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ANALYSIS OF STRAIGHTENING FORMULA

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ABSTRACT. The straightening formula has been an essential part of a proof showing that the set of standard bitableaux (or the set of standard monomials in minors) gives a free basis for a polynomial ring in a matrix of indeterminates over a field .The straightening formula expresses a nonstandard bitableau as an integral linear cobmbination of standard bitableaux. In this paper we analyse the exchanges in the process of straightening a nonstandard pure tableau of depth two.We give precisely the number of steps required to straighten a given violation of a nonstandard tableau.We also characterise the violation which is eliminated in a single step.

Keywords and Phrases. *Bitableaux, standard bitableaux, unitableaux of pure length and depth two, monomials in minors, Straightening formula, oddity function,violation, good violation, number of steps to straighten a nonstandard unitableau .*
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1. INTRODUCTION.

The straightening formula has been an intregral part of the theorem showing that the set of all standard monomials in minors of a matrix of indeterminates form a free basis of the polynomial ring in those indeterminates. It tranforms a nonstandard bitableau into an integral linear combination of the standard bitableaux, which makes sense only after using a correspondence between bitableaux and monomials in minors. The straightening formula is given in Rota-Doubillet -Stein [*1*] first and given again and exploited greatly in Desarmenien-Kung-Rota [*2*], DeConcini-Procesi [*3*], DeConcini - Eisenbud - Procesi [*4*]. Abhyankar [*5*] gives a proof of the above mentioned theorem by explicitly counting the dimension of a vector space generated by all the standard bitableaux of area V and length less than or equal to p, and deduces the result that the ideal generated by the p by p minors of a matrix of indeterminates is Hilbertian. In this exposition one also finds a form of the straightening formula which is very amenable for analysis. In a sequel of [*5*], Abhyankar has proved the Hilbertianness of a much more general determinantal ideal following the same strategy of counting and straightening, eliminating the proof of linear independence of standard bitableaux [*6*]. He also states the straightening formula in much general form and proposes the

Problem: Given a nonstandard bitableau T, if $T = \sum c_i T_i$ is the expression for T given by the Straightening formula where T_i 's are standard bitableaux; can one determine c_i's in terms of T ?

He defines the final integer function there which helps to give the coefficients c_1. He also gives a recursion satisfied by this fin function, and states a problem of finding fin in terms of a given nonstandard bitableau. More knowledge about the number of steps required to straighten a given nonstandard bitableau will help finding this fin function.

In this paper we analyse the formula as given in [5]. For an analysis of the straightening formula for a nonstandard unitableau it is enough to look at a nonstandard pure unitableau of depth two. We state a form of the straightening formula using an arbitrary violation. The proof of this form is identical to the proof in [5]. We give an exact number of steps required to eliminate the violation from all unitableaux obtained in the straightening. As a part of our proof we give a detailed analysis of exchanges in the straightening and characterise a violation which gets eliminated in a single step (a good violation).

2. NOTATION AND TERMINOLOGY.

Let $[\, X_{ij} \,]_{1\le i \le m, 1 \le j \le n}$ be a matrix all of whose entries are indeterminates over a field K. Let Y be an m by m+n matrix formed by keeping the first m columns of Y to be those of X and putting the (n+i)th column to be (m-i+1)th column of an m by m identity matrix for $1 \le i \le m$. Throughout the discussion we use the word "minor " with the meaning as "determinant of a minor". In the proof of a theorem, showing the set of standard monomials in maximal size minors of Y to be a free basis of K[Y], the spanning part of it is done by repeated applications of the straightening formula to a nonstandard unitableau of pure length m and bounded by m+n. Using this, one proves that the standard monomials in minors of X form a free basis of K[X] by invoking the correspondence between all minors of X and the maximal size minors of Y, and then the correspondence between tableaux and monomials in minors of X.

A univector of length m and bounded by p is an increasing sequence of m positive integers which are bounded by p. To a univector of length m and bounded by p there corresponds an m by m (a maximal size) minor of an m by p matrix of indeterminates which is formed by picking up the corresponding m columns. A unitableau of depth d is a sequence of d univectors, written as A(1)A(2)....A(d). By a pure unitableau of length m and bounded by p we mean a unitabeau each constituent univector has length m and is bounded by p. Given two univectors

$$A = (\, A_1 < A_2 < ... < A_p \,) \text{ and } B = (\, B_1 < B_2 < .... < B_q \,),$$

we say that

$$A \le B \quad \text{if } p \ge q \text{ and } A_i \le B_i \quad \text{for } 1 \le i \le q. \tag{2.1}$$

A unitableau A(1)A(2)...A(d) is standard if $A(i) \le A(i+1)$ for $1 \le i \le d-1$. A monomial in maximal size minors of an m by p matrix of indeterminates corresponding to a standard pure unitableau of length m and bounded by p is said to be standard.

