}}{5} \right) \) for at least one \( j \in \{n, n+1, n+2\} \)

**Remark.** Our method gives a way to try for similar results on T4, T5, etc.

**Proof.** Suppose \( \theta = [a_0, a_1, \ldots, a_n, \ldots] \), \( a_{n+2} = 2 \) and \( a_{n+3} = 1 \)

If \( a_{n+1} \geq 3 \), \( a_n(\theta) > 3 \) and we are through.

If \( a_{n+1} = 1 \); \( a_{n+1}(\theta) > [0, 2] + [2, 2] \) = 3 and we are through.

If \( a_{n+1} = 2 \) and \( a_{n+4} \geq 2 \) then \( a_{n+1}(\theta) > [0, 2, 1, 1] \) = 3 and we are through.

For the left out \( \theta \)'s: \( a_{n+1} = 2 = a_{n+2} \) and \( a_{n+3} = 1 = a_{n+4} \).

To deal with them, we put
\[
\alpha = [0, a_n, a_{n-1}, \ldots, a_1],
\beta = [0, a_{n+5}, a_{n+6}, \ldots],
\tau = \left( \frac{\sqrt{21}}{5} \right)
\]
and argue over all possible values of \( \beta \). We note:
\[
\alpha_{n+4}(\theta) > t \iff \alpha[5-(5t-3)\beta] > [(2t-7)\beta - 12]
\]
So \( \beta \leq \frac{12}{12t-7} \Rightarrow \alpha_{n+4}(\theta) > t \)
We next check:

\[ a_n(\theta) > t \iff a > f_1(\beta) = \frac{(5t-12) + (3t-7)\beta}{(5 + 3\beta)} \]

\[ a_{n+1}(\theta) > t \iff a < f_2(\beta) = \frac{(12-4t) + (7-2t)\beta}{(2t-5) - (3-t)\beta} \]

and

\[ f_2(\beta) - f_1(\beta) = A_1(\beta + \frac{251 - 5t}{146}) (\beta - \frac{5t - 9}{14}) \]

where \( A_1 = t(10-3t)(5+3\beta)^{-1} \) \((2t-5) - (3-t)\beta)^{-1}: (\beta > 0)\).

So \( \beta > \frac{5t-9}{14} \)

\[ \Rightarrow f_2(\beta) > f_1(\beta) \]

\[ \Rightarrow a > f_1(\beta) \text{ or } a < f_2(\beta) \]

\[ \Rightarrow a_n(\theta) > t \text{ or } a_{n+1}(\theta) > t. \]

Hence we confine attention to \( \frac{12}{12t-7} < \beta \leq \frac{5t-9}{14} \).

In this case \( a_{n+4}(\theta) > t \iff a > f_3(\beta) = \frac{(12t-7)\beta - 12}{5-(5t-3)\beta} \)

Also

\[ f_3(\beta) - f_2(\beta) = A_2(\beta + \frac{5t+19}{14}) (\beta - \frac{5t-9}{14}) \]

where \( A_2 = 2t(t-1)(5-(5t-3)\beta)^{-1} \cdot ((2t-5) - (3-t)\beta)^{-1}: (\beta > 0) \)

So if \( \frac{12}{12t-7} < \beta < \frac{5t-9}{14} \) then \( f_3(\beta) < f_2(\beta) \)

which implies \( a < f_2(\beta) \text{ or } a > f_3(\beta); \text{ equivalently} \)

\[ a_{n+1}(\theta) > t \text{ or } a_{n+4}(\theta) > t \text{ and we are through}. \]

Finally \( \beta = \frac{5t-9}{14} \)

\[ \Rightarrow f_1(\beta) = f_2(\beta) = f_3(\beta) = \frac{5t-9}{10} \text{ (an irrational number)} \]

\[ \Rightarrow a > f_1(\beta) \text{ or } a < f_2(\beta), \text{ (} a \text{ is rational) } \]

\[ \Rightarrow a_n(\theta) > t \text{ or } a_{n+1}(\theta) > t. \]

This completes the proof of the theorem.
REFERENCES

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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