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KELVIN PRINCIPLE FOR A CLASS OF SINGULAR EQUATIONS

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ABSTRACT. The classical Kelvin principle concerns invariance of solutions of the Laplace equation with respect to inversion in a sphere. By employing a hyperbolic-polar coordinate system, the principle is extended to cover a class of singular equations, which include the ultrahyperbolic equation.

KEY WORDS AND PHRASES. Kelvin principle, Laplace equation, ultrahyperbolic equation, Lorentzian distance.
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1. INTRODUCTION.

As is well known the classical Kelvin principle introduced in 1847 (Thomson [1]) concerns solutions of the Laplace equation. For solutions of some class of elliptic differential equations and their iterated forms in n independent variables, $n \geq 2$, the extension of Kelvin principle is usually proved using rectangular coordinates (Diaz and Martin [2], Germain and Bader [3], Huber [4], Weinstein [5]). In 1960 a generalization of Kelvin principle was established by Weinstein [5] for the equation

$$\sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial u}{\partial x_i} \right) = 0, \quad -\infty < k_i < \infty$$

using polar coordinates.

Following Weinstein method we shall give in this paper a new formulation of Kelvin principle for solutions of the class of singular partial differential equations

$$L(u) = \sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial u}{\partial x_i}\right) - \sum_{j=1}^{m} \left(\frac{\partial^2 u}{\partial y_j^2} + \frac{\beta_j}{y_j} \frac{\partial u}{\partial y_j}\right) + \frac{1}{r^2} P(u) = 0 \qquad (1.1)$$

where α_i $(1 \leq i \leq n)$ and β_j $(1 \leq j \leq m)$ are real parameters, r is the Lorentzian metric defined by

$$r^2 = \sum_{i=1}^{n} x_i^2 - \sum_{j=1}^{m} y_j^2 \qquad (1.2)$$

and P is a general linear operator of arbitrary order q in the variables z_1, z_2, ..., z_p vanishing for $u = 0$.

The domain of the operator L is the set of all real valued functions $u(x,y,z)$ of class $C^2(D) \cap C^q(\Omega)$, where $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_m)$ and $z = (z_1,\dots,z_p)$ denote points in R^n, R^m and R^p, respectively, and $D \times \Omega$ is a regularity domain of u in $R^{n+m} \times R^p$.

2. HYPERBOLIC-POLAR COORDINATE SYSTEM FOR EQUATION (1.1).

First let us consider the n+m-dimensional Laplace operator

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_{n+j}^2} \tag{2.1}$$

and introduce the polar coordinates

$$\begin{aligned}
x_1 &= r \cos\theta_1 \cos\theta_2 \dots \cos\theta_{m-1} \cos\theta_m \cos\theta_{m+1} \dots \cos\theta_{m+n-2} \cos\theta_{m+n-1} \\
x_2 &= r \cos\theta_1 \cos\theta_2 \dots \cos\theta_{m-1} \cos\theta_m \cos\theta_{m+1} \dots \cos\theta_{m+n-2} \sin\theta_{m+n-1} \\
x_3 &= r \cos\theta_1 \cos\theta_2 \dots \cos\theta_{m-1} \cos\theta_m \cos\theta_{m+1} \dots \sin\theta_{m+n-2} \\
&\dots\dots\dots\dots \\
x_n &= r \cos\theta_1 \cos\theta_2 \dots \cos\theta_{m-1} \cos\theta_m \sin\theta_{m+1} \\
x_{n+1} &= r \cos\theta_1 \cos\theta_2 \dots \cos\theta_{m-1} \sin\theta_m \\
&\dots\dots\dots\dots \\
x_{n+m-1} &= r \cos\theta_1 \sin\theta_2 \\
x_{n+m} &= r \sin\theta_1
\end{aligned} \tag{2.2}$$

where $0 \le \theta_j \le \pi$ for $j = 1,\dots, n+m-2$, $0 \le \theta_{n+m-1} \le 2\pi$ and

$$r = (x_1^2 + \dots + x_{n+m}^2)^{\frac{1}{2}} \tag{2.3}$$

Under this change of variables, the polar form of the Laplace operator is given by

$$\Delta u = \sum_{i=1}^{n+m} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n+m-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Phi_1(u) \tag{2.4}$$

where the operator Φ_1 depends only on the variables $\theta_1, \dots, \theta_{n+m-1}$ and r is the Euclidean distance given by (2.3).