For analysis one has to concentrate on unitableaux of pure length m and depth two. Let all unitableaux be bounded by p hereonwards. Let mom (X, AB) be a monomial in maximal size minors of an m by p matrix X of indeterminates over a field K. Given a unitableau AB of pure length m and depth two as

$$\begin{matrix} A_1 < A_2 < ... < A_m \\ B_1 < B_2 < ... < B_m \end{matrix} , \tag{2.2}$$

we say that the i-th column is straight if $A_i \le B_i$ and the i-th column is a violation if $A_i > B_i$ and we define the violation set as

$$V(AB) = \{ i : 1 \le i \le m , A_i > B_i \}, \tag{2.3}$$

and the oddity function for AB by putting for each i with $1 \le i \le m$,

$$N[AB](i) = \mathrm{card}(\{ k : 0 \le k \le i-1 , A_{i-k} > B_{i+k} \}) . \tag{2.4}$$

We note that AB is standard if $V(AB) = \emptyset$ and if $N[AB](i) = 0$ for $1 \le i \le m$. We say that the v-th column is good if $v \in V(AB)$ and $N[AB](v) = 1$.

3. STRAIGHTENING FORMULA.

We are giving here the straightening formula in [5] using any violation.

THEOREM 1: Let AB be a nonstandard pure unitableau of length m and depth two and bounded by p. Let $v \in V(AB)$, and let

$$\underline{A}[v] = \{A_k : v \le k \le m \} \text{ and } \underline{B}[v] = \{B_k : 1 \le k \le v \}. \tag{3.1}$$

We form a set $\underline{E}[v]$ as

$$\underline{E}[v] = \{ (a,b) : a \subset \underline{A}[v] , b \subset \underline{B}[v] , (A \setminus a) \cap b = \emptyset = (B \setminus b) \cap a \text{ and}$$

$$\mathrm{card}(a) = \mathrm{card}(b) \ne 0 \} \tag{3.2}$$

For $E = (a,b) \in \underline{E}[v]$, let sat (A , E) and sat (B, E) denote the univectors of length m formed by sets respectively $(A \setminus a) \cup b$ and $(B \setminus b) \cup a$, and sat [AB, E] denote a satellite unitableau corresponding to $E \in E[v]$ written as

$$\mathrm{sat}[AB , E] = \mathrm{sat}(A , E)\, \mathrm{sat}(B , E) . \tag{3.3}$$

We have

$$\mathrm{mom}(X , AB) = \sum_{E \in \underline{E}[v]} \#(AB , E)\, \mathrm{mom}(X , \mathrm{sat}[AB , E]) \tag{3.4}$$

where # (AB , E) is + or - according to the sign in the Laplace development and the signs required to form sat (A , E) and sat (B , E) for all $E \in \underline{E}[v]$.

PROOF. The proof of this theorem is the same as that in [5] which is done for $v = \min V(AB)$ there.

We say that a pure unitableau A*B* of length m occurs in the first step of straightening of AB with respect to v if

$$v \in V(AB) \text{ and } A^* = \mathrm{sat}(A , E) \text{ and } B^* = \mathrm{sat}(B, E) \text{ for some } E \in \underline{E}[v] , \tag{3.5}$$

we write it as $A^*B^* \varepsilon S[AB, v, 1]$. We say that $A^{**}B^{**}$ occurs in the second step of the straightening of AB with respect to v if

$$v \varepsilon V(A^*B^*) \text{ and } A^{**}B^{**} \varepsilon S[A^*B^*, v, 1] \quad (3.6)$$

for some $A^*B^* \varepsilon S[AB, v, 1]$ and we write it as $A^{**}B^{**} \varepsilon S[AB, v, 2]$; and we may define $S[AB, v, s]$ for $s \geq 1$ inductively.

4. ANALYSIS OF EXCHANGES.

In this section let AB be as given above and $v \varepsilon V(AB)$ and $A^*B^* \varepsilon S[AB, v, 1]$ for $(a, b) \varepsilon \underline{E}[v]$ as described in the Theorem 1. Let $N[AB](v) = q + 1$ and $\text{card}(a) = \text{card}(b) = r$. We note the condition

$$(A \setminus a) \cap b = \emptyset = (B \setminus b) \cap a \quad (4.0)$$

which we are going to use strongly .

THEOREM 2: If $\text{card}(a) = \text{card}(b) = r \geq q + 1$,

$$v \varepsilon V(A^*B^*) \text{ for all } A^*B^* \varepsilon S[AB, v, 1].$$

PROOF: Since every element of $\{A_1, A_2, \ldots, A_{v-q-1}\} \cup b$ is smaller than every element of $\{A_{v-q}, A_{v-q+1}, \ldots, A_m\} \setminus a$, and by (4.0) and the given ,

$$\text{card}(\{A_1, A_2, \ldots, A_{v-q-1}\} \cup b) = v - q - 1 + r \geq v, \quad (4.1)$$

we have $B^*_v \ \varepsilon \{A_1, A_2, \ldots, A_{v-q-1}\} \cup b$.

Since every element of $\{B_{v+q+1}, B_{v+q+2}, \ldots, B_m\} \cup a$ is greater than every element of $\{B_1, B_2, \ldots, B_{v+q}\} \setminus b$, and by (4.0) and the given,

$$\text{card}(\{B_{v+q+1}, B_{v+q+2}, \ldots, B_m\} \cup a) = m - v - q + r \geq m - v + 1, \quad (4.2)$$

we have $A^*_v \ \varepsilon \{B_{v+q+1}, B_{v+q+2}, \ldots, B_m\} \cup a$.

By noting that every element of $\{A_1, A_2, \ldots, A_{v-q-1}\} \cup b$ is smaller than or equal to every element of $\{B_{v+q+1}, B_{v+q+2}, \ldots, B_m\} \cup a$ as $A_{v-q-1} \leq B_{v+q+1}$ and $A_v > B_v$, we have $A^*v \leq B^*_v$, and so $v \varepsilon V(A^*B^*)$ for all $A^*B^* \varepsilon S[AB, v, 1]$.

COROLLARY 3: A violation v in V (AB) is good if and only if $v \notin V(A^*B^*)$ for all $A^*B^* \varepsilon S[AB, v, 1]$ if and only if $A_{v+1} \geq B_{v-1}$.

PROOF: It follows from above theorem by noting that v is good if $q = 0$ and $\text{card}(a) = \text{card}(b) \neq 0$ forces $r \geq 1$. The second part follows from the definition of $N[AB](v)$.

LEMMA 4: If $1 \leq \text{card}(a) = \text{card}(b) = r \leq q$, $v \varepsilon V(A^*B^*)$ for those $A^*B^* \varepsilon S[AB, v, 1]$.

PROOF: Since every element of $\{A_{v-q}, A_{v-q+1}, \ldots, A_m\} \setminus a$ is greater than every element of $\{A_1, A_2, \ldots, A_{v-q-1}\} \cup b$, and by (4.0) and $q - r \geq 0$,

$$\text{card}(\{A_1, A_2, \ldots, A_{v-q-1}\} \cup b) = v - q - 1 + r \leq v - 1, \qquad (4.3)$$

we have $A^*_v \ \varepsilon \ \{A_{v-q}, A_{v-q+1}, \ldots, A_m\} \backslash a$.

Since every element of $\{B_1, B_2, \ldots, B_{v+q}\} \backslash b$ is smaller than every element of $\{B_{v+q+1}, B_{v+q+2}, \ldots, B_m\} \cup a$,and by (4.0) ,

$$\text{card}(\{B_1, B_2, \ldots, B_{v+q}\} \backslash b) = v + q - r \geq v, \qquad (4.4)$$

we have $B^*_v \ \varepsilon \ \{B_1, B_2, \ldots, B_{v+q}\} \backslash b$.

By noting that every element of $\{A_{v-q}, A_{v-q+1}, \ldots, A_m\} \backslash a$ is greater than every element of $\{B_1, B_2, \ldots, B_{v+q}\} \backslash b$, we have

$$A^*_v > B^*_v \quad , \qquad (4.5)$$

and so $v \ \varepsilon \ V(A^*B^*)$ for those $A^*B^* \ \varepsilon \ S[AB, v, 1]$.

THEOREM 5: For all $A^*B^* \ \varepsilon \ S[AB, v, 1]$, for $1 \leq i \leq m$,

$$N[A^*B^*](i) \leq N[AB](i) .$$

There exists (a , b) in $\underline{E}$ [v] such that for the corresponding A^*B^*, we have

$$N[A^*B^*](v) = N[AB](v) - 1 . \qquad (4.6)$$

PROOF: Putting $N[AB](i) = n(i)$ for $1 \leq i \leq m$,by the definition of N , we have

$$A^*_{i-n(i)} \leq A_{i-n(i)} \leq B_{i+n(i)} \leq B^*_{i+n(i)}$$

since $A^*_k \leq A_k$ and $B^*_k \geq B_k$ for $1 \leq k \leq m$ as every element in A which is removed is being replaced by a smaller element in B and every element in B which is removed is being replaced by a greater element in A. From the inequality it follows that for $1 \leq i \leq m$,