Now in (2.1) and (2.2) let $x_{n+j} = iy_j$ and $\theta_j = i\phi_j$ for $j = 1, \dots, m$ with $i = \sqrt{-1}$ and let $\theta_{m+j} = \psi_j$ for $j = 1,\dots,n-1$. Since $(\partial/\partial x_{n+j})^2 u = -(\partial/\partial y_j)^2 u$, $\cos(i\phi_j) = \text{ch}\,\phi_j$ and $\sin(i\phi_j) = i\,\text{sh}\,\phi_j$, the operator (2.1) reduces to the ultrahyperbolic operator

$$\Box u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=1}^{m} \frac{\partial^2 u}{\partial y_j^2} \tag{2.5}$$

On the other hand, the polar coordinate system (2.2) takes the form

$$
\begin{aligned}
x_1 &= r\, ch\phi_1 ch\phi_2 \ldots ch\phi_{m-1} ch\phi_m \cos\psi_1 \ldots \cos\psi_{n-2} \cos\psi_{n-1} \\
x_2 &= r\, ch\phi_1 ch\phi_2 \ldots ch\phi_{m-1} ch\phi_m \cos\psi_1 \ldots \cos\psi_{n-2} \sin\psi_{n-1} \\
x_3 &= r\, ch\phi_1 ch\phi_2 \ldots ch\phi_{m-1} ch\phi_m \cos\psi_1 \ldots \sin\psi_{n-2} \\
&\ldots\ldots\ldots\ldots \\
x_n &= r\, ch\phi_1 ch\phi_2 \ldots ch\phi_{m-1} ch\phi_m \sin\psi_1 \\
y_1 &= r\, ch\phi_1 ch\phi_2 \ldots ch\phi_{m-1} sh\phi_m \\
y_2 &= r\, ch\phi_1 ch\phi_2 \ldots sh\phi_{m-1} \\
&\ldots\ldots\ldots\ldots \\
y_{m-1} &= r\, ch\phi_1 sh\phi_2 \\
y_m &= r\, sh\phi_1
\end{aligned}
\tag{2.6}
$$

where r is the Lorentzian distance given by (1.2). We refer to this as "polar-hyperbolic transformation". In the polar-hyperbolic coordinate system, the operator (2.5) assumes the form

$$
\Box u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=1}^{m} \frac{\partial^2 u}{\partial y_j^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n+m-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Phi_2(u) \tag{2.7}
$$

where Φ_2 depends only on the variables $\phi_1,\ldots,\phi_m,\psi_1,\ldots,\psi_{n-1}$ and r is given in (1.2).

For example, the polar forms of Δu for n = 2, 3 are given by

$$
\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}
$$

$$
\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \frac{\cos\theta}{\sin\theta}u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right)
$$

The corresponding forms for the hyperbolic equations are given by

$$
\Box u = u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u
$$

$$
\Box u = u_{rr} + \frac{2}{r}u_r - \frac{1}{r^2}\left(u_{\theta\theta} + \frac{sh\theta}{ch\theta}u_\theta - \frac{1}{ch^2\theta}u_{\phi\phi}\right)
$$

where $\Box u = u_{xx} + u_{yy} - u_{zz}$, $r^2 = x^2 + y^2 - z^2$ and $x = r\, ch\theta\cos\phi$, $y = r\, ch\theta\sin\phi$, $z = r\, sh\theta$.

3. A FUNDAMENTAL THEOREM AND MAIN RESULT.

In [5], using his main three recursion formulas, Weinstein gave the following theorem which will be used to establish our main result.

THEOREM 1. Let $v = v(r,\psi_1,\ldots,\psi_{n-1})$ satisfy the differential equation

$$
\frac{\partial^2 v}{\partial r^2} + \frac{k}{r}\frac{\partial v}{\partial r} = \frac{1}{r^2}\Phi(v) \tag{3.1}
$$

where k is a real or complex number and Φ is a linear differential operator vanishing for v = 0 which is independent of the variable r. Then $\rho^{1-k}w$ satisfies the same equation (3.1) in the variables $\rho, \psi_1, \ldots, \psi_{n-1}$, where $\rho = 1/r$ and $w(\rho,\psi_1,\ldots,\psi_{n-1}) = v(1/\rho,\psi_1,\ldots,\psi_{n-1})$

Using Theorem 1 we can now establish an extension of Kelvin principle to ultrahyperbolic equations :

THEOREM 2. Let $u = u(x_1,\ldots,x_n,y_1,\ldots,y_m,z_1,\ldots,z_p)$ be a solution of the equation (1.1). Then

$$w = r^{-\lambda} u(\frac{x_1}{r^2},\ldots,\frac{x_n}{r^2},\frac{y_1}{r^2},\ldots,\frac{y_m}{r^2}, z_1,\ldots,z_p) \tag{3.2}$$

is also a solution of the same equation (1.1), where

$$\lambda = n + m - 2 + \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j \tag{3.3}$$

and r is the Lorentzian distance defined by (1.2).

PROOF. Let us consider the polar-hyperbolic transformation (2.6) which can be written in the following form

$$\begin{aligned} x_i &= r\, f_i(\phi,\psi), \quad i = 1,\ldots,n \\ y_j &= r\, g_j(\phi,\psi), \quad j = 1,\ldots,m \end{aligned} \tag{3.4}$$

where the notations $f_i(\phi,\psi)$, $g_j(\phi,\psi)$ or without subscripts $f(\phi,\psi)$, $g(\phi,\psi)$ denote functions of $\phi_1,\ldots,\phi_m,\psi_1,\ldots,\psi_{n-1}$.

We note that from (3.4) we have

$$\frac{\partial x_k}{\partial x_j} = \frac{\partial x_k}{\partial r}\frac{\partial r}{\partial x_j} + \sum_{i=1}^{m} \frac{\partial x_k}{\partial \phi_i}\frac{\partial \phi_i}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial x_k}{\partial \psi_i}\frac{\partial \psi_i}{\partial x_j} = \delta_{jk}$$

$$\frac{\partial y_k}{\partial y_j} = \frac{\partial y_k}{\partial r}\frac{\partial r}{\partial y_j} + \sum_{i=1}^{m} \frac{\partial y_k}{\partial \phi_i}\frac{\partial \phi_i}{\partial y_j} = \delta_{jk}$$

where δ_{jk} is the Kronecker delta. From (1.2) we have

$$\frac{\partial r}{\partial x_j} = \frac{\partial x_j}{\partial r} \; (1 \le j \le n) \quad \text{and} \quad \frac{\partial r}{\partial y_j} = \frac{-y_j}{r} \; (1 \le j \le m)$$

and we may express the partial derivatives $\partial\phi_i/\partial x_j$, $\partial\psi_i/\partial x_j$ and $\partial\phi_i/\partial y_j$ as a quotient the denominator of which is the Jacobian of the transformation (2.6). It can be shown that

$$\frac{\partial(x_1,\ldots,x_n,y_1,\ldots,y_m)}{\partial(r,\phi_1,\ldots,\phi_m,\psi_1,\ldots,\psi_{n-1})} = r^{n+m-1} h(\phi,\psi)$$

The numerator of this quotient contains obviously only a factor r^{n+m-2}, hence

$$\frac{\partial\phi_i}{\partial x_j} = \frac{1}{r} F_{ji}(\phi,\psi), \quad \frac{\partial\psi_i}{\partial x_j} = \frac{1}{r} G_{ji}(\phi,\psi), \quad \frac{\partial\phi_i}{\partial y_j} = \frac{1}{r} H_{ji}(\phi,\psi)$$

On the other hand, since

$$\frac{\partial u}{\partial x_j} = \frac{x_j}{r}\frac{\partial u}{\partial r} + \frac{1}{r}\left(\sum_{i=1}^{m} \frac{\partial u}{\partial \phi_i} F_{ji}(\phi,\psi) + \sum_{i=1}^{n-1} \frac{\partial u}{\partial \psi_i} G_{ji}(\phi,\psi)\right)$$

$$\frac{\partial u}{\partial y_j} = -\frac{y_j}{r}\frac{\partial u}{\partial r} + \frac{1}{r}\sum_{i=1}^{m} \frac{\partial u}{\partial \phi_i} H_{ji}(\phi,\psi)$$

we see that

$$\frac{1}{x_j}\frac{\partial u}{\partial x_j} = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Psi_1(u), \quad 1 \le j \le n$$

$$\frac{1}{y_j}\frac{\partial u}{\partial y_j} = -\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Psi_2(u), \quad 1 \le j \le m \tag{3.5}$$

where the operators Ψ_1 and Ψ_2 depend only on the variables $\phi_1,\ldots,\phi_m,\psi_1,\ldots,\psi_{n-1}$. If we substitute the expressions (2.7) and (3.5) into (1.1), then our equation (1.1) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\left(n + m - 1 + \sum_{i=1}^{n}\alpha_i + \sum_{j=1}^{m}\beta_j\right)\frac{\partial u}{\partial r} + \frac{1}{r^2}\Psi(u) = 0 \tag{3.6}$$

Since $1-(n+m-1 + \Sigma_{i=1}^{n}\alpha_i + \Sigma_{j=1}^{m}\beta_j) = -\lambda$, by Theorem 1

$$w = r^{-\lambda}u\left(\frac{x_1}{r^2},\ldots,\frac{x_n}{r^2},\frac{y_1}{r^2},\ldots,\frac{y_m}{r^2},z_1,\ldots,z_p\right)$$

satisfies the equation (1.1). Here we note that, since

$$\rho^2 = \sum_{i=1}^{n}(x_i/r^2)^2 - \sum_{j=1}^{m}(y_j/r^2)^2 = \frac{1}{r^2}$$

the substitution $\rho = 1/r$ in the solution u means replacing the variables x_i and y_j by x_i/r^2 and y_j/r^2, respectively.