$$N[A^*B^*](i) \leq N[AB](i) . \qquad (4.7)$$

To prove the second part we have to first show that $A^*_{v-q} \leq B^*_{v+q}$ for all E in $\underline{E}$ [v] by letting $n(v) = q + 1$. Since every element of $\{A_1, A_2, \ldots, A_{v-q-1}\} \cup b$ is smaller than every element of $\{A_{v-q}, A_{v-q+1}, \ldots, A_m\} \backslash a$, and by (4.0) and $r - 1 \geq 0$,

$$\text{card}(\{A_1, A_2, \ldots, A_{v-q-1}\} \cup b) = v - q - 1 + r \geq v - q, \qquad (4.8)$$

we have $A^*_{v-q} \ \varepsilon \{A_1, A_2, \ldots, A_{v-q-1}\} \cup b$.

Since every element of $\{B_{v+q+1}, B_{v+q+2}, \ldots, B_m\} \cup a$ is greater than every element of $\{B_1, B_2, \ldots, B_{v+q}\} \backslash b$, and by (4.0),

$$\text{card}(\{B_{v+q+1}, B_{v+q+2}, \ldots, B_m\} \cup a) = m - v - q + r \geq m - v - q + 1, \qquad (4.9)$$

we have $B^*_{v+q} \varepsilon \{B_{v+q+1}, B_{v+q+2}, \dots, B_m\} \cup a$.

By noting that every element of $\{A_1, A_2, \dots, A_{v-q-1}\} \cup b$ is smaller than or equal to every element of $\{B_{v+q+1}, B_{v+q+2}, \dots, B_m\} \cup a$ as $A_{v-q-1} \le B_{v+q+1}$ and $A_v > B_v$, we have

$$A^*_{v-q} \le B^*_{v+q}, \tag{4.10}$$

and so $N[A^*B^*](v) < q = N[AB](v)$.

By putting $a = \{A_u\}$ and $b = \{B_w\}$ where $A_u = \min \underline{A}[v] \setminus B$ and $B_w = \max \underline{B}[v] \setminus A$, we note that for the corresponding A^*B^*,

$$A^*_k = A_{k-1} \quad \text{for} \quad v-q+1 \le k \le u \tag{4.11.1}$$

and

$$B^*_k = B_{k+1} \quad \text{for} \quad w \le k \le v+q-1 \tag{4.11.2}$$

as

$$\text{card}(\{A_1, A_2, \dots, A_{v-q-1}\} \cup b) = v-q \tag{4.12.1}$$

and

$$\text{card}(\{B_{v+q+1}, B_{v+q+2}, \dots, B_m\} \cup a) = m-v-q, \tag{4.12.2}$$

it follows that

$$A^*_{v-q+1} = A_{v-q} > B_{v+q} = B^*_{v+q-1}. \tag{4.13}$$

By the definition of N and knowing that $A^*_{v-q} \le B^*_{v+q}$, we have

$$N[A^*B^*](v) = N[AB](v) - 1. \tag{4.14}$$

The existence of u is assured by the definition of v and

$$\text{card}(\underline{A}[v]) > \text{card}(\{B_{v+1}, B_{v+2}, \dots, B_m\}). \tag{4.15.1}$$

The existence of w is assured by the definition of v and

$$\text{card}(\underline{B}[v]) > \text{card}(\{A_1, A_2, \dots, A_{v-1}\}). \tag{4.15.2}$$

THEOREM 6: If we apply the straightening formula to AB repeatedly using $v \varepsilon V(AB)$, for all $A^*B^* \varepsilon S[AB, v, N[AB](v)]$ we have that $v \notin V(A^*B^*)$. The number $N[AB](v)$ is the smallest integer s such that for all $A^*B^* \varepsilon S[AB, v, s]$ we have that $v \notin V(A^*B^*)$.

PROOF: It follows immediately from the previous theorem and recalling that

$$v \notin V(A^*B^*) \text{ if } N[AB](v) = 0. \tag{4.16}$$

From the above Theorem it is clear that starting with AB and applying the straightening formula $N[AB](v)$ times we can express AB (or mom (X , AB)) as an integral linear combination of unitableaux which do not have v in their violation sets. In this process we do not perform any cancellations as we go. With this in mind, talking of the number of steps required to straighten a nonstandard unitableau is not confusing. By the above theorem and noting that the oddity function drops at each step, it follows that a nonstandard unitableau AB can be straightened in at most $\Sigma_{1 \leq i \leq m} N[AB](i)$ steps.

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