4. REMARKS.

(i) We note that, since the r defined by (1.2) is not real for $\Sigma_{i=1}^{n} x_i^2 < \Sigma_{j=1}^{m} y_j^2$, the solutions of (1.1) is valid only in the domain D x Ω, where

$$D = D_n \times D_m = \{(x,y) : x \in D_n,\ y \in D_m,\ \sum_{i=1}^{n} x_i^2 > \sum_{j=1}^{m} y_j^2\}$$

is a hyperconoidal domain in R^{n+m}. Here D_n and D_m are the spherical domains centered at the origins of R^n and R^m, respectively, and $\Omega \subset R^p$ is the regularity domain of u with respect to the variable z.

(ii) In equation (1.1) if we have addition instead of subtraction of the two summations, then Theorem 2 remains valid, where

$$r^2 = \sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m} y_j^2$$

This includes Weinstein's [5] and Altin's [2] results.

(iii) In the special case Pu = γu where γ= const. the formula (3.2) gives the result obtained in [2] .

(iv) If we multiply both sides of the equation (1.1) by -1, we get the equation

$$-L(u) = \sum_{j=1}^{m} \left(\frac{\partial^2 u}{\partial y_j^2} + \frac{\beta_j}{y_j}\frac{\partial u}{\partial y_j}\right) - \sum_{j=1}^{n} \left(\frac{\partial^2 u}{\partial x_j^2} + \frac{\alpha_j}{x_j}\frac{\partial u}{\partial x_j}\right) + \frac{1}{r^2}P(u) = 0$$

where $r_1^2 = \Sigma_{j=1}^{m} y_j^2 - \Sigma_{j=1}^{n} x_j^2 = -r^2$. This shows that if $u(x_1,\ldots,x_n,y_1,\ldots,y_m,z_1,\ldots,z_p)$ is a solution of the equation (1.1), then by Theorem 2

$$w_1 = r_1^{-\lambda} u\left(\frac{x_1}{r^2},\ldots,\frac{x_n}{r^2},\frac{y_1}{r^2},\ldots,\frac{y_m}{r^2},z_1,\ldots,z_p\right)$$

is also a solution of the same equation (1.1), where λ is given by (3.3). It is clear that this solution is valid in a different domain where $r_1^2 > 0$, that is, $r^2 < 0$.

(v) It is clear that by a simple linear transformation, Theorem 2 also holds for a more general equation of the form

$$\sum_{i=1}^{n} a_i^2\left(\frac{\partial^2 u}{\partial \zeta_i^2} + \frac{\alpha_i}{\zeta_i-\zeta_i^o}\frac{\partial u}{\partial \zeta_i}\right) - \sum_{j=1}^{m} b_j^2\left(\frac{\partial^2 u}{\partial \eta_j^2} + \frac{\beta_j}{\eta_j-\eta_j^o}\frac{\partial u}{\partial \eta_j}\right) + \frac{1}{r^2}P(u) = 0$$

where a_i, b_i, α_i, β_i are real parameters ($a_j \neq 0$, $b_i \neq 0$), $\zeta^o = (\zeta_1^o,\ldots,\zeta_n^o)$ and $\eta^o = (\eta_1^o,\ldots,\eta_m^o)$ are fixed points in D_n and D_m, respectively. Here

$$w = r^{-\lambda} u\left(\frac{\zeta_1-\zeta_1^o}{a_1 r^2},\ldots,\frac{\zeta_n-\zeta_n^o}{a_n r^2},\frac{\eta_1-\eta_1^o}{b_1 r^2},\ldots,\frac{\eta_m-\eta_m^o}{b_m r^2},z_1,\ldots,z_p\right)$$

where

$$r^2 = \sum_{i=1}^{n}\left(\frac{\zeta_i-\zeta_i^o}{a_i}\right)^2 - \sum_{j=1}^{m}\left(\frac{\eta_j-\eta_j^o}{b_j}\right)^2$$

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